Solutions to Math 142 B Practice Exam 1-Corrected

1) a) A partition P of $[a, b]$ is a set of points x_i such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. An example is the regular partition of $[0, 1]$, given by $x_i = \frac{i}{n}, i = 0, 1, ..., n$.

b) Assume that f is a bounded function on $[a, b]$ and that P is a partition of $[a, b]$. Set

$$m_i = \inf\{f(x)|x \in [x_{i-1}, x_i]\}.$$

The lower Darboux sum is

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

As an example, consider the function $f(x) = 2$, for any x in the interval $[1, 5]$. Then for any partition P of $[1, 5]$ we have

$$L(f, P) = \sum_{i=1}^{n} 2(x_i - x_{i-1}) = 2 * 4 = 8.$$

c) The lower integral

$$\int_{a^-}^{b} f = \text{sup}\{L(f,P)|P = \text{partition of } [a,b]\}.$$

As an example, take the function from part b). Then the lower integral is 8.

d) A bounded function f on $[a,b]$ is integrable if and only if the lower integral equals the upper integral; i.e.,

$$\int_{a^-}^{b} f = \int_{a}^{b_-} f.$$

As an example, consider any constant function on any finite interval. Such a function is always integrable.

2) a) **False.** By the refinement lemma, the upper sums decrease as you refine; i.e., add more points to the partition.

b) **False.** Consider the function that is 1 on the irrationals and 0 on the rationals. We saw that this function is not integrable (p. 141 of the text).

c) **False.** This function was shown to be integrable by the theorem on page 157 of our text. For it is bounded on $[0,1]$ and continuous on $(0,1)$.

3) Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then f is integrable on $[a,b]$ if and only if there is a sequence $\{P_n\}$ of partitions of $[a,b]$ such that

$$\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0.$$

Moreover, for any such sequence of partitions,

$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

4) By the Archimedes-Riemann Theorem, there is a sequence $\{P_n\}$ of partitions of $[a, c]$ such that

$$\lim_{n \to \infty} (U(f, P_n) - L(f, P_n)) = 0.$$

By the refinement lemma, we let $Q_n = P_n \cup \{b\}$. Then let $Q_n^{'} = Q_n \cap [a, b]$ and $Q_n^{''} = Q_n \cap [b, c]$. It follows that

$$U(f, Q_n) = U(f, Q_n^{'}) + U(f, Q_n^{''})$$

$$L(f, Q_n) = L(f, Q_n^{'}) + L(f, Q_n^{''})$$

We know that

$$\begin{aligned} 0 &= \lim_{n \to \infty} (U(f, Q_n) - L(f, Q_n)) \\ & \lim_{n \to \infty} \left(U(f, Q_n^{'}) + U(f, Q_n^{''}) - L(f, Q_n^{'}) - L(f, Q_n^{''})\right) \\ &\geq \lim_{n \to \infty} \left(U(f, Q_n^{'}) - L(f, Q_n^{'})\right) \end{aligned}$$

By the squeeze lemma, $\lim_{n \to \infty} \left(U(f, Q_n^{'}) - L(f, Q_n^{'})\right) = 0$. Similarly $\lim_{n \to \infty} \left(U(f, Q_n^{''}) - L(f, Q_n^{''})\right) = 0$. Thus f is integrable on $[a, b]$ and on $[b, c]$.

5) Then for any partition, $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$, we see that $m_i = c = M_i$. It follows that

$$L(f, P) = U(f, P) = \sum_{i=0}^{n} c(x_i - x_{i-1}) = c(b - a),$$

since the last sum is telescoping and all but the first and last terms cancel. From this we see that the upper and lower integrals must both be $c(b - a)$. Since they are equal the function is integrable and the integral is $c(b - a)$.

6) Suppose that f is monotone increasing on $[a, b]$. Let $P_n = \{x_0, x_1, ..., x_n\}$ be a regular partition of $[a, b]$; $i.e.$,

$$x_i = a + \frac{i(b-a)}{n}, i = 0, 1, 2, ..., n.$$

By the fact that f is monotone increasing, we see that $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Thus

$$\begin{aligned} & U(f, P_n) - L(f, P_n) \\ = & \sum_{i=0}^{n} f(x_i)\frac{b-a}{n} - \sum_{i=0}^{n} f(x_{i-1})\frac{b-a}{n} \\ = & \frac{b-a}{n}\sum_{i=0}^{n} (f(x_i) - f(x_{i-1})) \\ = & \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

For the last equality, we used the fact that the preceding sum is telescoping. It follows that as n approaches infinity, $\frac{b-a}{n}(f(b) - f(a))$ approaches 0, since $(b-a)(f(b) - f(a))$ is a constant and $\frac{1}{n} \to 0$, as $n \to \infty$. Thus f is integrable, by the Archimedes-Riemann Theorem.

