The Logistic Equation
(a model for population growth)
\[ P'(t) = r P(t)(1 - P(t)/K) \]

Here \( r = 0.75 \) and \( K = 10 \).

We plot a solution curve corresponding to \( P(0) = 5 \).
The Logistic Equation with arrows for direction field and more solution curves
Here \( y(0)=5 \) for the left-most curve.

The logistic equation is an example of an autonomous ODE since the right hand side is independent of \( t \). This means if \( y(t) \) solves the ODE, so does \( y(t-c) \) for any constant \( c \). The graph of \( y(t-c) \) looks the same as that of \( y(t) \) except shifted to the right by \( c \). In the picture we shift a solution by 4, 8 and 12.
Exact Solution. The Logistic equation is separable.

\[ \frac{dy}{dt} = ry \left( 1 - \frac{y}{K} \right) \]

\[ \frac{dy}{\left( 1 - \frac{y}{K} \right)} = rdt \]

Partial fractions gives

\[
\left( \frac{1}{Y} + \frac{1}{K - y} \right) dy = rdt
\]

Solving

\[ \ln|y| - \ln|K-y| = rt + C' \]
\[ \ln|y/(K-y)| = rt + C' \]

exponentiating

\[ y/(K-y) = \pm e^{rx+C'} = \pm e^{rt} e^{C'} \]
\[ y/(K-y) = C \ e^{rt} \quad \text{where} \quad C = \pm e^{C'} \]
Now to find $y$

\[
\frac{y}{(K-y)} = C \ e^{rt}
\]

\[
y = C(K-y) \ e^{rt} = CKe^{rt} - Cy^{rt}
\]

\[
y + Cy^{rt} = CK \ e^{rt}
\]

\[
y(1 + Ce^{rt}) = CK \ e^{rt}
\]

\[
y = \frac{CKe^{rt}}{(1+C \ e^{rt})}
\]

Determine the constant $C$ from the initial condition. If the initial condition is $y(0) = K/2$, for example, then

\[
\frac{K}{2} = CK e^{r0} \ (1+C \ e^{r0})
\]

which implies

\[
1 = 2C/(1+C)
\]

\[
1 + C = 2C
\]

\[
1 = C
\]

\[
C = 1
\]

So for this initial condition, our solution is

\[
y = K \ \frac{e^{rt}}{1+e^{rt}} = K \ \frac{1}{e^{-rt} + 1}
\]

Long time behavior: As $t \rightarrow \infty$, $y \rightarrow K$
There are lots of population models. The simplest is $y'(t)=ry(t)$, $r>0$ constant. The solution is $y(t)=Ce^{rt}$. Here $C=y(0)$. This means the population grows exponentially.

The number of people on the earth in 1961 was around 3 billion and increasing at a rate of $r=.02$ per year. One can check whether this predicts the population now.

$$y(45) = 3 \times (10^9) \times (e^{0.02\times45}) \approx 7 \times 10^9$$
This isn’t too far off. As of May 4, 2006, the U.S. Census Bureau says the population of the world is about 6,513,823,130. See the website: http://www.census.gov/ipc/www/world.html
The equation predicts the population $3.6 \times 10^{15}$ by the year 2670. But the total surface area of the planet is only $1.8 \times 10^{15}$ square feet !!!! The logistic equation reflects the fact that the population cannot increase indefinitely.
Exact 1st Order ODEs
Suppose we have an equation
\[ u(x,y) = c, \quad c = \text{constant and } y = y(x) \]
Using the chain rule for functions of several variables, differentiate the equation with respect to \( x \) and get
\[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y'(x) = 0 \]
Such an ODE is called exact.
Definition. \( M+N(dy/dx) = 0 \) is exact \( \iff \)
there is a function \( u(x,y) \) so that
\[ M = u_x \quad \text{and} \quad N = u_y. \]
Assuming the partials continuous on some open rectangle, then \( u_{xy}=u_{yx} \) and this means \( M_y=N_x \). It turns out the converse is also true; meaning that (assuming \( M, N, M_y, N_x \) continuous in an open rectangle)
\[ M+N(dy/dx) = 0 \text{ is exact } \iff M_y=N_x. \]
There is a proof of this theorem in Section 2.6 of the text.
Example. \((3x^2+2xy)dx+(x^2+3y^2)dy=0\)

Here \(M = 3x^2+2xy\) and \(N= x^2+3y^2\)
\(M_y=2x=N_x\) so the ODE is exact.

How to find \(u\) such that \(M=u_x, N=u_y\)?

\[u_x = M = 3x^2+2xy\]

Integrate with respect to \(x\), holding \(y\) fixed

\[u(x,y) = \int (3x^2 + 2xy)dx + K(y)\]  \(1\)

\[u = x^3+x^2y+K(y).\]

To find \(K(y)\), recall

\[u_y = N = x^2+3y^2.\]  \(2\)

Formula (1) implies \(u_y = x^2 + K'(y).\)
So using (2) \(x^2 + 3y^2 = x^2 + K'(y).\)
This means \(3y^2 = K'(y).\)
It follows that \(K(y)=y^3+c'.\)

Therefore by (1) we can take

\[u = x^3+x^2y + y^3.\]

Solutions of our ODE are given implicitly by
\[x^3+x^2y+y^3=c\] where \(c\) is a constant.
Matlab draws the direction field for the exact equation \((3x^2+2xy)dx+(x^2+3y^2)dy=0\)

\[ y' = -\frac{(3x^2+2xy)}{(x^2+3y^2)} \]
Mathematica draws the contours for the implicit solution

\[ u = x^3 + x^2y + y^3. \]

Each line represents \( u = \) constant and the colors tell how big the constant is.
Compare

\[ y' = -\frac{3x^2 + 2xy}{x^2 + 3y^2} \]
We can extend the method of exact ODEs using an integrating factor.

If \( M \, dx + N \, dy = 0 \)

is not exact, sometimes one can multiply by some function \( v(x,y) \) so that

\[ v \, M \, dx + v \, N \, dy = 0 \]

is exact.

Example. \( y \, dx - (x+y^3) \, dy = 0 \)

is not exact since

\[ M_y = 1 \quad \text{and} \quad N_x = -1. \]

Rewrite the ODE as

\[ y \, dx - x \, dy = y^3 \, dy. \]

Recall that

\[ d\left( \frac{x}{y} \right) = \frac{y \, dx - x \, dy}{y^2}. \]

This makes us want to multiply the ODE by \( v = 1/(y^2), \) assuming \( y \neq 0. \)

That gives

\[ d(\frac{x}{y}) = y \, dy. \]

Integration yields

\[ \frac{x}{y} = (1/2) \, y^2 + c. \]

So we take

\[ u(x,y) = \frac{x}{y} - (1/2) \, y^2. \]

The implicit solution of our ODE is (for \( y \neq 0) \)

\[ \left( \frac{x}{y} \right) - (1/2) \, y^2 = c. \]