An additive version of Ramsey's theorem

Andy Parrish

April 17, 2011

Abstract

We show that, for every r, k, there is an $n = n(r, k)$ so that any r-coloring of the edges of the complete graph on $[n]$ will yield a monochromatic complete subgraph on vertices $\left\{a + \sum_{i \in I} d_i \mid I \subseteq [k]\right\}$ for some choice of $a, d_1, \ldots, d_k$. In particular, there is always a solution to $x_1 + \ldots + x_\ell = y_1 + \ldots + y_\ell$ whose induced subgraph is monochromatic.

1 Introduction

Given a set X and a number r, an r-coloring of X is any map $\chi : X \to [r]$, where $[r] = \{1, \ldots, r\}$ is the set of colors.

Ramsey's celebrated theorem [7] states that, given r, k, there is an $R = R(r, k)$ so that any r-coloring of the edges of the complete graph on R vertices contains a monochromatic complete graph on k vertcies. In addition to being an important result in itself, Ramsey is the namesake of a large field of research into Ramsey Theory, which more generally tells when a coloring of a large structure is guaranteed to have large monochromatic substructures. There are many great resources on Ramsey Theory, but the main source is due to Graham, Rothschild, and Spencer [2].

The first result in Ramsey theory was actually proved by Hilbert in 1892, predating Ramsey's theorem (1930) by several decades. Given natural numbers $a, d_1, \ldots, d_k$, define

$$H(a; d_1, \ldots, d_k) = \left\{ a + \sum_{i \in I} d_i \mid I \subseteq [k] \right\}.$$

We call such a set $H(a;d_1,\dots,d_k)$ a Hilbert cube of dimension k. Hilbert proved [4] that, given r,k natural numbers, there is a number $H=H(r,k)$ so that any r-coloring of $[H]$ contains a monochromatic Hilbert cube of dimension k.

It was further shown that finite-colorings of natural numbers would always contain monochromatic solutions to $x+y=z$ (Schur [8]), as well as long monochromatic arithmetic progressions (van der Waerden [10]). The holy grail of results of this type is Rado's theorem [6], which characterizes which systems of linear equations have monochromatic solutions under every finite-coloring of the naturals. Those which do are called *partition-regular*.

These results are philosophically related to Ramsey's theorem, but the graph theoretic and additive sides of Ramsey theory are largely distinct fields. In recent years, however, Deuber, Gunderson, Hindman, and Strauss proved a connecting result [1] — for any m, any sufficiently large graph either contains a $K_{m,m}$, or else it has an independent set with a prescribed additive structure. Later, Gunderson, Leader, Prömel, and Rödl showed [3] that for any m,k, large graphs must either contain a K_m or there must be an arithmetic progression of length k which is an independent set. There have been further results in this area, but none give a purely additive result.

Our goal is to find some additive property $\mathcal{P}$ for which we can guarantee that every finite edge-coloring of the complete graph on $[n]$ will contain a set of vertices with property $\mathcal{P}$ whose induced subgraph is monochromatic. In this paper, we consider properties where X has property $\mathcal{P}$ if X satisfies a particular system of linear equations.

We always demand solutions by distinct values, so that the monochromatic subgraphs are non-trivial. For example, if we solve $x+y=z$ by $x=y=3, z=6$, the corresponding graph has only a single edge, $\{3,6\}$. The induced graph has no choice but to be monochromatic.

Formally, given a matrix B and a number of colors r, we would like to know whether there is an $n=n(r,B)$ so that any r-edge-coloring of the complete graph on $[n]$ gives a vector $\vec{x}=(x_1,\dots,x_k)$ of distinct entries so that the values $\{x_1,\dots,x_k\}$ are monochromatic, and $B\vec{x}=\vec{0}$.

We consider this problem for many systems of equations known to be partition-regular. In Section 2, we give several negative results. In Section 3 we give an initial positive result: there is an n so that, any 2-coloring of the edges of the complete graph on $[n]$ gives a monochromatic 2-dimensional Hilbert cube. In Section 4, we prove a lemma about coloring k-ary trees which may be interesting in its own right. In Section 5 we extend our initial result to any number of colors and to Hilbert cubes of any size. We believe these are the first positive results in this direction.

2 Negative results

There are many families of equations for which monochromatic solutions can be easily avoided in this graph setting.

2.1 Arithmetic progressions

Van der Waerden's theorem [10] tells us that any finite-coloring of the naturals have arbitrarily long monochromatic arithmetic progressions. What can we say when coloring pairs of naturals? An arithmetic progression of length 3 is given by $a, a+d, a+2d$. We notice that the triple contains two differences: d and $2d$. This observation allows us to 2-color the complete graph on the naturals without a monochromatic 3-AP.

The coloring is simple. For a pair $\{x, y\}$, write $|x - y| = 2^p q$ where p, q are integers and q is odd. If p is even, color $\{x, y\}$ red. Otherwise, color it blue.

Now let $a, a+d, a+2d$ be a 3-AP. Write $d = 2^p q$. Then we see $2d = 2^{p+1} q$, so the edges $\{a, a + d\}$ and $\{a, a + 2d\}$ have different colors.

This coloring avoids 3-APs, so we certainly cannot hope for anything longer.

2.2 Schur's equation and generalizations

Schur's theorem [8] states that any finite-coloring of the naturals has a monochromatic solution to $x + y = z$. Additionally, it follows from Folkman's theorem that there is a monochromatic solution to $x_1 + \ldots + x_k = z$ for arbitrary k.

More generally, we consider equations of the form

$$a_1 x_1 + \ldots + a_k x_k = bz \tag{1}$$

with $a_1, \ldots, a_k \geq b > 0$.

We note that any solution to Equation 1 has $x_i \leq z$ for $i = 1, \ldots, k$. Using two colors, we can ensure that every graph induced by a solution to an equation of this form in the natural numbers contains both colors. We first show how to avoid $x + y = z$ as motivation for the approach, and then handle the general case.

