Zeta functions of buildings and Shimura varieties

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1. Modular curves and graphs.

Around 1965 Y. Ihara introduced a zeta function for certain kinds of discrete groups. There were two main cases:

a. A discrete cocompact subgroup

$$\Gamma \subset \mathrm{PGL}(2, \mathbf{Q}_p)$$

b. A discrete finite covolume subgroup

$$\Gamma \subset \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbf{Q}_p)$$

Let D be a quaternion algebra over $\mathbb{Q}$ which is unramified at p. Let G be the algebraic group given by D modulo its center (resp. by the elements of norm $= 1$ in D, mod center). These are forms of $\mathrm{PGL}(2)_{\mathbb{Q}}$ (resp. $\mathrm{PSL}(2)_{\mathbb{Q}}$)

Let $\Gamma = G(\mathbb{Z}[\frac{1}{p}])$ (an S-arithmetic group).

1. If D is definite we get example a. In this case, Γ acts on the Bruhat-Tits building (tree) X associated to $\mathrm{PGL}(2, \mathbf{Q}_p)$ and the quotient $X_\Gamma = \Gamma \backslash X$ is a finite graph.

2. If D is indefinite we get example b. In this case,

$$\Gamma_0 = \Gamma \cap (\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbf{Z}_p))$$

acts on the symmetric space (upper half-plane) $\mathbf{H}$ associated to $\mathrm{SL}(2, \mathbb{R})$ and the quotient $Y_\Gamma = \Gamma_0 \backslash \mathbf{H}$ is the Riemann surface of complex points of an algebraic curve $\mathcal{Y}_\Gamma$ defined over a number field (Shimura curve).

In case b. one can let Γ act on the product $\mathbf{H} \times X$ of the upper half-plane and the tree, and take the quotient (Borel - Serre).

Ihara defined his zeta functions as:

$$\zeta_\Gamma(u) = \prod_\gamma \frac{1}{1 - u^{\deg(\gamma)}}$$
$$= \exp\left(\sum_{m=1}^{\infty} N_m(\Gamma) u^m / m\right).$$

where the product ranges over $[\Gamma]_T$, the set of nontrivial primitive T-bolic Γ-conjugacy classes in Γ. Here,

$$N_m(\Gamma) = \sum_{\deg(\gamma)|m} \deg(\gamma).$$

In this definition, T is a maximal torus, and for γ to be T-bolic means that it is conjugate to a regular element of T.

In case a, one takes T to be the split torus consisting of diagonal elements

$$\mathbf{G}_m(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \mod \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right\}.$$

The T-bolic elements of Γ are hyperbolic in the usual terminology.

If Γ is torsion-free, then every element is hyperbolic, and the Euler product is over all the nontrivial primitive conjugacy classes in Γ.

In case b, one takes

$$T = \mathbf{SO}(2, \mathbb{R}) \times \mathbf{G}_m(\mathbb{Q}_p) \subset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbf{Q}_p)$$

so that to be T-bolic means that it is elliptic at the real place and hyperbolic at the p-adic place.

The reason for this choice is that the (scaled) Frobenius matrices of *ordinary* (ie., not supersingular) elliptic curves have this property.

Theorem (Ihara) In both these cases, the zeta function is a rational function of u.

In fact, Ihara proved much more.

In case b it is often the case that the zeta function is essentially equal to the zeta function of a modular curve over a finite field: There is a Shimura curve Y_Γ such that

$$\zeta_\Gamma(u) = (1-u)^h \cdot Z(Y_\Gamma/\mathbb{F}_{p^2}, u)$$

The curve Y_Γ parameterizes elliptic curves with additional structures. The "correction factor" $(1-u)^h$ is there since we have we have systematically omitted the h points of Y_Γ corresponding to supersingular elliptic curves because of the choice of the torus T.

