Math 20D Practice Exam 1 Solutions

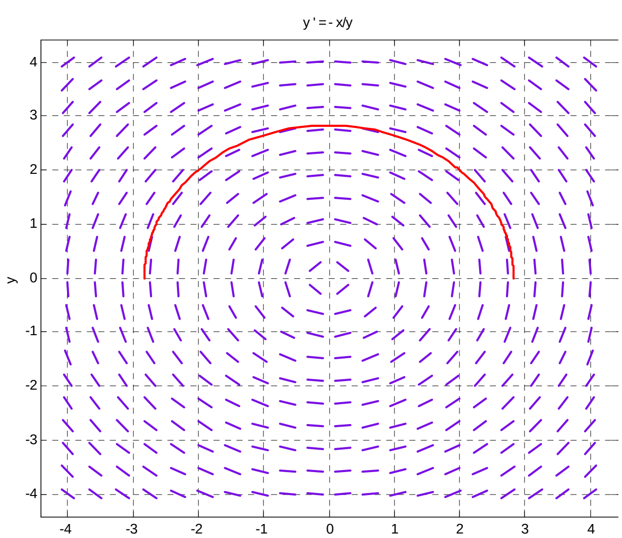
1) a) We want $u(x)$ such that $uy' - uxy = uy' + u'y$. So we must solve $\frac{du}{dx} = -xu$. This is separable:

$\frac{du}{u} = -xdx$. Integrate both sides to get $\ln|u| = -\frac{x^2}{2} + c$. Here we may as well take $c = 0$. So, exponentiating, we want $u(x) = e^{-\frac{x^2}{2}}$.

b) So now our ODE is $(uy)' = -ux$. So integrating gives $uy = -\int uxdx + C$. Then plug $u(x) = e^{-\frac{x^2}{2}}$. This gives $y(x) = -e^{\frac{x^2}{2}}\left(\int xe^{-\frac{x^2}{2}}dx + C\right)$. Thus $y(x) = -e^{\frac{x^2}{2}}\left(-e^{-\frac{x^2}{2}} + C\right) = 1 - Ce^{\frac{x^2}{2}}$. You might want to check this.

2) The ODE $y\frac{dy}{dx} + x = 0$ is separable. So we get $ydy = -xdx$. Integrate to see that

$x^2 + y^2 = c$. This is a circle centered at the origin of radius $\sqrt{c}$. So $y = \pm\sqrt{c - x^2}$. To fit the initial conditions, we need to plug $x = 2, y = 2$. This says $8 = c$. So your solution is $y = \sqrt{8 - x^2}$. We choose the $+\sqrt{}$ since $x = 2$ implies y is positive. The solution is defined when $x^2 \leq 8$; i.e., when $-\sqrt{8} \leq x \leq \sqrt{8}$. As our solutions are circles centered at the origin, we expect to see that the direction field circles about the origin. See the figure below.



Note: I do not expect you to draw direction fields on the exam. But I include the pictures to illustrate what is happening.

3) a) We measure $Q(t)$ in lb. and t in minutes. First note that

$Q'(t) =$ rate salt comes into tank $-$ rate salt leaves tank.

Thus $Q^{'}(t) = 0 - 4\frac{Q(t)}{100}$.

b) We see that $\frac{dQ}{dt} = -.04Q$ is separable. So $\frac{dQ}{Q} = -.04dt$. Integrate to get

$$\ln Q = -.04t + c.$$

Then exponentiate to get $Q = Ke^{-.04t}$. Then $Q(0) = 80$. This means if we plug $t = 0$, we get $80 = Ke^{0} = K$. So $K = 80$. Our final solution is