If $x + y = z$ then either x or y is smaller than their average, $\frac{1}{2}z$, and the other must be larger than their average. Thus, given a pair $\{u, v\}$ with $u < v$, we color it red if $u \leq \frac{1}{2}v$, and blue if $u > \frac{1}{2}v$. Now we see that whenever $x+y = z$, the largest of the three numbers must be z. Either x or y is smaller than $\frac{1}{2}z$, and the other is larger, so the pairs $\{x, z\}$ and $\{y, z\}$ have different colors. (Recall that we are only interested in solutions by distinct numbers).

In Equation 1, a similar logic applies. We see that $a_i x_i \leq bz$. Since $a_i \geq b > 0$, we get $x_i \leq z$ as before. Let $M = a_1 + \ldots + a_k$. Divide both sides of the equation by M to get

$$\frac{a_1}{M}x_1 + \ldots + \frac{a_k}{M}x_k = \frac{b}{M}z.$$

This says that the weighted average of the x_i's is $\frac{b}{M}z$. Again, one of the x_i's must be smaller than their average, and another must be larger. Thus, when $u < v$, we should color $\{u, v\}$ red if $u \leq \frac{b}{M}v$, and blue otherwise. We immediately see that one of the pairs $\{x_i, z\}$ must be red and another must be blue.

Remark:

The argument given above is really a greedy coloring. At step t, color the pairs $\{1, t\}, \ldots, \{t-1, t\}$ in a way that handles those solutions to Equation 1 with largest element t. Since we can manage all these solutions at once, we avoid all monochromatic solutions. The incredible thing to notice here is that this coloring is much stronger than needed. If $x_1, \ldots, x_k, z$ satisfy Equation 1, then the star connecting z to all of the x_i's is not even monochromatic. Forget about the clique! The strength of this technique suggests that we may be able to handle a larger family of equations.

On the other hand, this technique relies heavily on the numbers being positive. If we change the underlying set to $\mathbb{Z}$ or $\mathbb{Z}_p$, the approach falls apart.

2.3 Three variables, six colors

As with many problems in Ramsey theory, we may consider our conjecture as a hypergraph coloring problem. The vertex set is all pairs we are considering (be they pairs in $[n], \mathbb{N}, \mathbb{Z}, \mathbb{Z}_n$, etc). For each solution $(x_1, \ldots, x_k)$ to

$b_1x_1 + \ldots + b_kx_k = 0$, there is a hyperedge containing all pairs of the x_i's. If we properly color this $\binom{k}{2}$-uniform hypergraph (avoiding monochromatic hyperedges), then there are no monochromatic solutions to the equation. Thus we may apply theorems about hypergraph coloring.

For an equation in three variables, this hypergraph is *simple* — any two pairs are either disjoint (and have no hyperedges in common), or have the form $\{x, y\}, \{x, z\}$, leaving only $\{y, z\}$ to form a hyperedge.

Fix a, b, c, and consider the hypergraph formed as above by the equation

$$ax + by + cz = 0. \tag{2}$$

Consider a pair $\{u, v\}$. How many hyperedges can it be contained in? Well, there are 6 different ways of assigning the values u and v to the variables in Equation 2:

$$\begin{array}{rcl}
au + bv + cz = 0 & \Longrightarrow & z = -\frac{au+bv}{c} \\
av + bu + cz = 0 & \Longrightarrow & z = -\frac{av+bu}{c} \\
au + by + cv = 0 & \Longrightarrow & y = -\frac{au+cv}{b} \\
av + by + cu = 0 & \Longrightarrow & y = -\frac{av+cu}{b} \\
ax + bu + cv = 0 & \Longrightarrow & x = -\frac{bu+cv}{a} \\
ax + bv + cu = 0 & \Longrightarrow & x = -\frac{bv+cu}{a}
\end{array}$$

Thus we see that, so long as the numbers a, b, c are all invertible, each pair $\{u, v\}$ is contained in at most 6 hyperedges. In particular, if we are in $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{Z}_p$ for a prime p, then the degree is at most 6. The hypergraph version of Brooks' theorem [5] applies.

Theorem 2.1 *If H is a hypergraph with maximum degree Δ, then $\chi(H) \leq \Delta$ except in these cases:*

1. *$\Delta = 1$,*
2. *$\Delta = 2$ and H contains an odd cycle (an ordinary graph),*
3. *H contains a K_Δ (an ordinary graph).*

Since all of these cases are irrelevant — ours is a 3-uniform hypergraph, and we don't have any illusions that we can 1-color it — this tells us we can properly 6-color our hypergraph. By construction, this avoids monochromatic solutions to Equation 2.

Moreover, if for example $a = b$, then the six solutions reduce to three distinguishable ones, meaning 3 colors is enough.

Note 2.2 The case we avoided was solutions over $\mathbb{Z}_n$ with n composite and a, b, c not necessarily invertible. Taken to extremes, this case is quite degenerate. Consider, for example, $n = 2^r$, and $a = b = 2^{r-1}$. Any collection of even numbers then solves Equation 2. The problem of finding a solution which induces a monochromatic subgraph now reduces to the multicolor Ramsey's theorem for triangles.

3 Two colors, two dimensions

We will eventually prove the following result:

Theorem 3.1 *For all r, k, there is a number $n = n(r, k)$ so that any r-coloring of the edges of the complete graph on $[n]$ gives a Hilbert cube $H = H(a; d_1, \ldots, d_k)$ so that all edges in H are the same color, and the 2^k elements of H are distinct.*

We first prove the theorem for $r = k = 2$. Note that a 2-dimensional Hilbert cube is four numbers of the form $a, a+b, a+c, a+b+c$. We will then extend those ideas to any number of colors, and then to Hilbert cubes of any dimension.

The proof will rely on the Gallai-Witt theorem [11], and a consequence of Rado's theorem [6], both of which we state here.