For the case a Ihara's formula is

$$\zeta_\Gamma(u) = \frac{(1-u^2)^\chi}{\det(1 - Au + pu^2)}$$

where A (resp. χ) is the adjacency matrix (resp. Euler characteristic) of the finite graph $X_\Gamma = \Gamma\backslash X$, quotient of the Bruhat-Tits tree X. This formula was generalized by Hashimoto, Bass, Stark-Terras to more general kinds of graphs (and there is a theory of L-functions as well). The connection between Ihara's theory and graphs was first noticed by Serre.

Ihara also observed that the denominator in case a, namely

$$\det(1 - Au + pu^2)$$

could in some situations identified with the numerator of the zeta function of a modular curve.

For instance, let Γ be the group of p-integers of the definite quaternion algebra of discriminant a prime $l \neq p$. Then Ihara showed that A is a Brandt matrix for the quaternion algebra at the prime p. It decomposes as $A = (p+1)1 \oplus T_p$, where T_p is the Hecke operator acting on the space of cusp forms of weight 2 for $\Gamma_0(l)$.

This is explained by the "Jacquet-Langlands correspondence", now understood as a special case of the principle of functorality for automorphic forms.

2. An Example: $X_0(37)$

This is a genus 2 modular curve studied by Mazur and Swinnerton-Dyer. An equation is $y^2 = -x^6 - 9x^4 - 11x^2 + 37.$

The following computations were done with MAGMA.

Define the quaternion algebra:

$$D = \mathbb{Q}[1,\ i,\ j,\ k]$$

$i^2 = -2$, $j^2 = -37$. Its maximal order is

$$O_D = \mathbb{Z}\left[1,\ \frac{1+i+j}{2}, j,\ \frac{2+i+k}{4}\right].$$

The essential part of the zeta function of $X_0(37)$ mod p is given by the Brandt matrix $B(p)$, and these can be interpreted as adjacency matrices of (weighted) graphs.

For instance, let $p = 3$. The Brandt matrix is

$$B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

which has eigenvalues $4 = p + 1,\ 1,\ -3$, so that

$$Z\left(X_0(37)/\mathbb{F}_3,\, u\right) \;=\; \frac{(1 - u + 3u^2)(1 + 3u + 3u^2)}{(1 - u)(1 - 3u)}$$

The points of $X_0(37)$ over $\mathbf{F}_3$ are

$$(2, \pm 1), \quad (1, \pm 1), \quad (0, \pm 1)$$

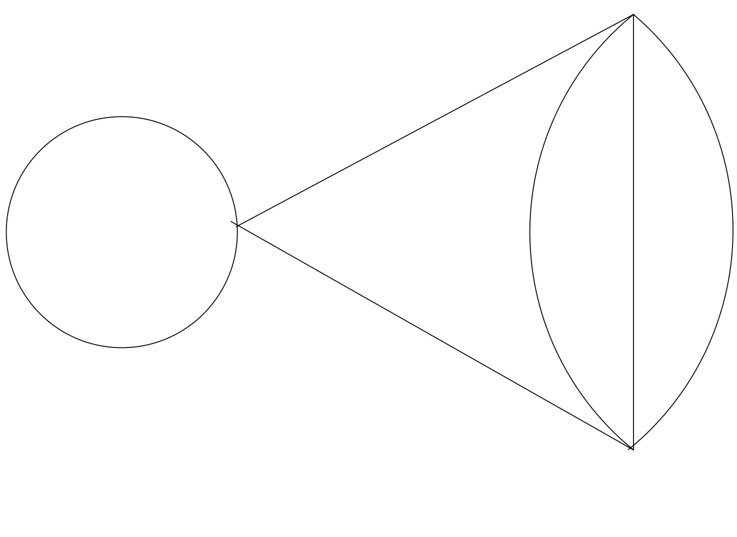
and the points over $\mathbf{F}_9$ are

$$\begin{array}{ccc} (2, \pm 1), & (1, \pm 1), & (0, \pm 1) \\ (2\sqrt{-1}, \pm 1), & (\sqrt{-1}, \pm 1), & \\ \infty_1 & \infty_2 & \end{array}$$

The unique point on the line at infinity $z = 0$ on the projectivization of our equation

$$y^2 z^4 = -x^6 - 9x^4 z^2 - 11x^2 z^4 + 37z^6$$

is singular. Resolving this singularity gives 2 points, but rational over $\mathbf{F}_9 = \mathbf{F}_3(\sqrt{-1})$.



