

Zeta Functions of Finite Graphs and Coverings, III

A. A. Terras

Math. Dept., U.C.S.D., La Jolla, CA 92093-0112

E-mail: aterras@ucsd.edu

and

H. M. Stark

Math. Dept., U.C.S.D., La Jolla, CA 92093-0112

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A graph theoretical analog of Brauer-Siegel theory for zeta functions of number fields is developed using the theory of Artin L-functions for Galois coverings of graphs from parts I and II. In the process, we discuss possible versions of the Riemann hypothesis for the Ihara zeta function of an irregular graph.

1. INTRODUCTION

In our previous two papers [12], [13] we developed the theory of zeta and L-functions of graphs and covering graphs. Here zeta and L-functions are reciprocals of polynomials which means these functions have poles not zeros. Just as number theorists are interested in the locations of the zeros of number theoretic zeta and L-functions, thanks to applications to the distribution of primes, we are interested in knowing the locations of the poles of graph-theoretic zeta and L-functions. We study an analog of Brauer-Siegel theory for the zeta functions of number fields (see Stark [11] or Lang [6]). As explained below, this is a necessary step in the discussion of the distribution of primes.

We will always assume that our graphs X are finite, connected, rank ≥ 1 with no danglers (i.e., degree 1 vertices). Let us recall some of the definitions basic to Stark and Terras [12], [13].

If X is any connected finite undirected graph with vertex set V and (undirected) edge set E , we orient its edges arbitrarily and obtain $2|E| = 2m$ oriented edges. We always use the following **oriented edge labelling**

$$e_1, e_2, \dots, e_m, e_{m+1} = e_1^{-1}, \dots, e_{2m} = e_m^{-1}. \quad (1)$$

"Primes" $[C]$ in X are equivalence classes of closed backtrackless tailless primitive paths C . Write $C = a_1 a_2 \cdots a_s$, where a_j is an oriented edge of X . The **length** of C is $\nu(C) = s$. **Backtrackless** means that $a_{i+1} \neq a_i^{-1}$, for all i . **Tailless** means that $a_s \neq a_1^{-1}$. The **equivalence class** $[C]$ is the set

$$[C] = \{a_1 a_2 \cdots a_s, a_2 a_3 \cdots a_s a_1, \dots, a_s a_1 \cdots a_{s-1}\}.$$

$[C]$ is **primitive** means $C \neq D^m$, for any integer $m \geq 2$ and path D in X .

Here r_X will denote the **rank** of the fundamental group of X . We have $r_X - 1 = |E| - |V|$. Then r_X is the number of edges deleted from X to form a spanning tree. We will call such deleted edges **"cut"** edges, since there should be no confusion with the other meaning of cut edge.

Next let us define an **unramified finite covering** graph Y over X (**written** Y/X) in the case that the graphs have no loops or multiple edges. In this case, Y covers X means that

there is a covering map $\pi : Y \rightarrow X$ such that π is an onto graph map and for each $x \in X$ and each $y \in \pi^{-1}(x)$, the set of points adjacent to y in Y is mapped by π 1-1, onto the set of points adjacent to x in X . We always consider connected coverings Y of the connected graph X obtained by viewing the d sheets of Y as copies of a spanning tree in X . If the graphs have loops and multiple edges, one must be a little more precise about the definition of covering graph. See Stark and Terras [13].

A d -sheeted (**unramified**) **graph covering** Y/X is **normal** iff there are d graph automorphisms $\sigma : Y \rightarrow Y$ such that $\pi\sigma(y) = \pi(y)$, for all $y \in Y$. Then $Gal(Y/X)$, the **Galois group** of Y/X , is the set of all these σ 's.

Recall that the **Ihara zeta function** of X is defined at $u \in \mathbb{C}$, for $|u|$ sufficiently small, by

$$\zeta_X(u) = \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-1}, \quad (2)$$

where $[C]$ runs over the primes of X .

As a power series in the complex variable u ,

$$\zeta_X(u) = \sum_{n=0}^{\infty} a_n u^n, \quad (3)$$

where each coefficient $a_n \geq 0$. Thus, by a classic theorem of Landau, both the series (3) and the product (2) will converge absolutely in a circle $|u| < R$ with a singularity (pole of order 1 for connected X) at $u = R$.

DEFINITION 1. R_X is the **radius of the largest circle of convergence** of the Ihara zeta function and $\omega_X = 1/R_X$.

When X is a $(q+1)$ -regular graph, $R_X = 1/q$ and $\omega_X = q$. As with the Dedekind zeta function, $\zeta_X(u)$ has a meromorphic continuation to the entire complex u -plane, but now $\zeta_X(u)^{-1}$ is entire. Thus our interest lies with the poles of $\zeta_X(u)$. In general, $\zeta_X(u)^{-1}$ is a polynomial which is essentially the characteristic polynomial of our edge matrix W_X (which we define next) and the largest eigenvalue of W_X is ω_X .

DEFINITION 2. We define the **0,1 edge matrix** W_X by orienting the edges of X and labeling them as in (1). Then W_X is the $2m \times 2m$ matrix with ij entry 1 if edge e_i feeds into e_j provided that $e_j \neq e_i^{-1}$, and ij entry 0 otherwise.

Recall from Stark and Terras [13] that

$$\zeta_X(u)^{-1} = \det(I - W_X u). \quad (4)$$

From this one can derive **Ihara's formula**

$$\zeta_X(u)^{-1} = (1 - u^2)^{r_X - 1} \det(I - A_X u + Q_X u^2), \quad (5)$$

where r_X is the rank of the fundamental group of X , A_X is the adjacency matrix of X , Q_X is the diagonal matrix whose j^{th} diagonal entry is $(-1 + \text{degree of } j^{\text{th}} \text{ vertex})$.

Kotani and Sunada [5] show that, if $q+1$ is the maximum degree of X and $p+1$ is the minimum degree of X , then every non-real pole u of $\zeta_X(u)$ satisfies the inequality

$$q^{-1/2} \leq |u| \leq p^{-1/2}. \quad (6)$$

Moreover, they show that every pole u of $\zeta_X(u)$ satisfies

$$R_X \leq |u| \leq 1.$$

Another result of Kotani and Sunada [5] says that

$$q^{-1} \leq R_X \leq p^{-1}, \quad p \leq \omega_X \leq q. \quad (7)$$

In particular, we can think of $\omega_X + 1$ as a zeta function average vertex degree of X which is precisely $q + 1$ when X is a $(q + 1)$ -regular graph. However Example 2 below shows that for irregular graphs, $\omega_X + 1$ is in general neither the arithmetic nor geometric mean of all the vertex degrees.

The main term in the graph analog of the prime number theorem is a power of ω_X . Further terms come from the same power of the reciprocals of the other poles of $\zeta_X(u)$ and thus the first step in discussing the error term is to locate the poles of $\zeta_X(u)$ with $|u|$ very near to R_X . In number theory, there is a known zero free region of a Dedekind zeta function which can be explicitly given except for the possibility of a single first order real zero within this region. This possible exceptional zero has come to be known as a "**Siegel zero**" and is closely connected with the famous Brauer-Siegel Theorem. There is no known example of a Siegel zero for Dedekind zeta functions.

Since $\zeta_X(u)^{-1}$ is a polynomial with a finite number of zeros, given X there is an $\epsilon > 0$ such that any pole of $\zeta_X(u)$ in the region $R_X \leq |u| < R_X + \epsilon$ must lie on the circle $|u| = R_X$. This gives us the graph theoretic analog of a "**pole free region**", $|u| < R_X + \epsilon$; the only exceptions lie on the circle $|u| = R_X$. We will show that $\zeta_X(u)$ is a function of u^δ with $\delta = \delta_X$ a positive integer from Definition 5 below. This will imply there is a δ -fold symmetry in the poles of $\zeta_X(u)$; i.e., $u = \varepsilon_\delta R$ is also a pole of $\zeta_X(u)$, for all δ^{th} roots of unity ε_δ . Any further poles of $\zeta_X(u)$ on $|u| = R$ will be called "**Siegel poles**" of $\zeta_X(u)$. Thus if $\delta = 1$, any pole $u \neq R$ of $\zeta_X(u)$ with $|u| = R$ will be called a Siegel pole.

In number fields, a Siegel zero "deserves" to arise already in a quadratic extension of the base field. This has now been proved in many cases (see Stark [8]). Our initial motivation for this paper was to carry over these results to zeta functions of graphs. This was accomplished, essentially by the same representation theoretic methods that were used for Dedekind zeta functions, and is presented in Theorem 2 below. In the process, we were led to study possible extensions of the meaning "Ramanujan graph" and the "Riemann hypothesis for graph zeta functions" for irregular graphs. We discuss these possibilities in Section 2.

A key reduction in the location of Siegel poles leads us to our first theorem which is purely combinatorial and is of independent interest. For this, we need three definitions.

DEFINITION 3. $\Delta_X = g.c.d. \{ \nu(C) \mid C = \text{closed backtrackless tailless path on } X \}$.

Note that Δ_X is even if and only if X is bipartite.