Theorem 3.2 (Gallai-Witt) *For all r, k, there exists $GW = GW(r, k)$ so that any r-coloring of $[GW] \times [GW]$ gives numbers x, y, d with the property that*

$$\{(x + id, y + jd) \mid i, j = 0, \ldots, k-1\}$$

are all the same color.

Theorem 3.3 (Corollary to Rado) *There is a number T so that any 2-coloring of $[T]$ gives distinct numbers $i, j, i+j, j-i$, all the same color.*

Note: Rado's theorem gives conditions for a system of linear equations to have monochromatic solutions by distinct numbers. It is a simple exercise to check that the above satisfies them.

Proof of Theorem 3.1 when $r = k = 2$:

Define $S = GW(T+1, 2)$, where T comes from Theorem 3.3. We will show that $n = 2S$ suffices.

Fix an 2-coloring $\chi : \binom{[n]}{2} \to [2]$. We would like to find a solution to $w + x = y + z$ which forms a monochromatic clique. We view χ as a coloring of the upper half of the lattice $[n] \times [n]$ — for $x < y$, the color of (x, y) is $\chi(\{x, y\})$.

Consider the top left quadrant of our grid: $\{1, \ldots, S\} \times \{S+1, \ldots, 2S\}$. Define $\chi' : [S] \times [S] \to [2]$ by

$$\chi'(a, b) = \chi(a, S + b).$$

Since $S = GW(T+1, 2)$, and χ' is a 2-coloring of $[S] \times [S]$, we may apply Gallai-Witt to find x, y, d so that all points of the form

$$\{(x + id, y + jd) \mid i, j = 0, \ldots, T\}$$

are the same color, say red, under χ. We will consider each subsquare of this large grid.

For now, consider a red square given by the points

$$(a, b) \quad (a+h, b) \quad (a, b+h) \quad (a+h, b+h).$$

We may rewrite the underlying numbers as $a, a+h, a+(b-a), a+h+(b-a)$ to see they form a Hilbert cube of dimension 2.

There are six edges in the graph on these four numbers, and we know that four of them are red. Thus, we only need to consider the edges $\{a, a+h\}$ and $\{b, b+h\}$. If these are both red (and the four values are distinct), then we have the desired monochromatic 4-clique. Thus, either we have our goal, or every red square gives us two points which cannot both be red.

Well, we have a great many red squares. Each has corner $(x + id, y + jd)$ and side-length ℓd, for every choice of i, j, ℓ with $i, j, i+\ell, j+\ell$ all in $\{0, \ldots, S\}$. The four underlying numbers are all distinct by the choice of our initial grid $\{1, \ldots, S\} \times \{S+1, \ldots, 2S\}$. The "final" edges of this square are $\{x + id, x + (i+\ell)d\}$ and $\{y + jd, y + (j+\ell)d\}$, so these two cannot both be red without reaching our goal.

All of our red squares will give us many interacting conditions, which we record in a graph. Let $G = (A, B, E)$ be a bipartite graph, where $A = B = \binom{\{0,\ldots,T\}}{2}$. We say $\{a, a'\} \sim \{b, b'\}$ if $\{x + ad, x + a'd\}$ and $\{y + bd, y + b'd\}$ are

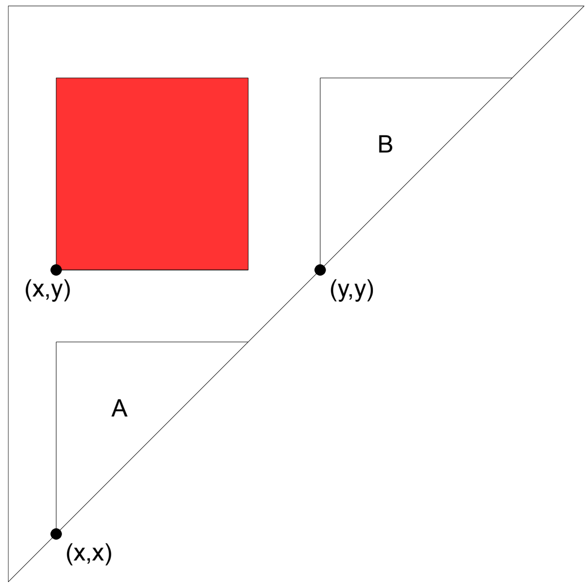

Figure 1: A large red grid, and the corresponding sets A and B

the final edges of some red square. There is an induced 2-coloring of both A and B — namely

$$\chi_A(\{i,j\}) = \chi(x+id, x+jd),$$
$$\chi_B(\{i,j\}) = \chi(y+id, y+jd)$$

We see immediately that $\{i, i+\ell\} \sim \{j, j+\ell\}$ so long as those numbers are all in $\{0,\ldots,T\}$. This means that each pair in A with difference ℓ is connected to every pair in B with that difference. This means that if one pair in A is red, all pairs in B with that difference must be blue (and vice versa). In fact, this is the entire structure of G.

Write $A = A_1 \cup A_2 \cup \ldots \cup A_T$, where A_ℓ contains all pairs in A of the form $\{i, i+\ell\}$. We now 2-color $[T]$, the index set of the A_ℓ's. Say $\phi(\ell) = \text{red}$ if *any* pair in A_ℓ is red. Otherwise, $\phi(\ell) = \text{blue}$, meaning that A_ℓ is entirely blue. Since ϕ is a 2-coloring of $[T]$, Theorem 3.3 tells us there are distinct numbers $i, j, i+j, j-i$ which are monochromatic.

Case 1: The numbers are red. This means each set $A_i, A_j, A_{i+j}, A_{j-i}$ contains a red pair. Therefore the corresponding sets in B, what we should call $B_i, B_j, B_{i+j}, B_{j-i}$, are all entirely blue. The proof continues as in case 2 below, but with all A's changed to B's, and all x's changed to y's.