$X_0(37)$ at the prime $p = 13$: Brandt matrix

$$B(13) = \begin{pmatrix} 2 & 6 & 6 \\ 6 & 3 & 5 \\ 6 & 5 & 3 \end{pmatrix}, \quad \text{eigenvalues} : 14 = p + 1,\ -2,\ -4$$

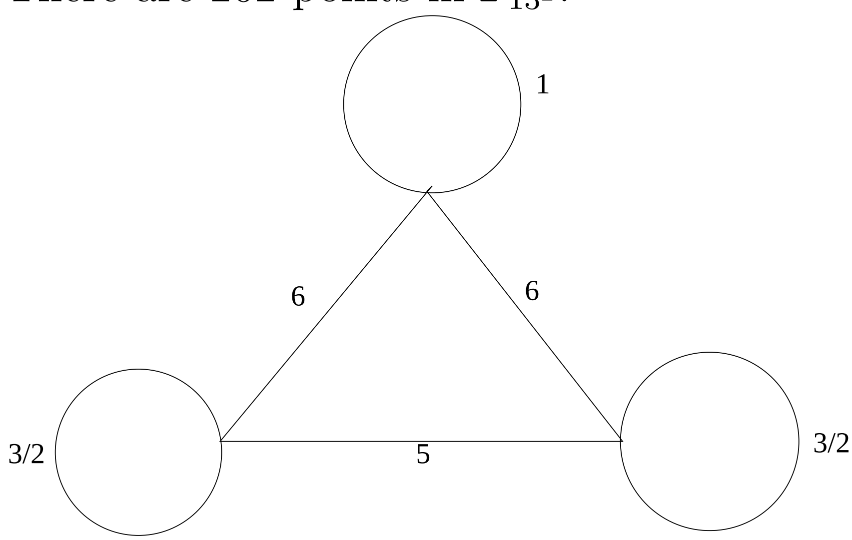
Zeta function:

$$Z\left(X_0(37)/\mathbb{F}_{13},\, u\right) \;=\; \frac{(1 + 2u + 13u^2)(1 + 4u + 13u^2)}{(1-u)(1-13u)}$$

Points in $\mathbb{F}_{13}$

$$\begin{array}{ccccc} (1, \pm 4) & (3, \pm 1) & (\pm 4, 0) & (5, \pm 1) & (6, \pm 4) \\ (7, \pm 4) & (8, \pm 1) & (12, \pm 4) & (10, \pm 1) & \infty_{1,2} \end{array}$$

There are 202 points in $\mathbb{F}_{13^2}$.



3. Zeta functions for buildings?

Let K be a nonarchimedian local field. Let G be the K-rational points of a reductive algebraic group over K. Let $\Gamma \subset G$ be a discrete cocompact subgroup of G. According to Bruhat and Tits, there is a contractible cell complex X on which Γ acts properly and cellularly, called the building. Its dimension r is the split rank of G, the dimension of a maximal split torus. The cells of various dimensions are in $1-1$ correspondence with the cosets G/B_I, where, $B =$ Iwahori subgroup, $W_I \subset W =$Weyl group,

$$B_I = BW_IB, \quad I \subset S \; = \text{generating reflections}$$

are the standard parahoric subgroups. Thus the cells of maximal dimension (chambers) are in one orbit and in a bijection with G/B. The vertices are in general in several orbits, corresponding to the maximal compact subgroups

Problem: Can one define zeta functions for the finite cell complexes $X_\Gamma = \Gamma\backslash X$ with good properties?

