Zeta Functions of Complexes

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1. The Ihara vertex zeta function of a graph

- $X$: connected undirected finite graph
- Count backtrackless and tailless cycles.

- Primitive cycle: not repeating another cycle more than once.
The Ihara vertex zeta function of $X$ is defined as

$$Z_0(X; u) = \prod_{[C]} \frac{1}{1 - u^{l(C)}} ,$$

where $[C]$ runs through all equiv. classes of primitive backtrackless and tailless cycles $C$, and $l(C)$ is the length of $C$.

Note that

$$u \frac{d}{du} \log Z_0(X; u) = \sum_{n \geq 1} N_n u^n ,$$

where $N_n$ is the number of backtrackless and tailless cycles of length $n$. Therefore

$$Z_0(X; u) = \prod_{[C]} \frac{1}{1 - u^{l(C)}} = \exp \left( \sum_{n \geq 1} \frac{N_n u^n}{n} \right) .$$
2. Properties of Ihara zeta function for regular graphs

• Ihara (1968): Let $X$ be a finite $(q+1)$-regular graph. Then its zeta function $Z_0(X, u)$ is a rational function of the form

$$Z_0(X; u) = \frac{(1 - u^2)\chi(X)}{\det(I - Au + qu^2I)},$$

where $\chi(X) = \#V - \#E$ is the Euler characteristic of $X$ and $A$ is the adjacency matrix of $X$.

• $X$ is Ramanujan if and only if $Z_0(X, u)$ satisfies RH, i.e. the nontrivial poles of $Z_0(X, u)$ all have absolute value $q^{-1/2}$. 
Recall

- The trivial eigenvalues of $X$ are $\pm(q + 1)$, of multiplicity one.
- $X$ is called a *Ramanujan graph* if the nontrivial eigenvalues $\lambda$ satisfy the bound

$$|\lambda| \leq 2\sqrt{q},$$

i.e. the roots of $1 - \lambda u + qu^2$ have absolute value $q^{-1/2}$.

**Alon-Boppana**: This eigenvalue bound is best possible.
3. The Hashimoto edge zeta function of a graph

Endow two orientations on each edge of a finite graph $X$. The neighbors of $\overrightarrow{e}$ are the directed edges starting from the ending vertex of $\overrightarrow{e}$ and not equal to the opposite of $\overrightarrow{e}$.

Associate the edge adjacency matrix $A_e$.

The Hashimoto edge zeta function $Z_1(X, u)$ counts backtrack-less and tailless oriented edge cycles, hence the same as $Z_0(X, u)$. Since $N_n = \text{Tr} A_e^n$, we get

$$Z_0(X, u) = Z_1(X, u) = \frac{1}{\det(I - A_eu)}.$$

Another viewpoint of the Ihara’s Theorem:

$$(1 - u^2)\chi(X) = \frac{\det(I - Au + qu^2I)}{\det(I - A_eu)}.$$
When \( q \) is a power of a prime \( p \), the \( (q + 1) \)-regular tree

\[ T = \text{PGL}_2(F)/\text{PGL}_2(\mathcal{O}_F), \]

where \( F \) is a \( p \)-adic field with \( q \) elements in its residue field and \( \mathcal{O}_F \) is its ring of integers.

vertices : \( \text{PGL}_2(\mathcal{O}_F) \)-cosets

vertex adjacency operator \( A \) : Hecke operator on

\[ \text{PGL}_2(\mathcal{O}_F) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \text{PGL}_2(\mathcal{O}_F). \]

directed edges : \( \mathcal{I} \)-cosets (\( \mathcal{I} \) is the Iwahori subgroup of \( \text{PGL}_2(\mathcal{O}_F) \))

edge adjacency operator \( A_e \) : Iwahori-Hecke operator on \( \mathcal{I} \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \mathcal{I} \).

\( X = X_\Gamma = \Gamma \backslash \text{PGL}_2(F)/\text{PGL}_2(\mathcal{O}_F) = \Gamma \backslash T \) for a torsion free discrete cocompact subgroup \( \Gamma \) of \( \text{PGL}_2(F) \).
4. The building on $\text{PGL}_3(F)$

- $F, \mathcal{O}_F, \pi$ : the same as before
- $G = \text{PGL}_3(F), \ K = \text{PGL}_3(\mathcal{O}_F)$
- The Bruhat-Tits building $\mathcal{B} = G/K$ is a 2-dim’l simplicial complex. The chambers are the 2-simplices, their edges are the 1-simplices, and the vertices are the 0-simplices.
- $\mathcal{B}$ is $(q+1)$-regular, namely, each edge is shared by exactly $q+1$ chambers.
- Topologically $\mathcal{B}$ is simply connected, so it is the universal cover of its finite quotients, called 3-hypergraphs/2-dim’l complexes.
5. Parametrizations of the simplices in $\mathcal{B}$

- $\sigma = \begin{pmatrix} 1 & \ \\ p & 1 \end{pmatrix}$. Have a filtration of $K$:

\[ K \supset E := K \cap \sigma K \sigma^{-1} \supset B := K \cap \sigma K \sigma^{-1} \cap \sigma^{-1} K \sigma. \]

- vertices $\leftrightarrow K$-cosets

Each vertex $gK$ has a type in $\mathbb{Z}/3\mathbb{Z}$ given by $\tau(gK) := \text{ord}_\pi(\det g) \mod 3$.