Case 2: The numbers are blue, so all pairs in $A_i, A_j, A_{i+j}, A_{j-i}$ are blue. We list the relevent blue pairs:

$$\begin{aligned} \text{In } A_i : &\quad \{0,i\},\{j,i+j\} \\ \text{In } A_j : &\quad \{0,j\},\{i,i+j\} \\ \text{In } A_{i+j} : &\quad \{0,i+j\} \\ \text{In } A_{j-i} : &\quad \{i,i+(j-i)\} = \{i,j\}. \end{aligned}$$

Taken together, we see that $0, i, j, i+j$ form a blue K_4 under χ_A. Recalling the relationship between χ and χ_A, this gives us a blue K_4 under χ with vertices $x, x+id, x+jd, x+(i+j)d$. This is the desired 2-dimensional Hilbert cube. □

4 Coloring binary trees

In order to acheive Theorem 3.1 for any number of colors, we will first require a Ramsey-type theorem for k-ary trees.

Notation 4.1 We use $[k]^*$ denote all finite sequences (strings) of elements of $[k] = \{1, \dots, k\}$. If $s, t \in \{1, \dots, k\}^*$, we use $s \cdot t$ to denote concatenation — all characters of s followed by all characters of t.

Def 4.2 A perfect k-ary tree $T_n^{(k)}$ of height n is the collection of nodes

$$T_n^{(k)} = \{s \in \{1, \dots, k\}^j \mid 0 \le j \le n\}.$$

We say λ, the empty string, is the root of the tree. A node s has k children, $s \cdot 1, \dots s \cdot k$. The child $s \cdot i$ together with all of its descendents forms the i^{th} subtree of s, rooted at $s \cdot i$. We see that s has k subtrees in all. The j^{th} level of $T_n^{(k)}$ consists of all those strings of length exactly j. The substrings of s are called the *ancestors* of s. The nodes at level n are called leaves. If s is a substring of t, we say that the *path* from s to t is the set of nodes r which are both superstrings of s and substrings of t (including s and t). The length of the path is the difference in lengths of s and t.

Since we are only interested in perfect k-ary trees in this paper, we may occasionally refer to them simply as "k-ary trees", or "trees" if k is implied.

Next, we define what it means to embed one k-ary tree into another.

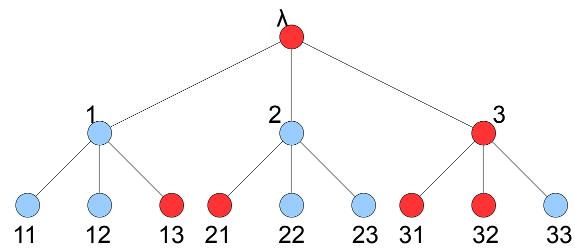

Figure 2: This 3-ary tree is 1-balanced. The nodes λ, 13, 21, and 31 are all red

Def 4.3 Let T, R be two k-ary trees. An embedding of T into R is a map φ from the nodes of T into the nodes of R with the following properties:

1. There is an increasing function x from levels of T to levels of R so that, if t is on the i^{th} level of T, then $\varphi(t)$ is on the $x(i)^{\text{th}}$ level of R.

2. If s, t are nodes in T , and s is contained in the i^{th} subtree of t, then $\varphi(s)$ is contained in the i^{th} subtree of $\varphi(t)$.

We now state the goal of this section:

Lemma 4.4 *For every k, c, n, there is a number $E = E(k, c, n)$ so that every c-coloring of the k-ary tree of height E yields a monochromatic embedding of the k-ary tree of height n.*

We say that a coloring is *n-balanced* if the conclusion holds.

Example 4.5 A coloring χ of the a tree T is 1-balanced if there is some node r, and strings $s_1, \ldots, s_k \in \{1, \ldots, k\}^j$ for some j, so that

$$r, r \cdot 1 \cdot s_1, \ldots, r \cdot k \cdot s_k$$

are all the same color. This corresponds to the embedding φ of $T_1^{(k)}$ into T given by $\varphi(\lambda) = r, \varphi(i) = r \cdot i \cdot s_i$.

We prove Theorem 4.4 by first finding $f(k, c) = E(k, c, 1)$, and repeating applying that result.

Lemma 4.6 *There is a function $f(k,c)$ so that, if $n \geq f(k,c)$, then every c-coloring of the perfect k-ary tree of depth n is 1-balanced.*

We take the proof slowly to delicately handle each part.

Proof:

When $c = 1$ the nodes $\lambda, 1, 2, \ldots, k$ must all be the same color, so they 1-balance the tree. Thus $f(k,1) = 1$.

Consider the case $k = 1$, so that each level has a unique node. By the pigeonhole principle, the $c+1$ nodes within levels $0, 1, \ldots, c$ must contain two with the same color. Thus $f(1,c) \leq c$. Of course, it is easy to see that this is the best possible bound.

We begin the same way for $k = 2$. Since we know $f(2,1) = 1$, we work by induction on c. We will show $f(2, c+1) \leq (c+1)(1 + f(2,c)) = n$.

Let $\chi : T_n^{(2)} \to [c+1]$ be a $(c+1)$-coloring of the tree of height n. Consider the path from the root to the node 1^n. The path contains $n+1$ nodes, so some color is represented at least

$$\left\lceil \frac{n+1}{c+1} \right\rceil = 2 + f(2,c)$$

times. Call the repeated color "red." Call the levels of these red nodes $j(-1), j(0), j(1), \ldots, j(f(2,c))$. Since we looked down the path of all 1s, the corresponding nodes are

$$\begin{aligned} r &= 1^{j(-1)} \\ s_0 &= 1^{j(0)} \\ &\vdots \\ s_{f(2,c)} &= 1^{j(f(2,c))}. \end{aligned}$$

Consider r along with with any of the other red nodes, s_i. These may be part of a balancing triple — if any descendent t of $r \cdot 2$ on level $j(i)$ is also red, then r, s_i, t balance the tree. Thus, if the tree is to be unbalanced, all of the levels $j(0), \ldots, j(f(2,c))$ within the second subtree of r must be entirely non-red. We will now use the definition of $f(2,c)$ to show that this tree is in fact 1-balanced by these non-red nodes.