DEFINITION 4. A vertex of X having degree ≥ 3 is called a **node** of X .

A graph X of rank ≥ 2 always has at least one node.

DEFINITION 5. If X has rank ≥ 2

$$\delta_X = g.c.d. \left\{ \nu(P) \mid \begin{array}{l} P = \text{backtrackless path in } X \text{ such that the} \\ \text{initial and terminal vertices are both nodes} \end{array} \right\}.$$

When a path P in the definition of δ_X is closed, the path will be backtrackless but may have a tail. However, in Section 4 we will give an equivalent definition of δ_X which does not involve paths with tails. The equivalent definition has the added advantage that it is visibly a finite calculation. The relation between δ_X and Δ_X is given by the following result which will be proved in Section 4.

THEOREM 1. *Suppose X has rank ≥ 2 . Then either $\Delta_X = \delta_X$ or $\Delta_X = 2\delta_X$.*

It is easy to see that if Y is a covering graph of X (of rank ≥ 2) we have $\delta_Y = \delta_X$ since they are the *g.c.d.s* of the same set of numbers. Therefore δ_X is a covering invariant. Because of this, Theorem 1 gives us the important

COROLLARY 1. *If Y is a covering of a graph X of rank ≥ 2 then*

$$\Delta_Y = \Delta_X \quad \text{or} \quad 2\Delta_X.$$

For a cycle graph X the ratio Δ_Y/Δ_X can be arbitrarily large. As we have stated, we consider the Brauer Siegel theorem to be a statement about the location of Siegel zeros (Siegel poles for graph theory zeta functions). The general case, Theorem 3 in Section 5, will be reduced to the more easily stated case where $\delta_X = 1$, where any pole of $\zeta_X(u)$ on $|u| = R$ other than $u = R$, is a **Siegel pole**,

THEOREM 2. *Suppose X has rank ≥ 2 , and $\delta_X = 1$. Let Y be a connected covering graph of X and suppose $\zeta_Y(u)$ has a Siegel pole μ . Then we have the following facts.*

1. *The pole μ is a first order pole of $\zeta_Y(u)$ and $\mu = -R$ is real.*
2. *There is a unique intermediate graph X_2 to Y/X with the property that for every intermediate graph \tilde{X} to Y/X (including X_2), μ is a Siegel pole of $\zeta_{\tilde{X}}(u)$ if and only if \tilde{X} is intermediate to Y/X_2 .*
3. *X_2 is either X or a quadratic (i.e., 2-sheeted) cover of X .*

In Theorem 2, \tilde{X} is intermediate to Y/X means that Y covers \tilde{X} and \tilde{X} covers X such that the composition of projection maps is consistent.

Our goal in this paper was to take the first steps in investigating the locations of the poles of Ihara zeta functions for irregular graphs. Having discovered Theorem 2, we saw that there is a purely graph theoretic equivalent statement which we state for completeness in Section 5 as Theorem 4. Knowing the result, a direct combinatorial proof of Theorem 4 was easily found. We give this proof in Section 6. In Section 6 we will also show that any graph X of rank ≥ 2 possesses a covering Y with a Siegel pole and thus the graph X_2 of Theorem 2 actually exists for all X . The analog of this result for number fields is an open question. One does not know any example of a number field whose Dedekind zeta function has a Siegel zero.

Remark 1. The authors wish to thank the referee for many useful comments.

2. RAMANUJAN GRAPHS AND THE RIEMANN HYPOTHESIS.

EXAMPLE 1. As just stated, in a $(q + 1)$ -regular graph with $q \geq 2$, every vertex is a node. Thus $\delta = 1$. By Theorem 1, when $q \geq 2$, Δ must then be either 1 or 2 for a regular graph. Consider the cube Y covering the tetrahedron $X = K_4$. Then $\Delta_X = 1, \Delta_Y = 2; \delta_X = 1, \delta_Y = 1$. The cube is the unique X_2 described in Theorem 2 in this case. The Ihara zeta function for the graph Y thus has a Siegel pole.

We now consider the twin problems of formulating the definitions of Ramanujan graphs and the Riemann hypothesis for irregular graphs. For regular graphs, we know that the two concepts are the same.

We begin with Ramanujan graphs. First define two constants associated to the graph X .

DEFINITION 6.

$$\begin{aligned} \rho_X &= \max \{ |\lambda| \mid \lambda \in \text{spectrum}(A_X) \}, \\ \rho'_X &= \max \{ |\lambda| \mid \lambda \in \text{spectrum}(A_X), |\lambda| \neq \rho_X \}. \end{aligned}$$

Lubotzky [8] has defined X to be **Ramanujan** if

$$\rho'_X \leq \sigma_X.$$

where σ_X is the **spectral radius of the adjacency operator on the universal covering tree of X** . Hoory [3] has proved that if \overline{d}_X denotes the **average degree of the vertices of X** , then

$$\sigma_X \geq 2\sqrt{\overline{d}_X - 1}.$$

This yields an easy test that detects some Ramanujan graphs, namely a graph X is certainly Ramanujan if we have the **Hoory inequality**

$$\rho'_X \leq 2\sqrt{\overline{d}_X - 1}. \quad (8)$$

Another possible definition of Ramanujan for X irregular would be the following. We say X satisfies the **naive Ramanujan inequality** if

$$\rho'_X \leq 2\sqrt{\rho_X - 1}. \quad (9)$$

Note that

$$\rho_X \geq \overline{d}_X.$$

This is easily seen using the fact that ρ_X is the maximum value of the Rayleigh quotient $\langle Af, f \rangle / \langle f, f \rangle$, while \overline{d}_X is the value when f is the vector all of whose entries are 1.

Now we turn to possible versions of the Riemann hypothesis. We will first make some comments about the number field situation in order to explain our choices of potential Riemann hypotheses. In number theory, given a number field K , the full Generalized Riemann Hypothesis (GRH) for the Dedekind zeta function $\zeta_K(s)$ corresponding to a number field K is equivalent to saying

RH-I. $\zeta_K(s) \neq 0$ for $1/2 < \operatorname{Re}(s) \leq 1$.

It is known that, except for a first order pole at $s = 1$, $\zeta_K(s) \neq 0$ for $\operatorname{Re}(s) \geq 1$. Because 75 years of effort have failed to prove that Siegel zeros (real zeros of $\zeta_K(s)$ very near $s = 1$, where "very near" depends upon K) do not exist (although it is known that given K , $\zeta_K(s)$ has at most one Siegel zero), researchers in the field have privately suggested the possibility of the weaker

RH-II. RH-I except possibly for one Siegel zero.

More recently, a further weakening has been proposed, the Modified Generalized Riemann Hypothesis (MGRH).

RH-III. $\zeta_K(s) \neq 0$ for $1/2 < \operatorname{Re}(s) \leq 1$ except for possible zeros, arbitrary in number, when s is real.

This was introduced because various spectral approaches to GRH would not detect real zeros and so would at best end up proving MGRH. Our first potential Riemann Hypothesis for graphs will be most analogous to RH-II. The reason for this can be traced to the definition of a Ramanujan graph by Lubotzky, Phillips and Sarnak [9] who wanted to allow nice regular bipartite graphs to be Ramanujan graphs and so defined Ramanujan graphs in terms of the second largest absolute value of an eigenvalue of the adjacency matrix of the regular graph, thereby allowing the largest absolute value to come from a plus and minus pair. The Riemann Hypothesis for regular graphs was defined in such a way that it was equivalent to a regular graph being a Ramanujan graph. We will carry over this definition to irregular graphs and see that thanks to Theorem 2, it will essentially be RH-II above for graphs.

In the $(q+1)$ -regular case the standard change of variables $u = q^{-s}$, $s \in \mathbb{C}$, turns $\zeta_X(u)$ into a Dirichlet series which is zero- and pole-free for $\operatorname{Re}(s) > 1$ and has a first order pole at $s = 1$. The Riemann hypothesis for regular graphs was then phrased (e.g. [12], p. 129) as $\zeta_X(u)$ has

no poles with $0 < \text{Re}(s) < 1$ except for $\text{Re}(s) = 1/2$. When X is a regular bipartite graph, $\zeta_X(u)$ has poles at $u = 1/q$, but also at $u = -1/q$. To include irregular graphs, the natural change of variable is $u = \omega_X^{-s}$ with ω_X from Definition 1. All poles of $\zeta_X(u)$ are then located in the "critical strip", $0 \leq \text{Re}(s) \leq 1$ with poles at $s = 0$ ($u = 1$) and $s = 1$ ($u = \omega_X^{-1} = R_X$). From this point of view, it is natural to say that the Riemann hypothesis for X should require that $\zeta_X(u)$ has no poles in the open strip $1/2 < \text{Re}(s) < 1$.

