Math 20D Homework 2 Solutions

2.2.2.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2}{y(1+x^3)} \\
y\,dy &= \frac{x^2\,dx}{1+x^3} \\
\int y\,dy &= \int \frac{x^2\,dx}{1+x^3} \\
\frac{y^2}{2} &= \frac{1}{3}\ln(1+x^3)+c \\
y^2 &= \frac{2}{3}\ln(1+x^3)+2c \\
\mathbf{y} &= \sqrt{\mathbf{\frac{2}{3}}\ln(\mathbf{1}+\mathbf{x^3})+\mathbf{C}}.
\end{aligned}$$

2.2.10.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1-2x}{y} \\
y\,dy &= (1-2x)\,dx \\
\int y\,dy &= \int (1-2x)\,dx \\
\frac{y^2}{2} &= -x^2+x+c \\
y^2 &= -2x^2+2x+2c \\
y &= \pm\sqrt{-2x^2+2x+C}
\end{aligned}$$

If $y(1) = -2$,

$$\begin{aligned}
-2 &= -\sqrt{2(1^2)-2(1)+C} \\
4 &= C \\
\mathbf{y} &= -\sqrt{-\mathbf{2x^2}+\mathbf{2x}+\mathbf{4}}.
\end{aligned}$$

This is defined iff the expression under the square root, $-2x^2+2x+4$, is positive, which happens when $x \in (-1,2)$. We exclude the boundary points, where the square root is zero, because $\frac{dy}{dx}$ is not defined there.

2.2.11.

$$\begin{aligned}
ye^{-x}\,dy &= -x\,dx \\
y\,dy &= -xe^{x}\,dx \\
\int y\,dy &= \int -xe^{x}\,dx \\
\frac{y^2}{2}+c &= \int -xe^{x}\,dx \\
u=-x, dv=e^{x}\,dx \Rightarrow du=-dx, v=e^{x} & \\
\frac{y^2}{2}+c &= -xe^{x}-\int e^{x}\,(-dx) \\
\frac{y^2}{2}+c &= -xe^{x}+e^{x} \\
y^2 &= -2xe^{x}+2e^{x}-2c \\
y &= \pm\sqrt{-2xe^{x}+2e^{x}+C}
\end{aligned}$$

If $y(0)=1$,

$$\begin{aligned}
1 &= \sqrt{-2(0)e^{0}+2e^{0}+C} \\
1^2 &= 2+C \\
-1 &= C \\
\mathbf{y} &= \sqrt{\mathbf{-2xe^{x}+2e^{x}-1}}.
\end{aligned}$$

This is defined iff $-2xe^{x}+2e^{x}-1$ is nonnegative. There is no closed algebraic expression for the resulting domain; it is approximately $x \in (-1.68, 0.77)$.

2.2.20.

$$\begin{aligned} y^2\,dy &= (1-x^2)^{-1/2}\arcsin x\,dx \\ \int y^2\,dy &= \int (1-x^2)^{-1/2}\arcsin x\,dx \\ \frac{y^3}{3}+c &= \int (1-x^2)^{-1/2}\arcsin x\,dx \\ u = \arcsin x, du = (1-x^2)^{-1/2}\,dx & \\ \frac{y^3}{3}+c &= \frac{(\arcsin x)^2}{2} \\ y^3 &= \frac{3}{2}(\arcsin x)^2 + C \\ y &= (\frac{3}{2}(\arcsin x)^2 + C)^{1/3} \end{aligned}$$

If $y(0) = 1$,

$$\begin{aligned} 1 &= (\frac{3}{2}(\arcsin 0)^2 + C)^{1/3} \\ 1 &= C \\ \mathbf{y} &= \mathbf{(\frac{3}{2}(\arcsin x)^2 + 1)^{1/3}} \end{aligned}$$

This is defined everywhere.