Consider the map from the nodes of $T = T_{f(2,c)}^{(2)}$ into our tree given by

$$\varphi(\lambda) = r \cdot 2^{j(0)-j(-1)} = 1^{j(-1)} \cdot 2^{j(0)-j(-1)},$$

$$\varphi(a_1 \dots a_{\ell-1} a_\ell) = \varphi(a_1 \dots a_{\ell-1}) \cdot (a_\ell)^{j(\ell)-j(\ell-1)}.$$

We make the following observations:

1. Nothing in the image of φ is red (unless the coloring is 1-balanced).

2. All nodes on level i of T are mapped to level $j(i)$ of our tree.

3. If t is contained in the i^{th} subtree of s, then $\varphi(t)$ is contained in the i^{th} subtree of $\varphi(s)$.

We color T by $\chi^*(s) = \chi(\varphi(s))$, the coloring induced by φ. Observation 1 tells us that χ^* is actually a c-coloring. By the definition of $f(2,c)$, we know that there are some nodes $w, w \cdot 1 \cdot s_1, w \cdot 2 \cdot s_2$ (with the latter two on the same level) which are all the same color under χ^*. Thus we see that $\varphi(w), \varphi(w \cdot 1 \cdot s_1)$, and $\varphi(w \cdot 2 \cdot s_2)$ must be the same color under χ. By observations 2 and 3, these nodes 1-balance the original tree.

Finally, for $k \geq 3$, we follow a very similar idea. We will show

$$f(k, c+1) \leq (c+1)(1 + (k-1)f(k,c)) = n.$$

Let $\chi : T_n^{(k)} \to [c+1]$ be a $(c+1)$-coloring of the k-ary tree of height n. Consider the path from the root to the node 1^n. The path contains $n+1$ nodes, so some color is represented at least

$$\left\lceil \frac{n+1}{c+1} \right\rceil = 2 + (k-1)f(k,c)$$

times. Call the repeated color "red." Call the levels of these red nodes $j(-1), j(0), j(1), \dots, j((k-1)f(k,c))$. Since we looked down the path of all 1s, the corresponding nodes are

$$r = 1^{j(-1)}$$

$$s_0 = 1^{j(0)}$$

$$\vdots$$

$$s_{(k-1)f(k,c)} = 1^{j((k-1)f(k,c))}.$$

Consider $j(-1)$ along with with any of the other red nodes, s_i. These may be part of a balancing set — if, for every $a = 2, \dots k$ some descendent t_a of

$r \cdot a$ on level $j(i)$ is also red, then $r, s_i, t_2, \ldots, t_k$ will 1-balance the tree. Thus, if the tree is to be unbalanced, each of the levels $j(0), \ldots, j((k-1)f(k,c))$ must be entirely non-red in at least one of the $k-1$ subtrees of r — if every subtree has a red node on the same level, the coloring is 1-balanced. By the pigeonhole principle, some subtree of r, say the p^{th} subtree, must be colored such that at least $1 + f(k,c)$ of the levels $j(0), \ldots, j((k-1)f(k,c))$ are entirely non-red. Label these levels $x(0), x(1), \ldots, x(f(k,c))$. We will now use the definition of $f(k,c)$ to show that this tree is in fact balanced by these non-red nodes.

Consider the map from $T = T^{(k)}_{f(k,c)}$ into our tree given by

$$\varphi(\lambda) = r \cdot p^{x(0)-x(-1)} = 1^{x(-1)} \cdot p^{x(0)-x(-1)},$$

$$\varphi(a_1 \ldots a_{\ell-1} a_\ell) = \varphi(a_1 \ldots a_{\ell-1}) \cdot (a_\ell)^{x(\ell)-x(\ell-1)}$$

We now make the same observations as before:

1. Nothing in the image of φ is red (unless the coloring is 1-balanced).
2. All nodes on level i of T are mapped to level $x(i)$ of our tree.
3. If t is contained in the i^{th} subtree of s, then $\varphi(t)$ is contained in the i^{th} subtree of $\varphi(s)$.

We color T by $\chi^*(s) = \chi(\varphi(s))$, the coloring induced by φ. Observation 1 tells us that χ^* is actually a c-coloring. By the definition of $f(k,c)$, we know that there are some nodes $w, w \cdot 1 \cdot s_1, \ldots w \cdot k \cdot s_k$ (with the last k on the same level) which are all the same color under χ^*. Thus we see that $\varphi(w), \varphi(w \cdot 1 \cdot s_1), \ldots, \varphi(w \cdot k \cdot s_k)$ must be the same color under χ. By observations 2 and 3, these nodes 1-balance the original tree. $\square$

Curiously, the bound given for $f(k,c)$ for $c, k \geq 2$ (as defined by the recurrence given, not necessarily the best bound) is exactly

$$f(k,c) = \lfloor e^{1/(k-1)}(k-1)^{c-1}c! \rfloor.$$

We may now prove the existence of $E(k,c,n)$.

Remark: The following proof feels a bit artificial. Perhaps the "correct" proof should be based on a weaker lemma, where we do not care about the color of the root.

Proof of Lemma 4.4:

We only show the result for $n = 2^\ell - 1$, since this implies all smaller values. The case $\ell = 1$ is Lemma 4.6.

Suppose $E(k, c', 2^\ell - 1)$ is known for all values c'. We will find a bound for $E(k, c, 2^{\ell+1} - 1)$.

Let $\chi_0 = \chi$ be a c-coloring of a large k-ary tree. We ignore the specific height for now, but will determine a bound at the end.