In terms of u , we would then put forward a "**graph theory Riemann hypothesis**" which says that there are no poles of $\zeta_X(u)$ strictly between the circles $|u| = R_X$ and $|u| = \sqrt{R_X}$. Because there is no functional equation for irregular graphs relating s to $1 - s$, this Riemann hypothesis makes no statement regarding poles in the open strip, $0 < \text{Re}(s) < 1/2$. This version makes no statement about poles on the circle $|u| = R_X$ other than $u = R_X$. Thanks to Theorem 2, at least when $\delta_X = 1$, such a pole would be a single first order pole at $u = -R_X$. We will discuss analogs of RH-III below.

There is another reason that this proposed Riemann hypothesis is natural. The main term in the analog of the prime number theorem which counts prime paths in X of length N is a constant times $\omega_X^N = (R_X^{-1})^N$, when Δ_X divides N . The remaining terms are of the form constant times $(\mu^{-1})^N$ where μ runs through the other poles of $\zeta_X(u)$. It is natural to hope that the remaining terms all combine to a sum which is $O(\sqrt{\omega_X^N})$ as $N \rightarrow \infty$. This requirement is equivalent to the Riemann hypothesis. (A Siegel pole has the effect of restricting this discussion to even N .)

In Example 4 below, we present one infinite family of examples of irregular graphs where the Riemann hypothesis holds and is best possible, but the family is admittedly artificial. One defect of the Riemann hypothesis is that, since there is no longer a functional equation, there is no reason to expect, for an irregular graph, that $\zeta_X(u)$ has any poles on the circle $|u| = R_X^{1/2}$, and thus there would be an infinitesimally bigger pole free region. Still, one could hope that in some sense, $R_X^{1/2}$ is the natural best radius with infinitely many truly distinct examples where the Riemann hypothesis holds with this radius. For instance, we could restrict X to run through graphs with fixed $q > p \geq 2$ (in particular every vertex is a node and $\delta_X = 1$). We stay away from $p = 1$ so as to avoid the possibility of doing to one example what we did in Example 4.

It appears from our examples that the Riemann hypothesis is not easily attained. For this reason, we put forward a possible weaker version which is almost as good when ω_X is close to q where $q + 1$ is the maximum degree of all the vertices of X . From (7), we know that $\frac{1}{q} \leq R_X$. Thus $\frac{1}{\sqrt{q}} \leq \sqrt{R_X}$ and we now ask whether the only possible poles of $\zeta_X(u)$ inside the circle $|u| = 1/\sqrt{q}$ are at radius R_X . We call this a "**graph theory weak Riemann hypothesis.**" Again, in the terminology from number theory zeta functions, this comes closest to RH-II. We will give two interesting families of examples where the graph theory weak RH holds and where there are poles at radius $1/\sqrt{q}$. In summary, we now have an analog of the Riemann hypothesis for irregular graphs X , regular or irregular,

(Graph theory Riemann hypothesis) $\zeta_X(u)$ is pole free for

$$R_X < |u| < \sqrt{R_X},$$

as well as a weakened version - reducing the pole free region,

(Weak graph theory Riemann hypothesis) $\zeta_X(u)$ is pole free for

$$R_X < |u| < 1/\sqrt{q}.$$

What about the graph theory analog of RH-III? The answer is surprising. The analog of RH-III is true for all regular graphs. That happens because, for $(q + 1)$ -regular graphs, $R = 1/q$ and, thanks to the relations between the poles and the eigenvalues of the adjacency matrix (see [12], p. 129), poles of $\zeta_X(u)$ not on the circle of radius $q^{-1/2}$ must be real.

But in fact the "modified weak Riemann hypothesis" (the analog of RH-III) is true for all graphs, regular and irregular. This is the content of a theorem in [5] quoted above in (6): if μ is a pole of $\zeta_X(u)$ and $|\mu| < q^{-1/2}$ then μ is real!

The proof is straightforward except on the circle $|u| = R_X$ which becomes quite interesting when $\delta_X > 1$. In this case, the methods of Example 4 apply and there is no contradiction on the circle $|u| = R_X$ to the claim that when μ is a pole of $\zeta_X(u)$ and $|\mu| < q^{-1/2}$ then μ is real. We leave this as an exercise to the reader.

Define

$$S_X = I - A_X u + Q_X u^2. \quad (10)$$

Ihara's formula (5) says that

$$\zeta_X(u)^{-1} = (1 - u^2)^{r_X - 1} \det(S_X).$$

Remark 2. Note that $u = 1$ is always a root of $\det(S_X) = 0$, with S_X defined in equation (10). This happens since

$$(-A_X + I + Q_X) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

3. EXAMPLES.

Our next example occupies most of this section.

EXAMPLE 2. Let K_n be the complete graph on n vertices; which is a regular graph of degree $n - 1$. For an (undirected) edge e of K_n , $K'_n = K_n - e$ denotes the graph obtained from K_n by removing the edge e . We will show that K'_n satisfies the naive Ramanujan inequality (9) and is Ramanujan in Lubotzky's sense. Moreover the Ihara zeta function of K'_n satisfies the weak Riemann hypothesis but not the full Riemann hypothesis. Indeed the weak Riemann hypothesis will be best possible for this example.

The proof of these statements requires two Lemmas. For any graph we define

$$\Lambda_X = \{u \in \mathbb{C} \mid \zeta_X(u)^{-1} = 0\}, \quad \text{counting multiplicity.} \quad (11)$$

LEMMA 1. *Here we use the notation set up in Example 2.*

1) The **Riemann hypothesis** holds for $\zeta_{K'_n}(u)$. The weak Riemann hypothesis holds for $\zeta_{K'_n}(u)$ but not the Riemann hypothesis.

2) If $P(u) = -1 + u(n - 4) + u^2(n - 3) + u^3(n - 3)(n - 2)$, then, with S_X as in (10), we have

$$\det(S_{K'_n}) = P(u)(u - 1)(1 + (n - 3)u^2)(1 + u + (n - 2)u^2)^{n-3},$$

while

$$\det(S_{K_n}) = (u - 1)(1 + u + (n - 2)u^2)^{n-1}(-1 + (n - 2)u).$$

Then, using the notation in formula (11),

$$\Lambda_{K'_n} \cap \Lambda_{K_n} \geq |\Lambda_{K_n}| - 7.$$

Proof. 2) First recall Ihara's determinant formula (5)

$$\zeta_X(u)^{-1} = (1 - u^2)^{r_X - 1} \det(S_X).$$

The factors $(1 - u^2)^{r_X - 1}$ are easily compared for $X = K_n$ and $X = K'_n$. The rank r_X of the fundamental group of X is

$$r_X - 1 = |E| - |V| = \frac{1}{2} \text{Tr}(Q - I).$$

From this, one has

$$r_{K'_n} = r_{K_n} - 1 = \frac{n(n-3)}{2}.$$

We turn to the comparison of $\det(S_{K_n})$ and $\det(S_{K'_n})$. We see that for $q = n - 2$,

$$S_{K_n} = \begin{pmatrix} f & g \\ {}^t g & h \end{pmatrix}, \quad \text{where } h = \begin{pmatrix} 1 + qu^2 & -u & -u & \cdots & -u \\ -u & 1 + qu^2 & -u & \cdots & -u \\ -u & -u & 1 + qu^2 & \cdots & -u \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -u & -u & -u & \cdots & 1 + qu^2 \end{pmatrix},$$

$$g = \begin{pmatrix} -u & -u & \cdots & -u \\ -u & -u & \cdots & -u \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 1 + qu^2 & -u \\ -u & 1 + qu^2 \end{pmatrix}.$$

Here h is $(n-2) \times (n-2)$, g is $(n-2) \times 2$ and f is 2×2 . Now suppose that we remove the edge e between the 1st and 2nd vertices of K_n to get K'_n . Again, with $q = n - 2$,

$$S_{K'_n} = \begin{pmatrix} f' & g \\ {}^t g & h \end{pmatrix}, \quad \text{where } f' = \begin{pmatrix} 1 + (q-1)u^2 & 0 \\ 0 & 1 + (q-1)u^2 \end{pmatrix},$$

with the same g and h as above.

The eigenvalues of A_{K_n} are $\left\{ q + 1, \underbrace{-1, \dots, -1}_{n-1} \right\}$. A set of eigenvectors is

$$e_1 = {}^t(1 \ 1 \ 1 \ 1 \ \cdots \ 1 \ 1), \quad e_2 = {}^t(1 \ 0 \ 0 \ 0 \ \cdots \ 0 \ -1),$$

$$e_3 = {}^t(0 \ 1 \ 0 \ 0 \ \cdots \ 0 \ -1), \dots, e_n = {}^t(0 \ 0 \ 0 \ 0 \ \cdots \ 1 \ -1),$$

where ${}^t v$ denotes the transpose of the vector v . These are also eigenvectors of Q_{K_n} and S_{K_n} . Only e_1, e_2, e_3 fail to be eigenvectors for $S_{K'_n}$. However, we can do slightly better. We replace e_3 by $e'_3 = e_3 - e_2$ which is still an eigenvector of $S_{K'_n}$ with the same eigenvalue as e_3 . But now e'_3 is an eigenvector of $S_{K'_n}$ (eigenvalue $1 + (n-3)u^2$).

