A SEPARABLE ODE

\[ y' = \frac{1 - 2x}{y} \]

\[ y(1) = -2 \]
dy/dx = (1-2x)/y; \quad y(1) = -2

Put all the y's on one side and all the x's on the other. Then find the interval on the x-axis where the solution is defined.

\[ y \, dy = (1-2x) \, dx \]

Integrate

\[ \int y \, dy = \int (1-2x) \, dx \]

\[ \frac{1}{2} y^2 = x - x^2 + C \]

\[ y^2 = 2x - 2x^2 + C' \]

Plug x=1 and y=-2 to find C'. Obtain C'=4.

\[ y = -\sqrt{-2x^2 + 2x + 4} \]

We choose the minus sign thanks to the initial condition.
Where does this make sense? You can’t take the square root of negative numbers if you want a real answer.

This means we need

\[-2x^2 + 2x + 4 \geq 0\]
\[-x^2 + x + 2 \geq 0\]
\[x^2 - x + 2 \leq 0\]
\[x^2 - x + 2 = (x-2)(x+1) \leq 0\]

This only happens if \(-1 \leq x \leq 2\).

The endpoints correspond to \(y=0\) where the differential equation \(y' = (1-2x)/y\) does not make sense.
Another Example: \( y' = \frac{3x^2}{(3y^2 - 4)}; \quad y(1) = 0 \)

Matlab behaved badly and bombed my computer when I tried to get it to draw the numerical solution curve thru (1,0). Non-linear problems cause Matlab trouble.
Still we can find out much about the solution, although not a very good formula.

\[(3y^2-4)dy = 3x^2 \, dx\]

Integrate to get \[y^3-4y = x^3 + C.\]

\[y(1)=0 \text{ implies } C=-1.\]

Our “Solution” is: \[y^3-4y = x^3 - 1\]

Unfortunately this does not lead to a simple formula for \(y\) as a function of \(x\). But as one class member said: “You can easily solve for \(x\).”

Mathematica will give a very complicated answer with many roots inside of roots including some complex roots when you solve for \(y\) as we are asked to do in Section 2.2.
To figure out what values of $x$ are legal, look back at the differential equation.

$$y' = \frac{3x^2}{(3y^2-4)}$$

Where is the denominator $= 0$?

**Answer:** Where $y^2 = 4/3$

$$y = \pm\frac{2}{\sqrt{3}}$$

For this value of $y$ you can solve for $x$ using our equation

$$y^3 - 4y = x^3 + -1$$

We see that $y^3 - 4y + 1 = x^3$.

Thus $x = (y^3 - 4y + 1)^{1/3}$.

Plug $y = \pm\frac{2}{\sqrt{3}}$ into this and see that the legal $x$-values lie between $-1.276$ and $1.598$. 
Here's the direction field and a solution starting at the point $x=1$, $y=0$, drawn in blue.

It does seem to stretch from $-1.3$ to $1.6$, where it encounters vertical tangent lines that send $y(x)$ off to infinity. The red point is the only thing produced by Matlab when asked for a solution thru $(1,0)$.