By induction, χ_0 gives a monochromatic embedding φ of a k-ary tree of height $2^\ell - 1$ into levels $0, 1, \ldots, E(k, c, 2^\ell - 1)$ of our large tree. Call the image T_λ, with color $\psi(\lambda)$. T_λ has $k^{2^\ell - 1}$ leaves, and each has k subtrees, so we have a total of $Y := k^{2^\ell}$ subtrees coming off of T_λ. The roots of these subtrees are given by

$$v_s = \varphi(t) \cdot i, \text{ where } s \in \{1, \ldots, k\}^{2^\ell} \text{ is written as } s = t \cdot i.$$

To each $t \in \{1, \ldots, k\}^*$ we associate a map $\chi_1(t)$ from $\{1, \ldots, k\}^{2^\ell}$ to $[c]$, given by

$$\chi_1(t)(s) = \chi(v_s \cdot t).$$

Note that there are "only" c^Y such maps $\chi_1(t)$. Since each t is mapped to one of c^Y elements, we treat χ_1 as a c^Y-coloring of a k-ary tree. The color of a node t is given by a list of colors, each one the color of the node corresponding to t in one of the Y subtrees of T_λ.

Because χ_1 is a c^Y-coloring of a k-ary tree, we know that there is an embedded k-ary tree contained within levels $0, 1, \ldots, E(k, c^Y, 2^\ell - 1)$ which is monochromatic under χ_1. Looking back to χ, this means we really have Y monochromatic trees, which we label T_s for $s \in \{1, \ldots, k\}^{2^\ell}$. Moreover, each T_s is in the same position relative to v_s. In particular, all the nodes at level i of any tree T_s are on the same level in the original tree. This means that, if all these trees were red, taking them all together with T_λ would give us our monochromatic embedded tree of height $2^{\ell+1} - 1$. Would that we were so lucky!

Instead, all we know is that, for each s, the entire tree T_s has some color $\psi(s)$.

We now have k^{2^ℓ} trees, each with $k^{2^\ell - 1}$ leaves, which in turn each have k subtrees. Altogether, that gives us $Y^2 = k^{2 \cdot 2^\ell}$ subtrees. We repeat the above argument to get a coloring χ_2 of the k-ary tree, corresponding colors in each subtree. We again find a large embedded tree which is monochromatic under

χ_2, and it again corresponds to many trees T_s, each with color $\psi(s)$ under χ. But this time

$$s \in \{1, \ldots, k\}^{2 \cdot 2^\ell} = \left(\{1, \ldots, k\}^{2^\ell}\right)^2.$$

We repeat this process, reaching $\chi_{f(Y,c)}$. The monochromatic trees here are T_s with color $\psi(s)$, where

$$s \in \{1, \ldots, k\}^{f(Y,c) \cdot 2^\ell} = \left(\{1, \ldots, k\}^{2^\ell}\right)^{f(Y,c)}.$$

This is quite nice! We may consider the trees $\{T_s\}$ to be the nodes of a large Y-ary tree, colored by ψ. Since ψ is a c-coloring, and this tree has height $f(Y, c)$, we get some monochromatic embedded subtree of height 1. Expanding the nodes as the full trees they are, and observing the relative structure, we find that these trees are in fact a monochromatic embedding of a k-ary tree of height $2^{\ell+1} - 1$, as desired.

In all, we needed to go a depth of

$$E(k, c, 2^\ell - 1) + E(k, c^Y, 2^\ell - 1) + \ldots + E(k, c^{Y^{f(Y,c)}}, 2^\ell - 1),$$

where again $Y = k^{2^\ell}$. This gives our bound for $E(k, c, 2^{\ell+1} - 1)$. □

5 The full result

In this section, we give the full proof of Theorem 3.1, first for any number of colors, but $k = 2$, and then for any k as well. As before, we view pairs of integers as ordered pairs (x, y) with $x < y$. When we have a grid $\{(x + id, y + jd)\}$ for a range of values i and j, we will say the grid is in position (x, y) with scale d.

5.1 Any colors, two dimensions

Proof of Theorem 3.1 when $k = 2$:

As in the proof of Lemma 4.4, we first give the arguments ignoring the numbers involved, and in the next section we determine a bound on $n(r, 2)$.

Begin with an r-coloring $\chi_0 = \chi$ of a large initial grid, G_λ. By Gallai-Witt, find a large monochromatic subgrid of color c_λ in position (x_0, y_0) with scale d_0.

As in the proof with two colors, this yields two grids, G_1 and G_2 of equal size, in positions (x_0, x_0) and (y_0, y_0) respectively, both with scale d_0. Note that these grids contain points on, above, and below the diagonal (x, x) — we only consider those points above the diagonal. As in the proof in Section 3, if two points in these grids of the form (x_0+id, x_0+jd) and (y_0+id, y_0+jd) are both the same color as the grid G_λ, then we get our monochromatic Hilbert cube of dimension 2. The colorings of G_1 and G_2 correspond to χ_A and χ_B from the initial proof. We consider the coloring of a new grid, where the point (i, j) is colored by

$$\chi_1(i, j) = (\chi_0(x_0 + id, x_0 + jd), \chi_0(y_0 + id, y_0 + jd)).$$

We now use Gallai-Witt with r^2 colors, to find a large subgrid under χ_1 with color (c_1, c_2) in position (x_1, y_1) with scale d_1. This grid really corresponds to two grids: one of color c_1 in position $(x_0 + x_1d_0, x_0 + y_1d_0)$, and the other of color c_2 in position $(y_0 + x_1d_0, y_0 + y_1d_0)$. Both grids have scale d_0d_1, and they are entirely contained in grids G_1 and G_2 respectively.

Again we pass to subgrids. The grid in G_1 yields two subgrids G_{11} and G_{12}, in positions $(x_0+x_1d_0, x_0+x_1d_0)$ and $(x_0+y_1d_0, x_0+y_1d_0)$ respectively, both with scale d_0d_1. Likewise G_2 give us two subgrids, G_{21}, and G_{22}. Now we have more ways to win: the colorings of G_{11} and G_{12} restrict each other, as do G_{21} and G_{22}, and both of G_{11}, G_{12} restrict both of G_{21}, G_{22}. Note that, whether the position of the grid involves x_0 or y_0 is determined by the first part of the subscript, and whether it involves x_1 or y_1 is dependent on the next part.