Now we compute the exact polynomial which is $\zeta_{K'_n}(u)^{-1}$. First we block lower triangularize the matrix $S_{K'_n}$ using the matrix

$$B = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix},$$

the columns of which are $e_1, e_2, e'_3, e_4, \dots, e_n$.

One finds that

$$nB^{-1}S_{K'_n}B = \begin{pmatrix} f'' & 0 \\ g'' & h'' \end{pmatrix},$$

where f'' is the 2×2 matrix below and h'' is the $(n-2) \times (n-2)$ matrix below:

$$\begin{aligned} f'' &= \begin{pmatrix} (u-1)((n(n-2)-2)u-n) & u(1-u) \\ 2(n-2)u(1-u) & n+(2n-2)u+(n-2)(n-1)u^2 \end{pmatrix}, \\ h'' &= n * \text{diag}(1+(n-3)u^2, \underbrace{1+u+(n-2)u^2, \dots, 1+u+(n-2)u^2}_{n-3}). \end{aligned}$$

Therefore if $P(u) = -1 + u(n-4) + u^2(n-3) + u^3(n-3)(n-2)$,

$$\det(S_{K'_n}) = P(u)(u-1)(1+(n-3)u^2)(1+u+(n-2)u^2)^{n-3}, \quad (12)$$

while

$$\det(S_{K_n}) = (u-1)(1+u+(n-2)u^2)^{n-1}(-1+(n-2)u).$$

So Λ_{K_n} and $\Lambda_{K'_n}$ have at least the following in common

$$\left\{ \text{roots of } (1-u)(1+u+(n-2)u^2)^{n-3}(1-u^2)^{r_{K_n}-2} \right\}.$$

The order is greater than or equal to $|\Lambda_{K_n}| - 7$.

1) We know from (7) that $p \leq \omega_{K'_n} \leq q$, where $p = n-3$ and $q = n-2$. To see that the poles of $\zeta_{K'_n}(u)$ satisfy the weak Riemann hypothesis but not the Riemann hypothesis, observe that by the preceding discussion, the poles of $\zeta_{K'_n}(u)$ have absolute values $1, p^{-1/2}, q^{-1/2}, \alpha, |\beta|$ where

$$\begin{aligned} P(u) &= -1 + u(n-4) + u^2(n-3) + u^3(n-3)(n-2) \\ &= (n-3)(n-2)(u-\alpha)(u-\beta)(u-\bar{\beta}). \end{aligned}$$

Then one can show that

$$\frac{1}{q} < \alpha = R < \frac{1}{\sqrt{q}} < \sqrt{R} < |\beta| < \frac{1}{\sqrt{p}}.$$

The poles with $|u| = q^{-1/2}$ contradict the Riemann hypothesis but not the weak Riemann hypothesis. ■

LEMMA 2. K'_n satisfies the naive Ramanujan inequality (9) and is Ramanujan in Lubotzky's sense.

Proof. The matrix B that we used in the proof of Lemma 1 will block upper triangularize the adjacency matrix \tilde{A} of the graph K'_n . We obtain

$$B^{-1}\tilde{A}B = \begin{pmatrix} \tilde{f} & 0 \\ \tilde{g} & \tilde{h} \end{pmatrix}, \quad \text{where } \tilde{f} \text{ is a } 2 \times 2 \text{ matrix, } \tilde{h} \text{ is diagonal } (n-2) \times (n-2).$$

The matrices \tilde{f} and \tilde{h} are:

$$\tilde{f} = \frac{1}{n} \begin{pmatrix} (n-2)(n+1) & -1 \\ 2(2-n) & 2(1-n) \end{pmatrix}, \quad \tilde{h} = \text{diag}(0, -1, -1, \dots, -1).$$

Then $\det \begin{pmatrix} (n-2)(n+1) - nu & -1 \\ 2(2-n) & 2(1-n) - nu \end{pmatrix} = 4n^2 - 2n^3 - n^3u + 3n^2u + n^2u^2$.

The roots of this polynomial are $\gamma = \frac{1}{2}n - \frac{3}{2} + \frac{1}{2}\sqrt{n^2 + 2n - 7}$ and its conjugate $\gamma' = \frac{1}{2}n - \frac{3}{2} - \frac{1}{2}\sqrt{n^2 + 2n - 7}$. We see that in fact we have a much stronger inequality than (9) since we have $|\gamma'| \leq 2$, while $|\gamma| \geq n-2$. Thus it is easily checked that $|\gamma'| \leq \sqrt{d-1}$ and thus, by the Hoory inequality (8), the graph K'_n is Ramanujan in Lubotzky's sense. ■

We turn to another family of graphs where we know the spectrum and zeta function zeros.

EXAMPLE 3. Let $X = K_{m,n}$ be the complete biregular bipartite graph on $m + n$ vertices. Thus the vertices are split into two sets V_1 with m vertices, each of degree n and V_2 with n vertices, each of degree m . $K_{m,n}$ satisfies our naive Ramanujan inequality (9) and it is also Ramanujan in Lubotzky's sense. The poles of the Ihara zeta function of $K_{m,n}$ satisfy the weak but not the strong Riemann hypothesis. The pole-free region of the weak Riemann hypothesis is, in fact, best possible.

The adjacency matrix of $K_{m,n}$ is

$$A = \begin{pmatrix} 0 & J_{m,n} \\ J_{n,m} & 0 \end{pmatrix},$$

where $J_{m,n}$ is the $m \times n$ matrix of ones. We assume $m \leq n$. Set $m = p + 1, n = q + 1$, with $p \leq q$.

Clearly

$$A^2 = \begin{pmatrix} nJ_{m,m} & 0 \\ 0 & mJ_{n,n} \end{pmatrix}.$$

The eigenvalues of $J_{k,k}$ are well known (and easily seen) to be $k, 0, \dots, 0$ (with $k - 1$ zeros). Thus the spectrum of A^2 is $\{mn, 0\}$ with mn being an eigenvalue of multiplicity 2 and 0 an eigenvalue of multiplicity $m + n - 2$. The spectrum of a bipartite graph comes in \pm pairs. Thus the spectrum of A is $\{-\sqrt{mn}, \sqrt{mn}, 0\}$ with $\pm\sqrt{mn}$ having multiplicity 1. Nothing can be more Ramanujan than this (using any of the inequalities we have given)! The spectral radius of the adjacency operator on the universal cover of any bipartite graph is $\sqrt{p} + \sqrt{q}$. See Godsil and Mohar [1].

The poles of $\zeta_X(u)$ are more complicated to find, but fortunately a nice result of Hashimoto (see [2], pp. 230, 260, 270) gives a recipe for biregular bipartite graphs which, in this instance, says

$$\begin{aligned} \zeta_X(u)^{-1} &= (1 - u^2)^{r-1} (1 + pu^2)^{n-m} \det((1 + pu^2)(1 + qu^2)I_m - nJ_{m,m}u^2) \\ &= (1 - u^2)^{pq} (1 - pqu^2)(1 + pu^2)^q (1 + qu^2)^p. \end{aligned}$$

Thus there are poles at $u = \frac{1}{\sqrt{pq}}$, $u = \frac{-1}{\sqrt{pq}}$ (a Siegel pole), $u = \pm 1$ and the remaining poles are on the circles (moving out from the origin), $|u| = \frac{1}{\sqrt{q}}$, $|u| = \frac{1}{\sqrt{p}}$. The first of these circles is the weak Riemann hypothesis circle. The full Riemann hypothesis circle is $|u| = \frac{1}{\sqrt[4]{pq}}$ and it contains no poles. So we see that the Riemann hypothesis is false while the weak Riemann hypothesis is true. See also Li and Solé [7] for information on biregular bipartite graphs.

Thus far, we have not presented an example of an irregular graph satisfying the full Riemann hypothesis. We now present an infinite family of such graphs.

EXAMPLE 4. Let X be a $(q + 1)$ -regular (connected) Ramanujan graph with $q > 1$. Thus $\zeta_X(u)$ satisfies the Riemann hypothesis. Let X_δ denote the graph that results from X by inserting $(\delta - 1)$ extra vertices on each edge of X so that any path on X of length n becomes a path on X_δ of length $n\delta$. Thus $\zeta_{X_\delta}(u) = \zeta_X(u^\delta)$. For this reason, the closest pole to the origin of $\zeta_{X_\delta}(u)$ is at radius $\frac{1}{q^{1/\delta}}$ and the Riemann hypothesis circle is at radius $\frac{1}{q^{2\delta}}$. Thus the Riemann hypothesis is satisfied for the graphs X_δ . The weak Riemann hypothesis is also satisfied here but for $\delta > 2$, the weak Riemann hypothesis circle at radius $\frac{1}{\sqrt{q}}$ is closer to the origin than the closest pole of $\zeta_{X_\delta}(u)$ and so the weak Riemann hypothesis bound carries no useful information at all for ζ_{X_δ} , when $\delta > 2$.

