1) a) Which of the following rings are integral domains? Give a brief explanation of your answer.
   b) Same as a) replacing “integral domains” with “fields.”
      i) \( \mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z} \} \), where \( i^2 = -1 \).
      ii) \( \mathbb{Z}/12\mathbb{Z} \).
      iii) \( M_2(\mathbb{Z}/2\mathbb{Z}) = 2 \times 2 \) matrices with entries in \( \mathbb{Z}/2\mathbb{Z} \).
      iv) \( \mathbb{Z} \oplus \mathbb{Z} = \{(a,b) \mid a, b \in \mathbb{Z} \} \), with \( (a,b)+(c,d)=(a+c,b+d) \) and \( (a,b)(c,d)=(ac,bd) \).
      v) \( \mathbb{Z}/11\mathbb{Z} \).
      vi) \( \mathbb{Q} \) = the rational numbers
      vii) \( C(\mathbb{R}) \) = the continuous real valued functions on \( \mathbb{R} \).

2) a) List all the zero divisors in the 7 rings \( R \) from problem 1 except that you should replace
   vii) \( C(\mathbb{R}) \) with \( C^\text{pw}(\mathbb{R}) \) - the piecewise continuous functions on \( \mathbb{R} \) (i.e., we allow a finite number of removable or jump discontinuities).
   b) List all the units in the same 7 rings as part a); i.e., find the unit group \( U(R) \).
   c) What is the relation between the zero divisors and the units of \( R \), if any?

3) Suppose that \( F \) is a field and \( S \) is a subset of \( F \). Develop a subfield test for \( S \) to be a subfield which is analogous to our subring test from Gallian Chapter 12. Prove that that the test works. Again make use of the 1-step subgroup test from Gallian, Chapter 3.

4) Show that there does not exist an integral domain with exactly 6 elements.
   Hint: You can use Theorems 13.3 and 13.4 in Gallian.

5) Suppose that \( F \) is a finite field with \( n \) elements. Define \( F^* \) to be the set of all non-0 elements of \( F \).
   a) Show that \( F^* \) is a group under multiplication.
   b) Show that \( x \) in \( F^* \) implies \( x^{n-1} = 1 \).

6) a) Consider \( \mathbb{Z}_5[i] = \{a+bi \mid a, b \in \mathbb{Z}_5 \} \), where \( i^2 = -1 \). Show that this ring is not a field.
   b) Consider \( \mathbb{Z}_7[i] = \{a+bi \mid a, b \in \mathbb{Z}_7 \} \), where \( i^2 = -1 \). Show that this ring is a field.
   c) Can you develop a more general version of this problem for \( \mathbb{Z}_p[i] \) where \( p \) is an odd prime according to whether \( p \) is congruent to 1 or 3 (mod 4)?