For instance, one would like a relation to the zeta functions of Shimura varieties similar to Ihara's results for $G = \mathrm{GL}(2)$. Recall that there is finite closed formula for the number of rational points $\#Y(\mathbf{F}_q)$ for a Shimura variety Y due to Langlands, Rapoport, Kottwitz.

There have been several proposals, due to Winnie Li, Anton Deitmar and myself.

4. Coxeter systems

Let $C^I(X_\Gamma)$ be the vector space spanned by the cells of type $I \subset S$. This is a right $H(G, B_I)$-module under convolution.

In particular, taking $I = \emptyset$, we have a finite-dimensional representation $R = C^\emptyset(X_\Gamma)$ of the Iwahori-Hecke algebra $H(G, B)$ and thus a Coxeter-Poincaré matrix $L(u, q, W_I, R)$ associated to it. This is due to Gyoja. The coefficients of the powers of t in the characteristic polynomial

$$\det(t1 - L(u, q, W_I, R))$$

are fundamental invariants of the discrete subgroup Γ and its action on the building X.

Special case: $G = \mathrm{SL}_2(K)$. Then we have the following formula for the zeta function of the finite graph:

Proposition. (H)

$$\frac{\det L(u,q,W,R_\Gamma)}{\det L(u,q,W_1,R_\Gamma)\det L(u,q,W_2,R_\Gamma)} = Z(X_\Gamma,u^2)$$

This suggests the following problems: Let R_Γ be the canonical representation of $H(G,B)$ on the cells of highest dimension of the quotient building $\Gamma\backslash X$. Define

$$L(X_\Gamma,u) = \prod_{I\subset S} \det L(u,q,W_I,R_\Gamma)^{(-1)^{\#I}}$$

It would be interesting to investigate the zeros and poles of this. Also, does it have an Euler product?

5. Product of graphs

In the REU program at LSU in 2005 a student of mine Zuhair Khandker defined two zeta functions for the 2-dimensional complex $X = X_1 \times X_2$ that is a product of graphs. Each of these is a rational function with an Euler product. One of them generalizes Ihara's formula:

$$\frac{(1-u^2)^{\chi}}{\det(1 - Au + Qu^2)}$$

The other generalizes Hashimoto's formula:

$$\frac{1}{\det(1 - Tu)}$$

6. Deitmar's zetas

K: a nonarchimedian local field.

$\Gamma \subset G$: a neat cocompact discrete subgroup.

$P = MAN \subset G$: a K-parabolic of rank one.

$\mathcal{E}_P(\Gamma)$: the set of all Γ-conjugacy classes $[\gamma]$ such that γ is in G conjugate to some element $a_\gamma m_\gamma$ of $A^+ M_{ell}$ ("P-bolic").

$\mathcal{E}_P^p(\Gamma) \subset \mathcal{E}_P(\Gamma)$: the set of primitive elements.

Consider the infinite product

$$Z_{P,\sigma,\chi,\omega}(T) = \prod_{[\gamma] \in \mathcal{E}_P^p(\Gamma)} \det\left(1 - T^{l_\gamma} \chi(a_\gamma) \omega(\gamma) \otimes \sigma(m_\gamma)\right)^{\chi_1(\Gamma_\gamma)}.$$

Theorem. $Z_{P,\sigma,\chi,\omega}(T)$ is a rational function of T. The proof is an application of the Selberg trace formula applied to a test function built out of Kottwitz's Euler-Poincaré functions. Also utilized is a result of Casselman's about the differentiable cohomology groups $H^i_d(M, \pi_{\bar{N}})$ for the Jacquet module $\pi_{\bar{N}}$ of an admissible representation π of G. **These results are limited to rank one parabolics. This is inadequate for application to higher dimensional Shimura varieties.**

For instance, Khandker's zetas are related to the zeta functions of products of modular curves, yet none of Deitmar's zetas give Khandker's - you need a rank two parabolic here.

Problem. Generalize Deitmar's zeta function to a higher rank parabolic P.

I have some ideas about this, utilizing Deitmar's Lefschetz formula for p-adic groups.

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