EXAMPLE 5. Random graph. Using Mathematica on a PC, we experimented with the locations of the poles of $\zeta_X(u)$ for a random (irregular) graph X . The results of a typical experiment with X having 80 vertices are found in Figure 1. This figure indicates the the roots

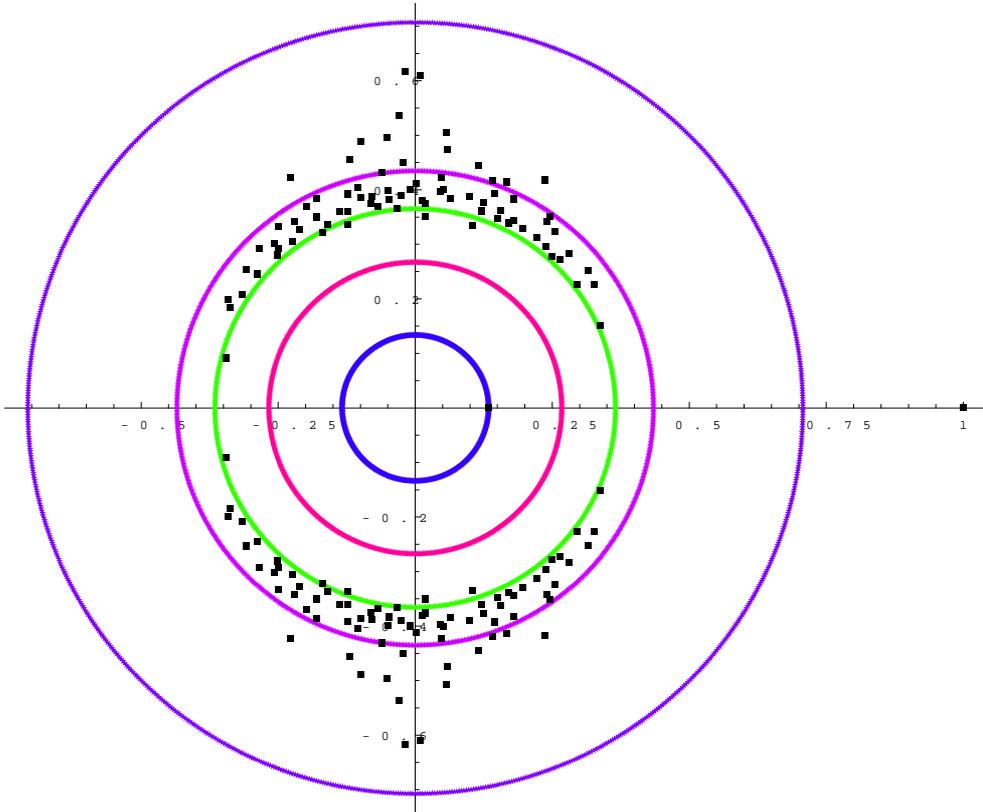


FIG. 1 All poles except -1 of $\zeta_X(u)$ for a random graph with 80 vertices are denoted by little boxes. The maximum degree was $q + 1 = 15$ while the minimum degree was $p + 1 = 3$. The 5 circles are centered at the origin and have radii $R, q^{-1/2}, \sqrt{R}, (pq)^{-1/4}, p^{-1/2}$. For this graph, the naive Ramanujan inequality is false as is the Riemann hypothesis, while the weak Riemann hypothesis is true.

of $\det(I - Au + Qu^2)$ with little black boxes. The 5 circles are all centered at the origin and have radii

$$R < q^{-1/2} < \sqrt{R} < (pq)^{-1/4} < p^{-1/2}.$$

Here $q + 1 = 15$ and $p + 1 = 3$ are the maximum and minimum degrees of vertices of X . In this experiment the naive Ramanujan inequality was false. The Riemann hypothesis was also false, while the weak Riemann hypothesis was true.

The exact command in Mathematica was `RandomGraph[80,1/10]`. This means that there was a probability of $1/10$ of an edge between 2 vertices. More information on the model used to generate random graphs in Mathematica can be found in Skiena [10].

Since A_X and Q_X do not necessarily commute when X is irregular, we cannot simultaneously diagonalize A_X and Q_X and write

$$\zeta_X(u)^{-1} = (1 - u^2)^{r-1} \prod_{\lambda \in \text{spectrum}(A_X)} (1 - \lambda u + q_\lambda u^2).$$

Thus we cannot easily relate the spectrum of A (the Ramanujan property) to the poles of $\zeta_X(u)$ (the Riemann and weak Riemann hypotheses).

4. PROOF OF THEOREM 1.

Before we prove Theorem 1, we need a Lemma.

LEMMA 3. *The invariant δ of Definition 5 equals $\delta' \triangleq$*

$$\text{g.c.d.} \left\{ \nu(P) \mid \begin{array}{l} P \text{ is backtrackless and the initial and terminal vertices of } P \text{ are} \\ \text{(possibly equal) nodes and no intermediate vertex is a node} \end{array} \right\}.$$

Proof. Clearly $\delta | \delta'$.

To show $\delta' | \delta$, note that anything in the length set for δ is a sum of elements of the length set for δ' . ■

The following Lemma shows that W_X from Definition 2 is irreducible (see Horn and Johnson [4], p. 362).

LEMMA 4. *Suppose X has rank at least 2. Given a directed edge e_1 starting at a vertex v_1 and a directed edge e_2 terminating at a vertex v_2 in X ($v_1 = v_2$, $e_1 = e_2$, $e_1 = e_2^{-1}$ are allowed), there exists a backtrackless path $P = P(e_1, e_2)$ from v_1 to v_2 with initial edge e_1 , terminal edge e_2 , and length $\leq 2|E|$. Equivalently, the matrix $(I + W_X)^{2|E|-1}$ has all positive entries.*

Remark 3. Lemma 4 is false for rank 1 graphs.

Proof. See Figure 2 which shows our construction of $P(e_1, e_2)$ in two cases. First we construct a path P without worrying about its length. This construction is hardly minimal, but it has the virtue of having relatively few cases to consider.

Choose a spanning tree T of X . Recall that by "cut" edge of X we mean an edge left out of T . We commence by creating two backtrackless paths $P_1 f_1$ and $P_2 f_2$ with initial edges e_1 and e_2^{-1} and terminal edges f_1 and f_2 such that f_1 and f_2 are cut edges (i.e., non-tree edges of X). If e_1 is a cut edge, we let P_1 have length 0 and $f_1 = e_1$ (i.e., $P_1 f_1 = e_1$). If e_1 is not a cut edge, we take P_1 to be a backtrackless path in the tree with initial edge e_1 which proceeds along T until it is impossible to go any further along the tree. Symbolically we write $P_1 = e_1 T_1$, where T_1 is a path along the tree, possibly of length zero. Let v'_1 be the terminal vertex of P_1 . With respect to the tree T , v'_1 is a dangler (vertex of degree 1), but X has no danglers. Thus there must be a directed cut edge in X , which we take to be f_1 , with initial vertex v'_1 . By construction, $P_1 f_1$ is backtrackless also since P_1 is in the tree and f_1 isn't.

Likewise, if e_2 is a cut edge, we let P_2 have length 0 and $f_2 = e_2^{-1}$ (i.e., $P_2 f_2 = e_2^{-1}$). If e_2 is not a cut edge, then as above we create a backtrackless path $P_2 f_2 = e_2^{-1} T_2 f_2$ where T_2 is in the tree, possibly of length 0 and f_2 is a cut edge. In all cases, we let v'_1 and v'_2 be the initial vertices of f_1 and f_2 .

Now, if we can find a path P_3 beginning at the terminal edge of f_1 and ending at the terminal vertex of f_2 such that the path $f_1 P_3 f_2^{-1}$ has no backtracking, then $P = P_1 f_1 P_3 f_2^{-1} P_2^{-1}$ will have no backtracking, with e_1 and e_2 as its initial and terminal edges, respectively. Of course, creating the path $f_1 P_3 f_2^{-1}$ is the original task of this Lemma. However, we now have the additional information that f_1 and f_2 are cut edges of the graph X .

We now have two cases. Case 1 is the case that $f_1 \neq f_2$, which is pictured at the top of Figure 2. In this case we can take $P_3 = T_3 =$ the path within the tree T running from the terminal vertex of f_1 to the terminal vertex of f_2 . Then, even if the length of T_3 is 0, the path $f_1 T_3 f_2^{-1}$ has no backtracks and we have created P .

Case 2 is $f_1 = f_2$. Thus f_1 and f_2 are the same cut edge of X . The worst case scenario in this case would have $e_2 = e_1^{-1}$, $T_2 = T_1$, $f_2 = f_1$. See the lower part of Figure 2. Since X has rank at least 2, there is another cut edge f_3 of X with $f_3 \neq f_1$ or f_1^{-1} . Let T_3 be the path along the tree T from the terminal vertex of f_1 to the initial vertex of f_3 and let T_4 be the path along the tree T from the terminal vertex of $f_2 = f_1$ to the terminal vertex of f_3 . Then $P_3 = T_3 f_3 T_4^{-1}$ has the desired property that $f_1 P_3 f_2^{-1}$ has no backtracking, even if T_3 and/or T_4 have length 0. Thus we have created in all cases a backtrackless path P with initial edge e_1 and terminal edge e_2 .