The next step, which we briefly state, is to define a grid-coloring χ_2 with r^4 colors corresponding to each of the four grids $G_{11}, G_{12}, G_{21}, G_{22}$. We find a subgrid of color $(c_{11}, c_{12}, c_{21}, c_{22})$ under this coloring, which corresponds to four grids, which further restrict one another.

Continue this for $f(2, r) + 1$ steps, so that the final grids are indexed by strings of length $f(2, r)$. The "large" monochromatic grid we find under $\chi_{f(2,r)-1}$ need only be a 2×2 grid, giving G_s a single off-diagonal point for all s of length $f(2, r)$. The color of this point is c_s.

We now recognize the map $s \mapsto c_s$ as an r-coloring of the perfect binary tree of height $f(2, r)$. By the definition of f, this coloring must be 1-balanced, meaning there is a node σ and two children $s = \sigma \cdot 1 \cdot u$ and $t = \sigma \cdot 2 \cdot v$, all the same color, where $u, v \in \{1, 2\}^\ell$ for some ℓ. Call this color red.

Write $\sigma = \sigma_0\sigma_1 \ldots \sigma_{k-1}$. Since σ is red, the monochromatic grid found in

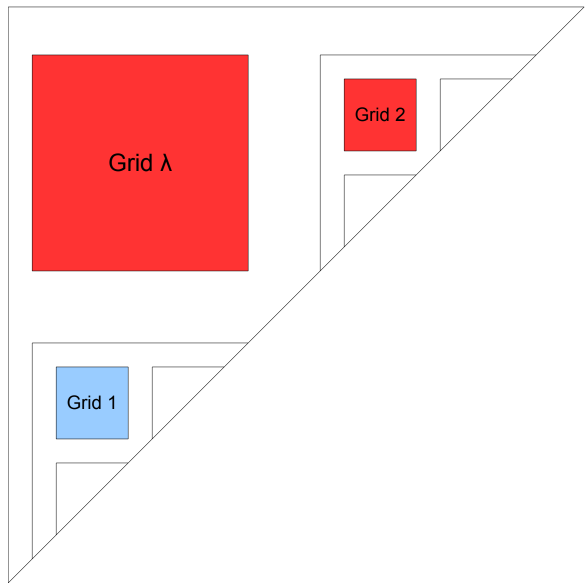

Figure 3: The sequence of subgrids

grid G_{σ_k} is red. Let

$$z_i(\sigma) = \begin{cases} x_i & \text{if } \sigma_i = 1 \\ y_i & \text{if } \sigma_i = 2. \end{cases}$$

Then the grid G_σ is in position $(X(\sigma), Y(\sigma))$, where

$$\begin{aligned} X(\sigma) &= z_0(\sigma) + d_0(z_1(\sigma) + d_1(\dots(z_{k-1} + d_{k-1}x_k)\dots)), \\ Y(\sigma) &= z_0(\sigma) + d_0(z_1(\sigma) + d_1(\dots(z_{k-1} + d_{k-1}y_k)\dots)). \end{aligned}$$

and has scale $D = d_0 d_1 \cdots d_k$. Note that the only difference between X and Y is the x_k and y_k respectively in the inner-most term.

Now we look at the grids G_s and G_t. We will only use a single point from these grids. Define z_i, X, and Y in the same way as above for s and t. Noting that

$$s_0 = \sigma_0, s_1 = \sigma_1, \dots, s_{k-1} = \sigma_{k-1}, s_k = 1 \text{ and}$$

$$t_0 = \sigma_0, t_1 = \sigma_1, \dots, t_{k-1} = \sigma_{k-1}, t_k = 2,$$

we see that G_s is in position $(X(s), Y(s))$ with

$$\begin{aligned} X(s) &= X(\sigma) + D(x_k + d_k(\dots(z_{k+\ell-1}(s) + d_{k+\ell-1}x_{k+\ell})\dots), \\ Y(s) &= X(\sigma) + D(x_k + d_k(\dots(z_{k+\ell-1}(s) + d_{k+\ell-1}y_{k+\ell})\dots), \end{aligned}$$

and similarly G_t is in position $(X(t), Y(t))$ with

$$\begin{aligned} X(t) &= Y(\sigma) + D(y_k + d_k(\ldots(z_{k+\ell-1}(t) + d_{k+\ell-1}x_{k+\ell})\ldots)), \\ Y(t) &= Y(\sigma) + D(y_k + d_k(\ldots(z_{k+\ell-1}(t) + d_{k+\ell-1}y_{k+\ell})\ldots)). \end{aligned}$$

We claim that $X(s), X(t), Y(s), Y(t)$ form our Hilbert cube. Indeed, writing $a = X(s)$, $b = X(t) - X(s) = Y(t) - Y(s)$, and

$$c = Dd_k \cdots d_{k+\ell}(y_{k+\ell+1} - x_{k+\ell+1}),$$

we see that they have the form $a, a+b, a+c, a+b+c$ respectively.

Now consider the colors of the six points among these values (still only considering points above the line (x, x)). Since the points $(X(s), Y(s))$ and $(X(t), Y(t))$ are in G_s and G_t respectively, we know that both points are red.