- The type of an edge $gK \to g'K$ is $\tau(g'K) - \tau(gK) = 1$ or $2$.

- type one edges $\leftrightarrow E$-cosets

- chambers $\leftrightarrow B$-cosets such that $gB$, $g\sigma B$ and $g\sigma^2 B$ represent the same chamber.
6. Operators on $\mathcal{B}$

The $K$-double cosets define Hecke operators acting on $L^2(G/K)$. They are polynomials in

$$A_1 = K \begin{pmatrix} 1 & \pi \\ \pi & 1 \end{pmatrix} K \quad \text{and} \quad A_2 = K \begin{pmatrix} 1 & \\ \pi & \pi \end{pmatrix} K.$$

The $B$-double cosets define Iwahori-Hecke operators acting on $L^2(G/B)$. Denote by $L_B$ the operator supported on $Bt_2\sigma^2B$, where

$$t_2 = \begin{pmatrix} \pi^{-1} \\ \pi \end{pmatrix}.$$
7. Finite quotients of $\mathcal{B}$

$\Gamma$: a torsion free discrete subgroup of $G$ with compact quotient.

$X = X_\Gamma = \Gamma \backslash G / K = \Gamma \backslash \mathcal{B}$

Two assumptions on $\Gamma$:

(I) $\text{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$ so that $\Gamma$ identifies vertices of the same type.

(II) $\Gamma$ is *regular*, that is, the centralizer in $\Gamma$ of any nonidentity element $\gamma \in \Gamma$ is a torus.

Division algebras of degree 9 yield many such $\Gamma$’s.

**Goal:** Find a closed form expression of the vertex zeta function of $X$ analogous to graph zeta functions.

Previously considered by Deitmar, and Deitmar-Hoffman, partial results.
8. The main results

The type one vertex zeta function on $X$ is defined as

$$Z_{0,1}(X, u) = \prod_{[\mathcal{C}]} \frac{1}{1 - ul_A(\mathcal{C})},$$

where $[\mathcal{C}]$ runs through the equiv. classes of primitive tailless type one cycles in $X$.

Such cycles traveled in reverse direction are type two cycles with algebraic length doubled. Define

$$Z_0(X, u) = Z_{0,1}(X, u)Z_{0,2}(X, u) = Z_{0,1}(X, u)Z_{0,1}(X, u^2).$$
Figure 3: tailless
Figure 4: with tail
Main Theorem

(1) $Z_0(X, u)$ is a rational function given by

$$Z_0(X, u) = \frac{(1 - u^3)\chi(X)}{\det(I - A_1u + qA_2u^2 - q^3u^3I) \det(I + L_Bu)},$$

where $\chi(X) = \#V - \#E + \#C$ is the Euler characteristic of $X$.

(2) $X$ is a Ramanujan complex if and only if $Z_0(X, u)$ satisfies the RH.
Recall that

- The trivial eigenvalues of $I - A_1 u + q A_2 u^2 - q^3 u^3 I$ are 1, $q^{-1}$, $q^{-2}$ as well as their multiples by the cubic roots of 1.

- $X$ is a 2-dim’l Ramanujan complex if the nontrivial eigenvalues of $I - A_1 u + q A_2 u^2 - q^3 u^3 I$ all have absolute value $q^{-1}$.

Li (2004): Such bounds for eigenvalues of $A_1$ and $A_2$ are best possible.

In this case the nontrivial roots of $\det(1 + L Bu)$ have absolute value $q^{-1/2}$, proved by Kang-Li-Wang.
9. A sketch of the proof

- Partition the homotopy geodesic cycles on $X$ into sets parametrized by the conjugacy classes $[\gamma]$ of $\Gamma$; the set indexed by $[\gamma]$ consists of cycles $\kappa_\gamma(gK)$ with starting point $gK \in C_\Gamma(\gamma) \backslash G/K$ and ending point $\gamma gK$. Here $C_\Gamma(\gamma)$ is the centralizer of $\gamma$ in $\Gamma$.

- Give an algebraic criterion of tailless cycles.

- Compute the number of type one cycles of given length in each set $[\gamma]$, with and without tails. Need to separate $\gamma$ into three cases, according to the field generated by its eigenvalues being $F$, quadratic ramified or unramified extension of $F$. ($\Gamma$ does not contain elements whose eigenvalues generate a cubic extension of $F$.) Very complicated.
• Put together, this allows us to express
\[
\frac{(1 - u^3)\chi(X)}{\det(I - A_1u + qA_2u^2 - q^3u^3I)}
\]
as \(L_{0,1}(X, u)\) times "something extra".

• Show that the zeta function of the type one tailless closed galleries in \(X\) is \(L_2(X, u) = 1/\det(I - L_Bu)\). The boundaries of such galleries are type one tailless cycles in \(X\). Characterize the chambers which will lie in a tailless type one closed gallery. This allows us to compare \(L_2(X, -u)\) with \(L_{0,2}(X, u)\), leading to the conclusion that "something extra" is \(L_{0,2}(X, u)/L_2(X, -u)\).