2.3.4. Let $S(t)$ be the amount of salt in the tank at time t before the tank overflows, in pounds. Then

$$\begin{aligned} \frac{dS}{dt} &= \text{(rate at which salt is entering tank)} - \text{(rate at which it is leaving)} \\ &= \text{(3 gal water/min} \cdot \text{1 lb salt/gal water)} - \text{(2 gal water/min} \cdot \frac{\text{salt in tank}}{\text{water in tank}}) \\ &= 3 - 2 \cdot \frac{S}{200+t} \end{aligned}$$

Solving this equation,

$$\begin{aligned} S' + \frac{2}{200+t}S &= 3 \\ \frac{d}{dt}[S \cdot (200+t)^2] &= 3(200+t)^2 \\ S(200+t)^2 &= \int 3(200+t)^2\,dt \\ &= (200+t)^3 + c \\ S &= 200 + t + C(200+t)^{-2} \end{aligned}$$

Since $S(0) = 100$,

$$\begin{aligned} 100 &= 200 + 0 + C(200+0)^{-2} \\ -100 &= \frac{C}{40000} \\ -4000000 &= C \\ \mathbf{S} &= \mathbf{200 + t - \frac{4000000}{(200+t)^2}} \end{aligned}$$

When the tank is on the point of overflowing, at $t = 300$, the salt concentration is

$$\begin{aligned} \frac{S(300)}{500} &= \frac{500 - \frac{4000000}{500^2}}{500} \\ &= \frac{484}{500} \\ &= \mathbf{0.968} \text{ lb/gal water.} \end{aligned}$$

The theoretical limiting concentration, obtained by taking the limit of $S(t)$ as $t \to \infty$, is 1 lb/gal water. 0.968 is already very close.

2.3.7.a. Let $S(t)$ be the value of the investment at time t; then $S(t) = S_0 e^{rt}$. We wish to find T for which $S(T) = 2S(0)$.

$$\begin{aligned} S_0 e^{rT} &= 2S_0 e^{r\cdot 0} \\ e^{rT} &= 2 \\ rT &= \ln 2 \\ \mathbf{T} &= \frac{\ln \mathbf{2}}{\mathbf{r}}. \end{aligned}$$

2.3.7.b. $T = \frac{\ln 2}{0.07} \approx \mathbf{9.90}$ years.
2.3.7.c.

$$\begin{aligned} 8 &= \frac{\ln 2}{r} \\ r &= \frac{\ln 2}{8} \\ &\approx \mathbf{8.66}\%. \end{aligned}$$

2.3.10.a. We calculate the size of the mortgage that would be paid off at the end of 20 years if the maximum \$800/month is spent.

Let $M(t)$ be the unpaid balance of the mortgage at time t (in years). (Months would be fine too, as long as you use consistent units.) Then

$$\begin{aligned}
\frac{dM}{dt} &= \text{interest} - \text{payment} \\
&= 0.09M - 9600 \\
M' - 0.09M &= -9600 \\
\frac{d}{dt}[M \cdot e^{-0.09t}] &= -9600e^{-0.09t} \\
Me^{-0.09t} &= \frac{9600}{0.09}e^{-0.09t} + c \\
M &= \frac{320000}{3} + Ce^{0.09t}
\end{aligned}$$

Since $M(20) = 0$,

$$\begin{aligned}
0 &= \frac{320000}{3} + Ce^{1.8} \\
-\frac{320000}{3} &= Ce^{1.8} \\
-\frac{320000e^{-1.8}}{3} &= C \\
C &\approx 17631.89 \\
M &\approx \frac{320000}{3} - 17631.89e^{0.09t} \\
\mathbf{M(0)} &\approx \mathbf{89034.78}.
\end{aligned}$$

2.3.10.b. The total amount paid over the term of the mortgage is \$800/month times 240 months, \$192000. \$89034.78 of that goes to the mortgage principal, so the rest, **\$102965.22**, is interest.

2.3.12.a. If t is in years, $Q(5730) \approx 0.5Q(0)$, since the half-life is about 5730 years. The solution to the differential equation $Q' = -rQ$ is $Q = Q_0e^{-rt}$. So,

$$\begin{aligned}
Q_0e^{-5730r} &= 0.5Q_0e^{-0\cdot r} \\
e^{-5730r} &= 0.5 \\
-5730r &= \ln 0.5 \\
r &= \frac{\ln 2}{5730} \\
\mathbf{r} &\approx \mathbf{0.000121}.
\end{aligned}$$

2.3.12.b. $\mathbf{Q(t) = Q_0e^{-0.000121t}}$.

2.3.12.c.

$$\begin{aligned} Q(t) &= 0.2Q_0 \\ Q_0e^{-0.000121t} &= 0.2Q_0 \\ e^{-0.000121t} &= 0.2 \\ -0.000121t &= \ln 0.2 \\ t &= \frac{\ln 5}{0.000121} \\ \mathbf{t} &\approx \mathbf{13305} \text{ years.} \end{aligned}$$

2.4.2. We divide both sides by $t(t-4)$ (getting $y' + \frac{1}{t(t-4)}y = 0$) to convert the equation into a form Theorem 2.4.1 applies to. Applying the theorem, we observe that $\frac{1}{t(t-4)}$ is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, and $(4, \infty)$. The initial condition involves $t_0 = 2$, so we conclude that a solution of the initial value problem is certain to exist on $(\mathbf{0}, \mathbf{4})$.