Now we recognize that these values are given by

$$\begin{aligned} X(s) &= X(\sigma) + iD, \\ Y(s) &= X(\sigma) + jD, \\ X(t) &= Y(\sigma) + iD, \\ Y(t) &= Y(\sigma) + jD, \end{aligned}$$

so the four points we need look like

$$\begin{aligned} (X(s), X(t)) &= (X(\sigma) + iD, Y(\sigma) + iD) \\ (X(s), Y(t)) &= (X(\sigma) + iD, Y(\sigma) + jD) \\ (Y(s), X(t)) &= (X(\sigma) + jD, Y(\sigma) + iD) \\ (Y(s), Y(t)) &= (X(\sigma) + jD, Y(\sigma) + jD). \end{aligned}$$

By design, these fall nicely into the grid G_σ, so these points are red as well. □

5.2 Upper bounds

The process repeats to a depth of $f(2, r)$, at which point we have $2^{f(2,r)}$ grids, meaning $r^{2^{f(2,r)}}$ colors. At this level, we are looking for a square, so these grids must have size

$$S_{f(2,r)} = 2.$$

At the prior level, our $2^{f(2,r)-1}$ grids must have monochromatic subgrids of size $S_{f(2,r)}$, and the joint coloring has $r^{2^{f(2,r)-1}}$ colors. Thus

$$S_{f(2,r)-1} = 2GW(S_{f(2,r)}, r^{2^{f(2,r)-1}}),$$

where the factor of 2 allows us to take the top-left quadrant of the grid. As before, this ensures distinct values in the x and y components. Repeating this reasoning, we find that

$$S_k = 2GW(S_{k+1}, r^{2^k}),$$

which leaves us with this bound for the size of the initial grid:

$$n(r, 2) \leq S_0 = 2GW(S_1, r).$$

5.3 Any colors, any dimensions

We have now done all of the hard work. In order to prove the full result at this point, we only need to reconsider the proof for $k = 2$.

Theorem 3.1 *For all r, k, there is a number $n = n(r, k)$ so that for any r-coloring of the edges of the complete graph on $[n]$, there is a Hilbert cube $H = H(a; b_1, \ldots, b_k)$ so that all edges within H are monochromatic.*
Proof:

Let χ be an r-coloring of a large grid. Repeat the process from the proof in Section 5.1, only now continuing until we have a tree of height $E(2, r, k-1)$.

By Lemma 4.4, there is an embedded tree of height $k-1$ which is entirely, say, red. Call the embedding φ, so the nodes are labeled $\varphi(s)$ for $s \in \{1, 2\}^j$ for $0 \leq j < k$.

Let G_s denote the red grid corresponding to the node $\varphi(s)$.[1] Say this grid is in position $(X(s), Y(s))$. If i is the length of s, then the scale of G_s is $d_0 d_1 \cdots d_i$.

For each $s \in \{1, 2\}^{k-1}$, consider the red point $(X(s), Y(s)) \in G_s$. We claim that the 2^k values

$$\{X(s) \mid s \in \{1, 2\}^{k-1}\} \cup \{Y(s) \mid s \in \{1, 2\}^{k-1}\}$$

have the form $a + \sum_{i \in I} b_i$ and comprise an entirely red clique.

As we saw in the previous proof, for s on level $\ell - 1$, and $s \cdot 1, s \cdot 2$ on level ℓ,

$$X(s \cdot 2) - X(s \cdot 1) = Y(s) - X(s) = d_0 d_1 \cdots d_{\ell-1}(y_\ell - x_\ell).$$

[1] In the previous proof, we would have called this $G_{\varphi(s)}$, but here we have no need to refer to the nodes outside of our monochromatic tree.

Inspired by this, we define

$$b_\ell = Y(s) - X(s)$$

for s on level $\ell - 1$.

Now set $a = X(1^{k-1})$. Let $s = s_1 \cdots s_{k-1} \in \{1,2\}^{k-1}$. Let $I = \{i \mid s_i = 2\} \subseteq [k-1]$. This gives us $X(s) = a + \sum_{i \in I} b_i$ and $Y(s) = a + b_k + \sum_{i \in I} b_i$.

This tells us the numbers we are looking at really do have the desired form. We only need to check that all the edges among these values are red.

Let s be any string on level $k-1$. By virture of $(X(s), Y(s))$ being a point in the grid G_s, we know that edge is red. Now let t be another string on level $k-1$, and assume $s < t$ lexicographically. Let σ be the longest initial string that s and t agree on — their closest common ancestor. Since $s < t$, we must have that $s = \sigma \cdot 1 \cdot u$ and $t = \sigma \cdot 2 \cdot v$ for some u and v of the same length.

As we saw in the previous proof, since G_σ is red, we immediately get that $(X(s), X(t)), (X(s), Y(t)), (Y(s), X(t)), (Y(s), Y(t))$ are all red.

By considering all possible s, t on level $k-1$, this argument says that all edges among these values are red, so we have reached our goal. □

Along the same lines as Section 5.2, we may define the recurrence

$$T_{E(2,r,k)} = 2, \text{ and}$$

$$T_k = 2GW(T_{k+1}, r^{2^k}),$$

to get an upper bound of

$$n(r,k) \leq T_0 = 2GW(T_1, r).$$

5.4 Additional results

Theorem 3.1 immediately gives several nice consequences.

By considering subsets of Hilbert cubes, it is easy to see that, for large n, any edge-coloring of the complete graph on $[n]$ will always have solutions to equations of the form $x_1 + \ldots + x_\ell = y_1 + \ldots + y_\ell$ which induce monochromatic subgraphs.

Combining Theorem 3.1 with Szemerédi's celebrated theorem on arithmetic progressions [9], we get the following nice corollary.

Corollary 5.1 *For any $\delta > 0$, and naturals r, k, there is a number $n = n(r, \delta)$ so that for any set $A \subseteq \mathbb{N}$ of upper density δ, and any r-coloring of the edges of the complete graph on A, there is a Hilbert cube $H = H(a; d_1, \ldots, d_k)$ contained in A so that all edges within H are monochromatic.*

On the other hand, our theorem also inspires another negative result. A Hilbert cube of dimension 2 is simply a set satisfying $w - x = y - z$. We consider a similar equation, $a(w - x) = b(y - z)$, for $a \neq b$ fixed. To avoid this equation, color pairs based on their difference. Write $|w - x| = \left(\frac{b}{a}\right)^k p$ for k as large as possible, and color $\{w, x\}$ by the parity of k. Since $(w - x)$ and $(y - z)$ will always be different by a factor of $\frac{b}{a}$, this will assure the edges $\{w, x\}$ and $\{y, z\}$ have different colors.

6 Acknowledgment

The author would like to thank Ron Graham for his guidance, which helped to strengthen the main theorem.

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