You can create a path P of length $\leq 2|E|$ as follows. If an edge is repeated, it is possible to delete all the edges in between the 1st and 2nd versions of that edge as well as the 2nd version of the edge without harming the properties of P .

Look at the e, f entry of $(I + W_X)^{2|E|-1}$. Take a backtrackless path P starting at e and ending at f . We have just shown that we can assume that the length of P is $\nu = \nu(P) \leq 2|E|$. Look at the e, f entry of the matrix $W^{\nu-1}$ which is a sum of terms of the form $w_{e_1 e_2} \cdots w_{e_{\nu-1} e_\nu}$, where each e_{i_j} denotes an oriented edge and $e_1 = e, e_\nu = f$. The term corresponding to the path P will be positive and the rest of the terms are non-negative. ■

EXAMPLE 6. Consider the graph in Figure 3. The shortest possible $P = P(e_1, e_1^{-1})$ is the path $P = (e_1 \cdots e_n) e_{n+1} (e_1 \cdots e_n)^{-1}$ of length $2n + 1 = 2|E| - 1$.

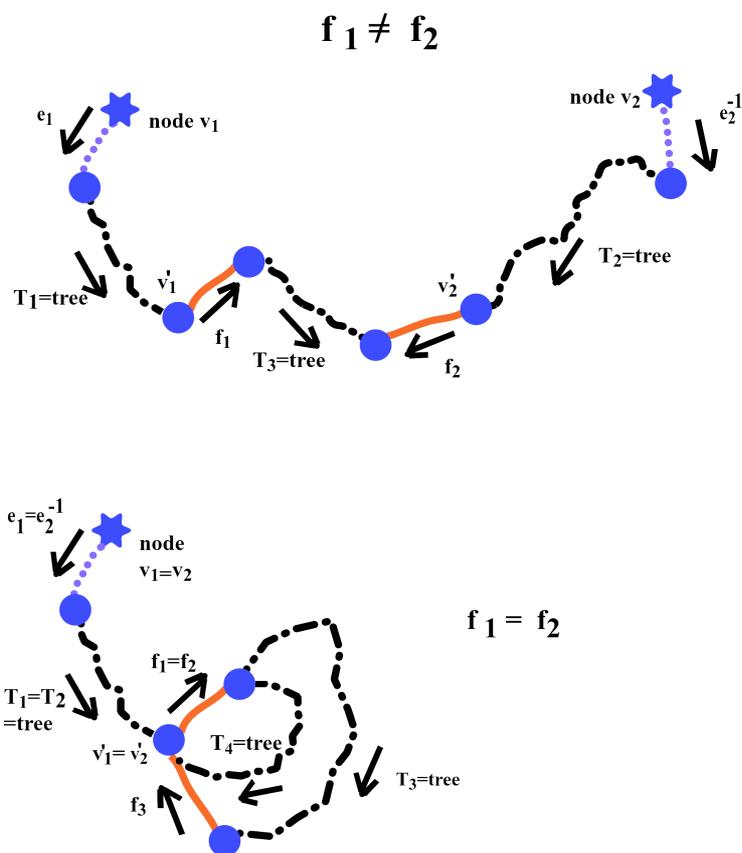


FIG. 2 The paths in Lemma 4. Here dashed paths are along the spanning tree of X . The edges e_1 and e_2 may not be edges of X cut to get the spanning tree T . But f_1, f_2 and (in the second case) f_3 are cut or non-tree edges. Note that the lower figure does not show the most general case as f_3 need not touch $f_1 = f_2$.

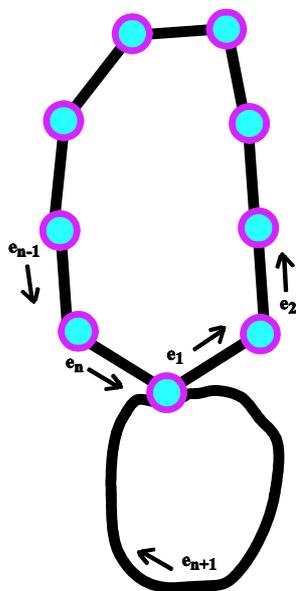


FIG. 3 Example of a graph with shortest path $P(e_1, e_1^{-1})$ from Lemma 4 having length $2|E| - 1$.

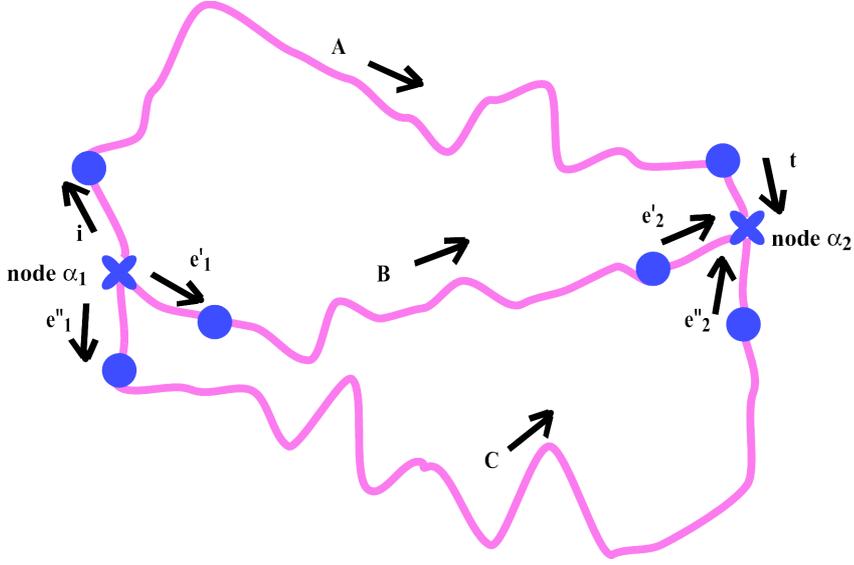


FIG. 4 The paths A, B, C in the proof of Theorem 1 when the nodes α_1 and α_2 are distinct.

Proof of Theorem 1.

Proof. Theorem 1 says that if Δ_X is odd then $\Delta_X = \delta_X$ and otherwise either $\Delta_X = \delta_X$ or $\Delta_X = 2\delta_X$. First observe that $\delta|\Delta$ since every cycle in a graph X of rank ≥ 2 has a node (otherwise X would not be connected). Second we show that $\Delta|2\delta$.

Note that in the special case that X has a loop, the vertex of the loop must be a node (when the rank is ≥ 2) and thus $\Delta = \delta = 1$. **So we will assume for the rest of the proof that X does not have loops.**

By Lemma 3 we may consider only backtrackless paths A between arbitrary nodes α_1 and α_2 without intermediate nodes. There are two cases. In the first case, $\alpha_1 \neq \alpha_2$. Let e'_1 be an edge out of α_1 not equal to the initial edge i of A (or i^{-1} since there are no loops) and e'_2 be an edge into α_2 not equal to the terminal edge t of A (or t^{-1}). Let $B = P(e'_1, e'_2)$ from Lemma 4.

Suppose e''_1 is another edge out of α_1 such that $e''_1 \neq i$, $e''_1 \neq e'_1$ (or their inverses). Likewise suppose e''_2 is another edge into α_2 such that $e''_2 \neq t$, $e''_2 \neq e'_2$ (or their inverses). Let $C = P(e''_1, e''_2)$ from Lemma 4. See Figure 4.

Then $AB^{-1}, AC^{-1}, BC^{-1}$ are backtrackless tailless paths from α_1 to α_1 .

We have

$$\begin{aligned}\Delta|\nu(AB^{-1}) &= \nu(A) + \nu(B), \\ \Delta|\nu(AC^{-1}) &= \nu(A) + \nu(C), \\ \Delta|\nu(BC^{-1}) &= \nu(B) + \nu(C).\end{aligned}$$

It follows that Δ divides $2\nu(A)$ since

$$2\nu(A) = (\nu(A) + \nu(B)) + (\nu(A) + \nu(C)) - (\nu(B) + \nu(C)).$$

Now we consider the case that $\alpha_1 = \alpha_2$. Then A is a backtrackless path from α_1 to α_1 without intermediate nodes. This implies that A has no tail, since then the other end of the tail would have to be an intermediate node. Therefore Δ divides $\nu(A)$ and hence Δ divides $2\nu(A)$. So now, in all cases, Δ divides $2\nu(A)$ and hence $\Delta|2\delta$. ■

5. SIEGEL POLES

LEMMA 5. *Suppose Y is an n -sheeted covering of X . The maximal absolute value of an eigenvalue of W_Y is the same as that for W_X . This common value is $R_Y^{-1} = R_X^{-1} = \omega_Y = \omega_X$.*

Proof. First note that from Stark and Terras [12], we know $\zeta_X^{-1}(u)$ divides $\zeta_Y^{-1}(u)$. It follows that $R_Y \leq R_X$. Then a standard estimate from the theory of zeta functions of number fields works for graph theory zeta functions as well. For all real $u \geq 0$ such that the infinite product for $\zeta_X(u)$ converges, we have $\zeta_Y(u) \leq \zeta_X(u)^n$, with n equal to the number of sheets of the covering. Thus $R_X \leq R_Y$. See Lang [6] (p. 160) for the idea of the proof of the inequality relating ζ_Y and ζ_X which comes from the product formula and the behavior of primes in coverings. ■

The referee notes that there is another proof of the preceding Lemma. We will discuss this after the proof of Theorem 2. See Remark 4.

