# How to find a TU cooperative strategy

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Consider the bimatrix we looked at in class:

$$\begin{pmatrix} (3,6) & (4,-2) \\ (7,1) & (0,3) \end{pmatrix}.$$

We may write Player I's payoff matrix as

$$A = \begin{pmatrix} 3 & 4 \\ 7 & 0 \end{pmatrix}$$

and Player II's payoff matrix as

$$B = \begin{pmatrix} 6 & -2 \\ 1 & 3 \end{pmatrix}.$$

We know the optimal cooperative strategy is to pick the entry with maximum sum. This means Player I uses row 1, and Player II uses column 1. This maximizes total utility: $\sigma = 3 + 6 = 9$. We still need to determine the actual payoff (and the side payment to acheive that).

To do this, we find the threat point (or disagreement point). Each player will choose a strategy to show that they deserve more than the other player. This means each player will try to choose a strategy to maximize their gain over the other player. This becomes a zero-sum game — if Player I has two more than Player II, then Player II has two less than Player I. This leads us to the difference matrix:

$$D = A - B = \begin{pmatrix} -3 & 6 \\ 6 & -3 \end{pmatrix}.$$

We now solve this as a zero-sum game. The solution in this case is

$$\mathbf{p} = (1/2, 1/2) \qquad \mathbf{q} = (1/2, 1/2).$$

The game value is $\mathbf{p}^T D\mathbf{q} = \frac{3}{2}$. The threat point is $(\mathbf{p}^T A\mathbf{q}, \mathbf{p}^T B\mathbf{q}) = (7/2, 2)$ — the payout if both players use their threat strategies. This determines that Player I should receive $\frac{7}{2} - 2 = \frac{3}{2}$ (the value of the game $D$) more than Player II does.

This gives the final payoff of

$$\left(\frac{\sigma}{2} + \frac{3/2}{2}, \frac{\sigma}{2} - \frac{3/2}{2}\right) = (5.25, 3.75),$$

which you may check is the only way to have two numbers add to 9 with a difference of $\frac{3}{2}$.

Since the payoff from the original game is $(3, 6)$, Player II must make a sidepayment of 2.25 to Player I. Equivalently, Player I makes a sidepayment of $-2.25$ to Player II.