7) Let $F : [a, b] \to \mathbb{R}$ be continous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose, in addition, that the derivative $F^{'} : (a, b) \to \mathbb{R}$ is continuous and bounded (thus integrable on $[a, b]$). Then

$$\int_a^b F^{'} = F(b) - F(a).$$

Proof . Let $P_n = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. We apply the mean value theorem (text, p. 103) to see that there is a point c_i in (x_{i-1}, x_i) so that $\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}} = F'(c_i)$. Then set

$$\begin{aligned} m_i &= \inf\{F'(x) | x \in [x_{i-1}, x_i]\} \\ M_i &= \sup\{F'(x) | x \in [x_{i-1}, x_i]\}. \end{aligned}$$

It follows that $m_i \leq F'(c_i) \leq M_i$. Thus

$$L(F', P) \leq \sum_{i=1}^{n} F'(c_i)(x_i - x_{i-1}) \leq U(F', P).$$

The mean value formula $\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}} = F'(c_i)$ implies that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$. If we substitute this into the middle sum above, we see that

$$L(F', P) \leq \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \leq U(F', P).$$

But the middle sum here is telescoping and thus equals $F(b) - F(a)$. We know that our upper and lower sums both approach the same limit since $F^{'}$ is integrable. The limit is $\int_a^b F^{'}$ which is both $\leq$ and $\geq$ $F(b) - F(a)$ and, therefore

$$\int_a^b F^{'} = F(b) - F(a).$$

8) a) $m_i(f) \le f(x) \le g(x)$, for all $x \in [x_{i-1}, x_i]$. Thus $m_i(f)$ is a lower bound for all the $g(x), x \in [x_{i-1}, x_i]$. It follows that the glb or **inf** of these $g(x)$'s is larger than or equal to this lower bound,i.e., $m_i(g) \ge m_i(f)$.

b) Since $x_i - x_{i-1} \ge 0$, by part a),

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) = L(g, P). \end{aligned}$$

c) Part b) says that $L(g, P)$ is an upper bound on all the $L(f, P)$. Thus the lub or **sup** must be less than or equal to it. This says that for all partitions P of $[a, b]$

$$\int_{a^-}^{b} f = \textbf{sup}\{L(f, P) | P = \textbf{partition of } [a, b]\} \le L(g, P).$$

But then $\int_{a^-}^{b} f$ is a lower bound on all the $L(g, P)$. This means it must be less than or equal to the **sup** of the $L(g, P)$. Thus

$$\int_{a^-}^{b} f \leq \int_{a^-}^{b} g.$$

9) First recall the Extreme Value Theorem for continuous functions (text, p. 60). If P_n is a regular partition of $[a, b]$; i.e., $x_i = \frac{(b-a)i}{n}, i = 0, 1, .., n$, then $m_i = \inf\{f(x)|x \in [x_{i-1}, x_i]\} = f(u_i)$, for some $u_i \in [x_{i-1}, x_i]$. The Extreme Value Theorem also says $M_i = \sup\{f(x)|x \in [x_{i-1}, x_i]\} = f(v_i)$, for some $v_i \in [x_{i-1}, x_i]$. It follows that

$$L(f, P_n) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}) = \frac{b-a}{n} \sum_{i=1}^{n} f(u_i).$$

Similarly

$$U(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(v_i).$$

Therefore

$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} \left(f(v_i) - f(u_i)\right).$$

Now we need the theorem saying that a continuous function on $[a, b]$ is uniformly continuous; i.e., for every $\varepsilon > 0$, there exists a $\delta > 0$ (independent of x, y) so that for any $x, y \in [a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. The text proved this on p. 68 and we use the theorem on p. 73, to get the ε, δ—version of uniform continuity. So let us take n so large that $\frac{b-a}{n} < \delta$. (That is, $n > \frac{b-a}{\delta}$). This means that $|v_i - u_i| < \delta$, which implies $|f(v_i) - f(u_i)| < \varepsilon$. Therefore $n > \frac{b-a}{\delta}$ implies

$$\begin{aligned} & U(f, P_n) - L(f, P_n) \\ = \;& \frac{b-a}{n} \sum_{i=1}^{n} \left(f(v_i) - f(u_i)\right) \\ < \;& \frac{b-a}{n} n\varepsilon = (b-a)\varepsilon. \end{aligned}$$

Replace ε by $\frac{\varepsilon}{b-a}$ and find a new value of δ for this ε, to see that for $n > \frac{b-a}{\delta}$, we have

$$U(f, P_n) - L(f, P_n) < \varepsilon.$$

By the Archimedes-Riemann Theorem, this implies that f is integrable on $[a, b]$.