LEMMA 6. $\zeta_Y(u) = f(u^d)$ if and only if d divides Δ_Y .

Proof. Clearly $\zeta_Y(u)$ is a function of u^{Δ_Y} and therefore of u^d for all divisors d of Δ_Y . Conversely suppose $\zeta_Y(u)$ is a function of u^d . Thus, in the power series (3), d divides n for all n with $a_n > 0$. But if P is a prime cycle of Y with length $\nu(P) = n$, then $a_n \geq 1$ and hence for all prime cycles P , d divides $\nu(P)$. Therefore, by Definition 3, d divides Δ_Y . ■

Proof of Theorem 2.

Proof. We first reduce the Theorem to the case that Y/X is normal. To see that this is possible, let \tilde{Y} be a normal cover of X containing Y . Since $\zeta_{\tilde{Y}}(u)^{-1}$ is divisible by $\zeta_Y(u)^{-1}$ and both graphs have the same R (by Lemma 5), as well as the same δ , it follows that a Siegel pole of $\zeta_Y(u)$ is a Siegel pole of $\zeta_{\tilde{Y}}(u)$. Once the Theorem is proved for normal covers of X , the graph X_2 which we obtain will be contained in Y as well as in every graph intermediate to Y/X whose zeta function has the Siegel pole and we will be done. From this point on, we assume Y/X is normal. Let G be the Galois group of Y/X .

Recall formula (4), $\zeta_X(u)^{-1} = \det(I - W_X u)$ and Definition 2 of the 0,1 edge matrix W_X . Poles of $\zeta_X(u)$ are thus reciprocal eigenvalues of W_X . Note that for graphs of rank ≥ 2 the edge matrix W_X satisfies the hypotheses of the Perron-Frobenius theorem, namely that W_X is irreducible. See Horn and Johnson [4], pp. 360 and 508. For W_X is irreducible iff $(I + W_X)^{2|E|-1}$ has all positive entries and this is true by Lemma 4.

By Lemma 5, the Perron-Frobenius Theorem (see Horn and Johnson [4], pp. 360 and 508) now says that if there are d poles of $\zeta_Y(u)$ on $|u| = R_Y = R_X = \frac{1}{\omega}$, then these poles are equally spaced first order poles on the circle and further $\zeta_Y(u)$ is a function of u^d . By Lemma 6, Δ_Y has to be divisible by d . But $\delta = \delta_X = \delta_Y = 1$ implies $\Delta_Y = 1$ or 2. Therefore $d = 1$ or 2. If there is a Siegel pole, $d > 1$. Thus if there is a Siegel pole, $d = 2$, $\Delta_Y = 2$ and the equal spacing result says the Siegel pole is $-R_X$ and it is a pole of order one.

From Stark and Terras [13], we know that the Ihara zeta function of Y (a Galois cover of X) factors as follows as a product of Artin L-functions corresponding to irreducible representations of $G = \text{Gal}(Y/X)$:

$$\zeta_Y(u) = \prod_{\pi \in \hat{G}} L(u, \pi)^{d_\pi}. \quad (13)$$

Therefore $L(u, \pi)$ has a pole at $-R_X$ for some π and $d_\pi = 1$. Moreover π must be real or $L(u, \bar{\pi})$ would also have a pole at $-R_X$.

So either π is trivial or it is first degree and $\pi^2 = 1$, $\pi \neq 1$. Then we say π is **quadratic**.

Case 1. π is trivial.

Then $\Delta_X = 2$ just like $\Delta_Y = 2$. Every intermediate graph then has poles at $-R_X$ as well.

Case 2. $\pi = \pi_2$ is quadratic.

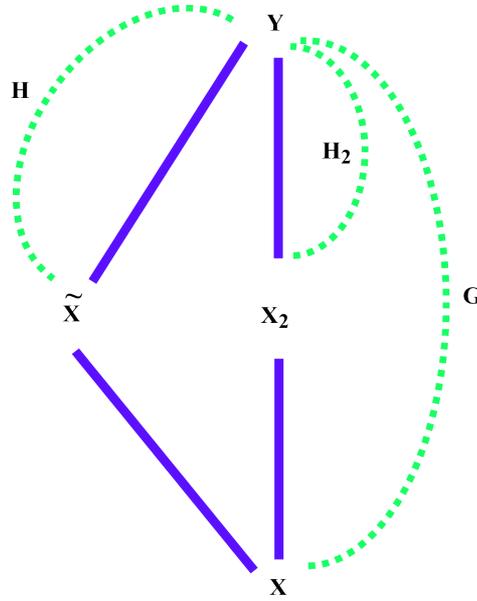


FIG. 5 The covering graphs in Theorem 2. Galois groups are indicated with dotted lines.

No other $L(u, \pi)$ has $-R_X$ as pole since it is a first order pole of $\zeta_Y(u)$. Suppose

$$H_2 = \{x \in G \mid \pi_2(x) = 1\} = \ker \pi_2.$$

Then $|G/H_2| = 2$ which implies there is a graph X_2 corresponding to H_2 by the Galois theory developed in Stark and Terras [13] and X_2 is a quadratic cover of X .

Consider the diagram of covering graphs with Galois groups indicated next to the covering lines in Figure 5. Then

$$\zeta_{\tilde{X}}(u) = L(u, \text{Ind}_H^G 1) = \prod_{\kappa \in \hat{G}} L(u, \kappa)^{m_\kappa}.$$

Now $L(u, \kappa)$ appears m_κ times in the factorization and Frobenius reciprocity says

$$m_\kappa = \langle \chi_{\text{Ind}_H^G 1}, \kappa \rangle = \langle 1, \kappa|_H \rangle \leq \deg \kappa.$$

Let $\kappa = \pi_2$, which has $\deg \kappa = 1$. This implies $\zeta_{\tilde{X}}(u)$ has $-R_X$ as a (simple) pole if and only if $\pi_2|_H = \text{identity}$. Note that $-R_X$ is not a pole of any $L(u, \pi)$, for $\pi \neq \kappa$. We have $\pi_2|_H = \text{identity}$ if and only if $H \subset H_2 = \ker \pi_2$, which is equivalent to saying \tilde{X} covers X_2 .

The proof is complete. X_2 is unique because each version of X_2 would cover the other. ■

Remark 4. Our proof of Lemma 5 is a standard technique in number theory. The referee points out that ω_X is the Perron-Frobenius eigenvalue of W_X (see Horn and Johnson [4]) so that the corresponding eigenvector e_X has all positive entries. This eigenvector lifts to an eigenvector e_Y of W_Y with all positive entries and the same eigenvalue ω_X . Thus ω_X is the Perron-Frobenius eigenvalue of W_Y ; therefore $\omega_X = \omega_Y$ and $R_X = R_Y$.

Note that if \tilde{X} is intermediate to Y/X in Theorem 2, then $\Delta(\tilde{X}) = 1$ or 2 and the Perron-Frobenius Theorem says $\zeta_{\tilde{X}}(u)$ is a function of u^d , where d is the number of poles of $\zeta_{\tilde{X}}(u)$ on the circle $|u| = \omega^{-1}$. Thus the \tilde{X} with $\Delta(\tilde{X}) = 2$ are exactly the \tilde{X} with $\zeta_{\tilde{X}}(u)$ having $-\omega^{-1}$

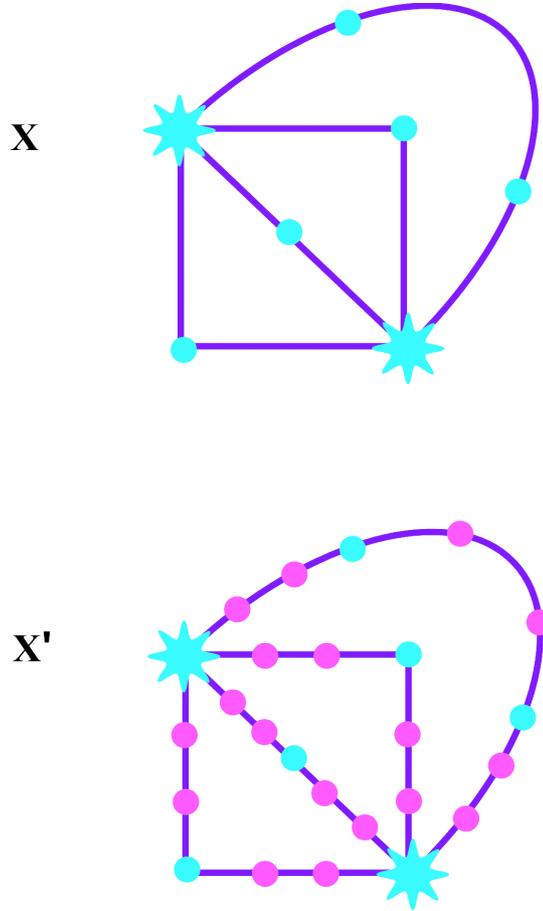


FIG. 6 The graph X is the deflation of X' with $\delta_{X'} = 3$, giving $\delta_X = 1$.

as a Siegel pole and these are the \tilde{X} which cover X_2 . Since $\Delta(\tilde{X}) = 2$ is the condition for \tilde{X} to be bipartite, this says that \tilde{X} is bipartite. \tilde{X} is not quadratic unless $\tilde{X} = X_2$. All remaining intermediate graphs \tilde{X} to Y/X have $\Delta_{\tilde{X}} = 1$.