2.4.7. $f = \frac{t-y}{2t+5y}$ is continuous everywhere except the line $y = -0.4t$, while $\frac{\partial f}{\partial y} = \frac{(2t+5y)(-1)-(t-y)(5)}{(2t+5y)^2} = \frac{-7t}{(2t+5y)}$ is continuous in the same region. So, the hypotheses of Theorem 2.4.2 are satisfied **everywhere in the *ty*-plane except the line $\mathbf{y = 0.4t}$**.

2.4.22.a. Plugging in $y_1(t) = 1 - t$, we get

$$\begin{aligned} \frac{d}{dt}[1-t] &= \frac{-t + (t^2 + 4(1-t))^{1/2}}{2} \\ -1 &= \frac{-t + (t^2 + 4 - 4t)^{1/2}}{2} \\ -1 &= \frac{-t + |t-2|}{2} \end{aligned}$$

Both sides are equal iff $t - 2$ is nonnegative, so $\mathbf{y_1}$ **is valid for $\mathbf{t \geq 2}$**.

Plugging in $y_2(t) = -t^2/4$, we get

$$\begin{aligned} \frac{d}{dt}[-t^2/4] &= \frac{-t + (t^2 + 4(-t^2/4))^{1/2}}{2} \\ -t/2 &= \frac{-t + 0}{2} \end{aligned}$$

Both sides are always equal, so $\mathbf{y_2}$ **is valid everywhere**.

2.4.22.b. $f(t, y) = \frac{-t+(t^2+4y)^{1/2}}{2}$ is not continuous in a rectangle around $(2, -1)$, since the term under the square root becomes negative if y increases. So Theorem 2.4.2 does not apply.

2.4.22.c. Substituting $ct + c^2$ for y, we get

$$\begin{aligned}\frac{d}{dt}[ct + c^2] &= \frac{-t + (t^2 + 4(ct + c^2))^{1/2}}{2} \\ c &= \frac{-t + |t + 2c|}{2}\end{aligned}$$

Both sides are equal iff $t + 2c$ is nonnegative, i.e. $t \geq -2c$. We can immediately see that plugging in $c = -1$ yields $ct + c^2 = y_1(t)$. There is no t^2 term, so no choice of c will give $ct + c^2 = y_2(t)$.

2.4.25. We know that $y_1' = -p(t)y_1$ since y_1 is a solution to the first equation, and $y_2' = g(t) - p(t)y_2$ since y_2 is a solution to the second equation. So, plugging in $y = y_1(t) + y_2(t)$ into equation (ii),

$$\begin{aligned}\frac{d}{dt}[y_1 + y_2] + p(t)(y_1 + y_2) &= -p(t)y_1 + g(t) - p(t)y_2 + p(t)y_1 + p(t)y_2 \\ &= g(t).\end{aligned}$$

2.5.3. **Critical points: $\mathbf{y = 0, 1, 2}$.**
$\frac{dy}{dt}$ is negative for $y < 0$, positive for $0 < y < 1$, negative for $1 < y < 2$, and positive for $y > 2$. So **$\mathbf{y = 1}$ is asymptotically stable, while $\mathbf{y = 0}$ and $\mathbf{y = 2}$ are unstable**.

2.5.4. **Critical point: $\mathbf{y = 0}$.**
$\frac{dy}{dt}$ is negative for $y < 0$ and positive for $y > 0$. So **$\mathbf{y = 0}$ is an unstable equilibrium point**.