$$Q(t) = 80e^{-.04t}.$$

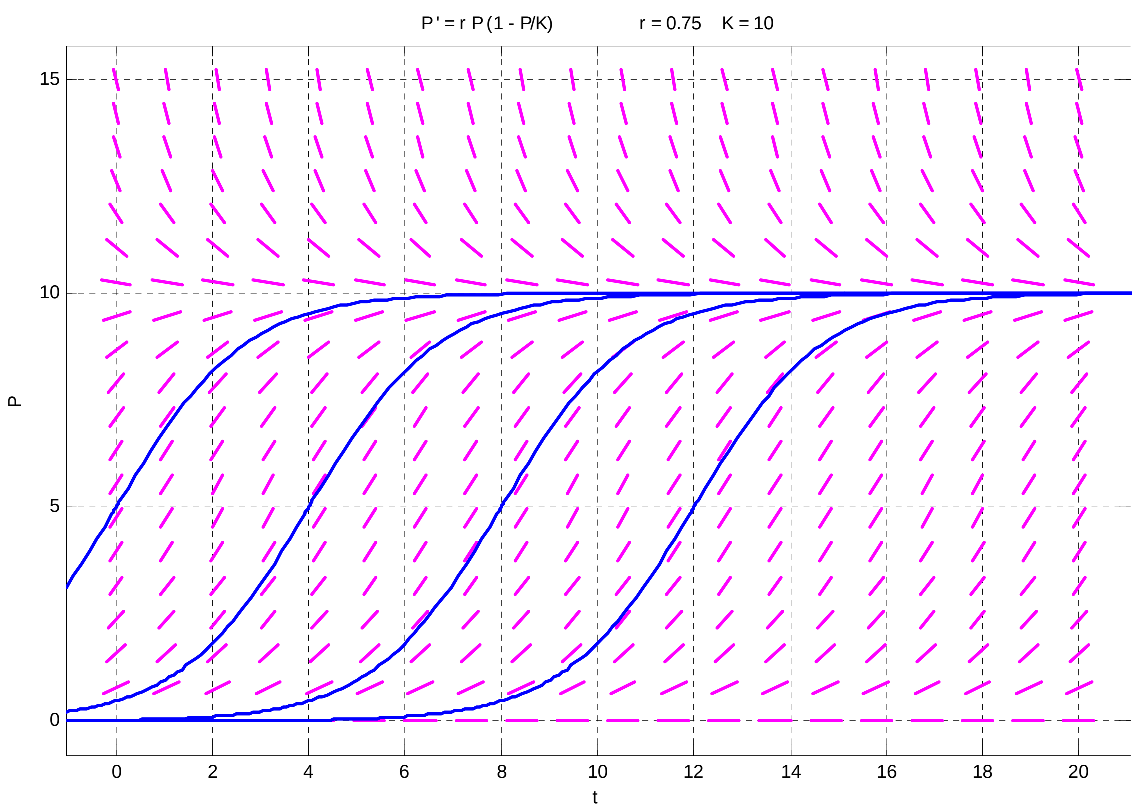
c) Now half the salt is 40 lb. so $40 = Q(t) = 80e^{-.04t}$. This implies, upon division by 80, that $\frac{1}{2} = e^{-.04t}$. Next take logs, $\ln\frac{1}{2} = \ln e^{-.04t} = -.04t$. Thus

$$t = \frac{\ln 2}{.04} minutes.$$

4) a) True. Here $M = y \cos x + 2xe^y$ and $N = \sin x + x^2 e^y - 1$. Thus $M_y = \cos x + 2xe^y = N_x$.

b) This follows from the comforting existence and uniqueness theorem on page 70 of our text. It implies the solutions curves Matlab draws cannot cross. Otherwise there would be 2 solutions satisfying the initial condition corresponding to the point of intersection.

c) This is true. To see it, think about the slopes of the direction fields near $y = K$ and $y = 0$. If y is near K but $y > K$, then y' is negative and so the solution is directed toward $y = K$. If y is near K but $y < K$, then $y' < 0$ and so the solution is again directed towards $y = K$. That means $y = K$ is a **stable** equilibrium. If y is near 0, but $y > 0$, then $y' > 0$ so that the solution is direction away from $y = 0$. If y is near 0 but $y < 0$, then $y' < 0$, so that the solution is directed away from $y = 0$. That means $y = 0$ is an **unstable** equilibrium.



5) a) Substitute $y = e^{rt}$ and find that $r^2 + r - 2 = (r+2)(r-1) = 0$. So roots are $r = 1, -2$. Thus our fundamental solutions are $y_1 = e^t$ and $y_2 = e^{-2t}$.

b) The Wronskian is $W(y_1, y_2)(t) = y_1 y_2^{'} - y_2 y_1^{'} = e^t(-2e^{-2t}) - e^{-2t}e^t = -3e^{-t} \neq 0$, for all real numbers t.

c) $y(t) = c_1 e^t + c_2 e^{-2t}$; $y'(t) = c_1 e^t + -2c_2 e^{-2t}$. To satisfy the initial conditions: $y(0) = c_1 e^0 + c_2 e^{-2\cdot 0} = 2$ & $y'(0) = c_1 e^0 + -2c_2 e^{-2\cdot 0} = 3$. This gives

$$c_1 + c_2 = 2$$

$$c_1 - 2c_2 = 3$$

Subtract to get $3c_2 = -1$ so that $c_2 = \frac{-1}{3}$. Then $c_1 = 2 - c_2 = 2 + \frac{1}{3} = \frac{7}{3}$. Our solutions is

$y(t) = \frac{7}{3}e^t - \frac{1}{3}e^{-2t}$.