Every graph X of rank ≥ 2 has a covering Y with zeta function having a Siegel pole; we will give a construction in Section 6. This is probably not the case for algebraic number fields.

COROLLARY 2. *Under the hypotheses of Theorem 2 with X_2 the unique graph defined in that Theorem, the set of intermediate bipartite covers to Y/X is precisely the set of graphs intermediate to Y/X_2 .*

For the next result, we need some definitions.

DEFINITION 7. The **inflation** $I^\delta(X)$ is defined by putting $\delta - 1$ vertices on every edge of X .

DEFINITION 8. The **deflation** $D_\delta(X)$ is obtained from X by collapsing δ consecutive edges between consecutive nodes to one edge.

See Figure 6 for examples.

The following theorem gives the analog of the preceding theorem for arbitrary δ .

THEOREM 3. *Suppose X is connected, not a cycle, with no dangles and $\delta = \delta_X = \Delta_X$. Suppose that Y covers X and Y is connected with $\Delta_Y = 2\Delta_X = 2\delta$. Then we have the following results.*

1. *There is a unique intermediate quadratic cover X_2 to Y/X such that $\Delta_{X_2} = 2\delta$.*
2. *Further, if \tilde{X} is any graph intermediate to Y/X then $\Delta_{\tilde{X}} = 2\delta$ if and only if \tilde{X} is intermediate to Y/X_2 .*

Proof. When $\delta > 1$, this is proved by deflation. The deflated graph $D_\delta(X) = X'$ contains all the information on X and its covers. This graph X' has $\delta_{X'} = 1$ and $\zeta_X(u) = \zeta_{X'}(u^\delta)$. Every single Y/X has a corresponding Y' covering X' such that

$$\zeta_Y(u) = \zeta_{Y'}(u^{\delta_X}).$$

There is also a relation between all the Artin L-functions

$$L_{Y/X}(u, \pi) = L_{Y'/X'}(u^{\delta_X}, \pi).$$

where π is a representation of $Gal(Y/X) = Gal(Y'/X')$. Theorem 3 now follows from Theorem 2 which contains the case $\delta = 1$ of Theorem 3. ■

Note that in Theorem 3 if $\delta = 1$, the \tilde{X} with $\Delta_{\tilde{X}} = 2\delta$ are the bipartite covering graphs intermediate to Y/X . and, in particular, X_2 is bipartite. Even when $\Delta_X = \delta_X = \delta$ is odd, the \tilde{X} with $\Delta_{\tilde{X}} = 2\delta$ are precisely the bipartite covering graphs intermediate to Y/X . But, if $\Delta_X = \delta_X = \delta$ is even, then every graph intermediate to Y/X , including X itself, is bipartite, and thus being bipartite does not determine which quadratic cover of X is X_2 . Note also that when the rank of X is ≥ 2 , we have proved the following purely graph theoretic equivalent theorem.

THEOREM 4. *Suppose X is a finite connected graph of rank ≥ 1 and that Y is a bipartite covering graph of X . Then we have the following facts.*

1. *When X is bipartite, every intermediate covering \tilde{X} to Y/X is bipartite.*
2. *When X is not bipartite, there is a unique quadratic covering graph X_2 intermediate to Y/X such that any intermediate graph \tilde{X} to Y/X is bipartite if and only if \tilde{X} is intermediate to Y/X_2 .*

6. COMBINATORIAL PROOF OF THEOREM 4

Here we give a combinatorial proof of Theorem 4. Danglers do not matter, and if we did not insist on all our graphs and covering graphs being connected, the theorem would hold for rank 0 graphs as well.

Combinatorial Proof of Theorem 4.

Proof. If X is bipartite, we need to show that any cover \tilde{X} of X is bipartite. The set V of vertices of X may be written as $V = A \cup B$, $A \cap B = \emptyset$, such that no vertices in A (or B) are adjacent. We label the vertices in set A with 1 and those in set B with 2. Then label the vertices of \tilde{X} above vertex v of X with the same label as that of vertex v .

If X is not bipartite, we know there exists a connected covering graph X_2/X such that the Galois group of X_2/X has order 2 by Stark and Terras [13]. But why does there exist a unique bipartite X_2 ? We construct such an X_2 and in the process we will discover that the entire process is forced. Take a spanning tree T in X . Label the vertices of T either 1 or 2 so that adjacent vertices within T always have different labels. Sheet 1 of X_2 is then labeled with the same labels as those of T and sheet 2 of X_2 has vertices with the same labels as T except that these are given primes. Then use α to relabel vertices in X_2 with labels 1, 2'. And use β to relabel vertices of X_2 with labels 1', 2. Since X is not bipartite, there exists a cut edge e of X joining 2 to 2 or 1 to 1. All such edges e (edges g and h as in the example shown in Figure 7) must lift to edges \tilde{e} going between sheets and, since there is at least one such edge, X_2 is now connected. Any cut edges f that connect a vertex labeled 1 in X to a vertex labeled 2 (edge i in Figure 7) must lift to edges \tilde{f} that start and terminate in the same sheet. Thus X_2 is uniquely determined by X .

Any graph covering X_2 is bipartite. Thus what remains to be proved is that if \tilde{X} is intermediate to Y/X and \tilde{X} is bipartite, then \tilde{X} is intermediate to Y/X_2 . Suppose \tilde{X} is such a graph. Since it is bipartite, we can label the vertices by α and β such that each edge of \tilde{X} has one vertex labeled α and one labelled β . Let π be the projection map from \tilde{X} to X and suppose \tilde{v} is a vertex of \tilde{X} projecting down to the vertex v of X . By construction, there are two vertices v' and v'' , say, in X_2 projecting to v , and one of these vertices is labeled α , the other β . We project \tilde{v} in \tilde{X} to whichever of v' and v'' in X_2 has the same label as \tilde{v} . This gives our projection map from \tilde{X} to X_2 .

If \tilde{e} is an edge of \tilde{X} with initial vertex \tilde{v} , then \tilde{e} projects to an edge e in X with initial vertex v and then lifts uniquely to edges e' and e'' in X_2 with initial vertices v' and v'' . Since the initial and terminal vertices of \tilde{e} , e' , and e'' all have opposite labels, this shows that our projection map from \tilde{X} to X_2 extends to edges and completes the proof that \tilde{X} is intermediate to Y/X_2 . ■

Remark 5. The referee notes that there is a simpler construction of X_2 . The vertices are $V \times \mathbb{Z}_2$ and the edges are between (u, a) and $(v, 1 - a)$, if (u, v) is an edge in X and $a \in \mathbb{Z}_2$. We prefer our construction of the covering graph using the spanning trees as sheets where it is easier to decide that the sheets are connected and hence that the covering graphs are connected.

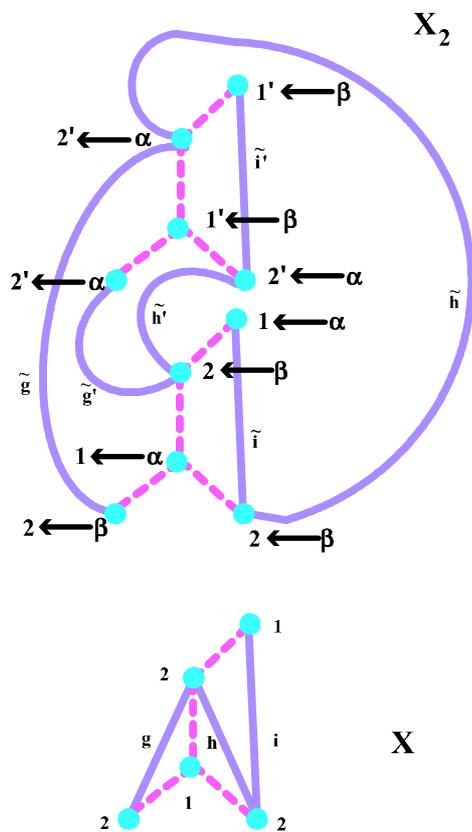


FIG. 7 Construction of Unique Bipartite Quadratic Cover X_2 of a Non-Bipartite Graph X . The labels on X_2 are α and β . The spanning tree of X is dashed.

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