1) a) We want \( u(x) \) such that \( uy' - uxy = uy' + u'y \). So we must solve \( \frac{du}{dx} = -xu \). This is separable:

\[
\frac{du}{u} = -xdx. \quad \text{Integrate both sides to get} \quad \ln |u| = -\frac{x^2}{2} + c. \quad \text{Here we may as well take} \quad c = 0. \quad \text{So, exponentiating, we want} \quad u(x) = e^{-\frac{x^2}{2}}.
\]

b) So now our ODE is \( (uy)' = -ux \). So integrating gives \( uy = -\int ux \, dx + C \). Then plug \( u(x) = e^{-\frac{x^2}{2}} \). This gives \( y(x) = -e^{\frac{x^2}{2}} \left( \int x e^{-\frac{x^2}{2}} \, dx + C \right) \). Thus

\[
y(x) = -e^{\frac{x^2}{2}} \left( -e^{-\frac{x^2}{2}} + C \right) = 1 - Ce^{\frac{x^2}{2}}. \quad \text{You might want to check this.}
\]
2) The ODE \( y \frac{dy}{dx} + x = 0 \) is separable. So we get \( ydy = -xdx \). Integrate to see that

\[ x^2 + y^2 = c. \]

This is a circle centered at the origin of radius \( \sqrt{c} \). So \( y = \pm \sqrt{c - x^2} \). To fit the initial conditions, we need to plug \( x = 2, y = 2 \). This says \( 8 = c \). So your solution is \( y = \sqrt{8 - x^2} \). We choose the \( + \sqrt{c} \) since \( x = 2 \) implies \( y \) is positive. The solution is defined when \( x^2 \leq 8 \); i.e., when \( -\sqrt{8} \leq x \leq \sqrt{8} \). As our solutions are circles centered at the origin, we expect to see that the direction field circles about the origin. See the figure below.
Note: I do not expect you to draw direction fields on the exam. But I include the pictures to illustrate what is happening.
3) a) We measure $Q(t)$ in lb. and $t$ in minutes. First note that

$$Q'(t) = \text{rate salt comes into tank} - \text{rate salt leaves tank}.$$ 

Thus $Q'(t) = 0 - 4 \frac{Q(t)}{100}$.

b) We see that $\frac{dQ}{dt} = -0.04Q$ is separable. So $\frac{dQ}{Q} = -0.04dt$. Integrate to get

$$\ln Q = -0.04t + c.$$ 

Then exponentiate to get $Q = Ke^{-0.04t}$. Then $Q(0) = 80$. This means if we plug $t = 0$, we get $80 = Ke^0 = K$. So $K = 80$. Our final solution is

$$Q(t) = 80e^{-0.04t}.$$ 

c) Now half the salt is 40 lb. so $40 = Q(t) = 80e^{-0.04t}$. This implies, upon division by 80, that $\frac{1}{2} = e^{-0.04t}$. Next take logs, $\ln \frac{1}{2} = \ln e^{-0.04t} = -0.04t$. Thus

$$t = \ln \frac{2}{.04} \text{ minutes}.$$
4) a) True. Here \( M = y \cos x + 2xe^y \) and \( N = \sin x + x^2e^y - 1 \). Thus \( M_y = \cos x + 2xe^y = N_x \).

b) This follows from the comforting existence and uniqueness theorem on page 70 of our text. It implies the solutions curves Matlab draws cannot cross. Otherwise there would be 2 solutions satisfying the initial condition corresponding to the point of intersection.
c) This is true. To see it, think about the slopes of the direction fields near $y = K$ and $y = 0$. If $y$ is near $K$ but $y > K$, then $y'$ is negative and so the solution is directed toward $y = K$. If $y$ is near $K$ but $y < K$, then $y' < 0$ and so the solution is again directed towards $y = K$. That means $y = K$ is a **stable** equilibrium. If $y$ is near $0$, but $y > 0$, then $y' > 0$ so that the solution is direction away from $y = 0$. If $y$ is near $0$ but $y < 0$, then $y' < 0$, so that the solution is directed away from $y = 0$. That means $y = 0$ is an **unstable** equilibrium.
5) a) Substitute $y = e^{rt}$ and find that $r^2 + r - 2 = (r + 2)(r - 1) = 0$. So roots are $r = 1, -2$. Thus our fundamental solutions are $y_1 = e^t$ and $y_2 = e^{-2t}$.

b) The Wronskian is $W(y_1, y_2)(t) = y_1y_2' - y_2y_1' = e^t(-2e^{-2t}) - e^{-2t}e^t = -3e^{-t} \neq 0$, for all real numbers $t$.

c) $y(t) = c_1e^t + c_2e^{-2t}$; $y'(t) = c_1e^t - 2c_2e^{-2t}$. To satisfy the initial conditions: $y(0) = c_1e^0 + c_2e^{-2\cdot0} = 2$ & $y'(0) = c_1e^0 - 2c_2e^{-2\cdot0} = 3$. This gives

$$c_1 + c_2 = 2$$

$$c_1 - 2c_2 = 3$$

Subtract to get $3c_2 = -1$ so that $c_2 = -\frac{1}{3}$. Then $c_1 = 2 - c_2 = 2 + \frac{1}{3} = \frac{7}{3}$. Our solutions is

$$y(t) = \frac{7}{3}e^t - \frac{1}{3}e^{-2t}.$$