2.5.7.a. The only zero of $k(1 - y)^2$ is $y = 1$, so **$\mathbf{y = 1}$ is the only critical point of the differential equation**.
2.5.7.b. $\frac{dy}{dt}$ is positive for $y < 1$, and also positive for $y > 1$.
2.5.7.c.

$$\begin{aligned}\frac{dy}{(1 - y)^2} &= k\,dt \\ \frac{1}{1 - y} &= kt + c \\ \frac{1}{kt + c} &= 1 - y \\ y &= 1 - \frac{1}{kt + C}\end{aligned}$$

Since $y(0) = y_0$,

$$\begin{aligned} y_0 &= 1 - \frac{1}{k \cdot 0 + C} \\ y_0 &= 1 - \frac{1}{C} \\ \frac{1}{C} &= 1 - y_0 \\ C &= \frac{1}{1 - y_0} \\ \mathbf{y} &= \mathbf{1} - \frac{\mathbf{1}}{\mathbf{kt} + \frac{\mathbf{1}}{\mathbf{1-y_0}}}. \end{aligned}$$

If $y_0 < 1$, we have $y = 1 - \frac{1}{kt+C}$ for some positive C, so as $t \to \infty$ the fraction will approach zero, and y will approach 1. If, on the other hand, $y_0 > 1$, we have $y = 1 + \frac{1}{C-kt}$ for some positive C, and y will increase until a vertical asymptote at $t = \frac{C}{k}$.

2.5.8. **Critical point: $\mathbf{y = 1}$.**
$\frac{dy}{dt}$ is negative for $y < 1$, and also negative for $y > 1$, so **$\mathbf{y = 1}$ is a semistable equilibrium point**.

2.5.15.a. Solving the differential equation, we get

$$\begin{aligned} \frac{dy}{dt} &= ry[1 - (y/K)] \\ \frac{dy}{y[1-(y/K)]} &= r\,dt \\ \int \frac{dy}{y[1-(y/K)]} &= \int r\,dt \\ \int \frac{-K}{y(y-K)}\,dy &= \int r\,dt \\ \int \frac{1}{y} - \frac{1}{y-K}\,dy &= \int r\,dt \\ \ln|y| - \ln|y-K| &= rt + c \\ \ln|y| &= rt + c + \ln|y-K| \\ |y| &= |y-K|e^{rt+c} \end{aligned}$$

$0 < y < K$ at all times, so

$$\begin{aligned} y &= (K-y)e^{rt+c} \\ y(1+e^{rt+c}) &= Ke^{rt+c} \\ y &= \frac{CKe^{rt}}{1+Ce^{rt}} \end{aligned}$$

Since $y_0 = K/3$,

$$\begin{aligned} K/3 &= \frac{CKe^{r\cdot 0}}{1+Ce^{r\cdot 0}} \\ K/3 &= \frac{CK}{1+C} \\ \frac{1}{3} &= \frac{C}{1+C} \\ 1+C &= 3C \\ 1 &= 2C \\ C &= \frac{1}{2} \end{aligned}$$

We now wish to find τ, where $y(\tau) = 2y_0 = 2K/3$.

$$\begin{aligned} \frac{0.5Ke^{r\tau}}{1+0.5e^{r\tau}} &= 2K/3 \\ 1.5e^{r\tau} &= 2+e^{r\tau} \\ 0.5e^{r\tau} &= 2 \\ e^{r\tau} &= 4 \\ r\tau &= \ln 4 \\ \tau &= \frac{\ln \mathbf{4}}{\mathbf{r}} \end{aligned}$$

If $r = 0.025$ per year, $\tau \approx$ **55.45 years**.
2.5.15.b. If $y_0/K = \alpha$,

$$\begin{aligned} \alpha K &= \frac{CK}{1+C} \\ \alpha &= \frac{C}{1+C} \\ \alpha + \alpha C &= C \\ \alpha &= C(1-\alpha) \\ C &= \frac{\alpha}{1-\alpha} \end{aligned}$$

Then, $y(T)/K = \beta$ when

$$\begin{aligned}
\frac{CKe^{rT}}{1+Ce^{rT}} &= \beta K \\
Ce^{rT} &= \beta(1+Ce^{rT}) \\
Ce^{rT}(1-\beta) &= \beta \\
Ce^{rT} &= \frac{\beta}{1-\beta} \\
e^{rT} &= \frac{\beta}{1-\beta} \cdot 1 - \alpha\alpha \\
rT &= \ln\frac{\beta}{1-\beta} + \ln\frac{1-\alpha}{\alpha} \\
T &= \frac{1}{r}(\ln\frac{\beta}{1-\beta} - \ln\frac{\alpha}{1-\alpha}).
\end{aligned}$$

As $\alpha \to 0$, the second natural log goes to negative infinity, sending T to positive infinity. Similarly, as $\beta \to 1$, the first natural log goes to positive infinity, sending T to positive infinity.

For $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$, we get $T = 40(\ln 9 - \ln\frac{1}{9}) \approx$ **175.78 years**.