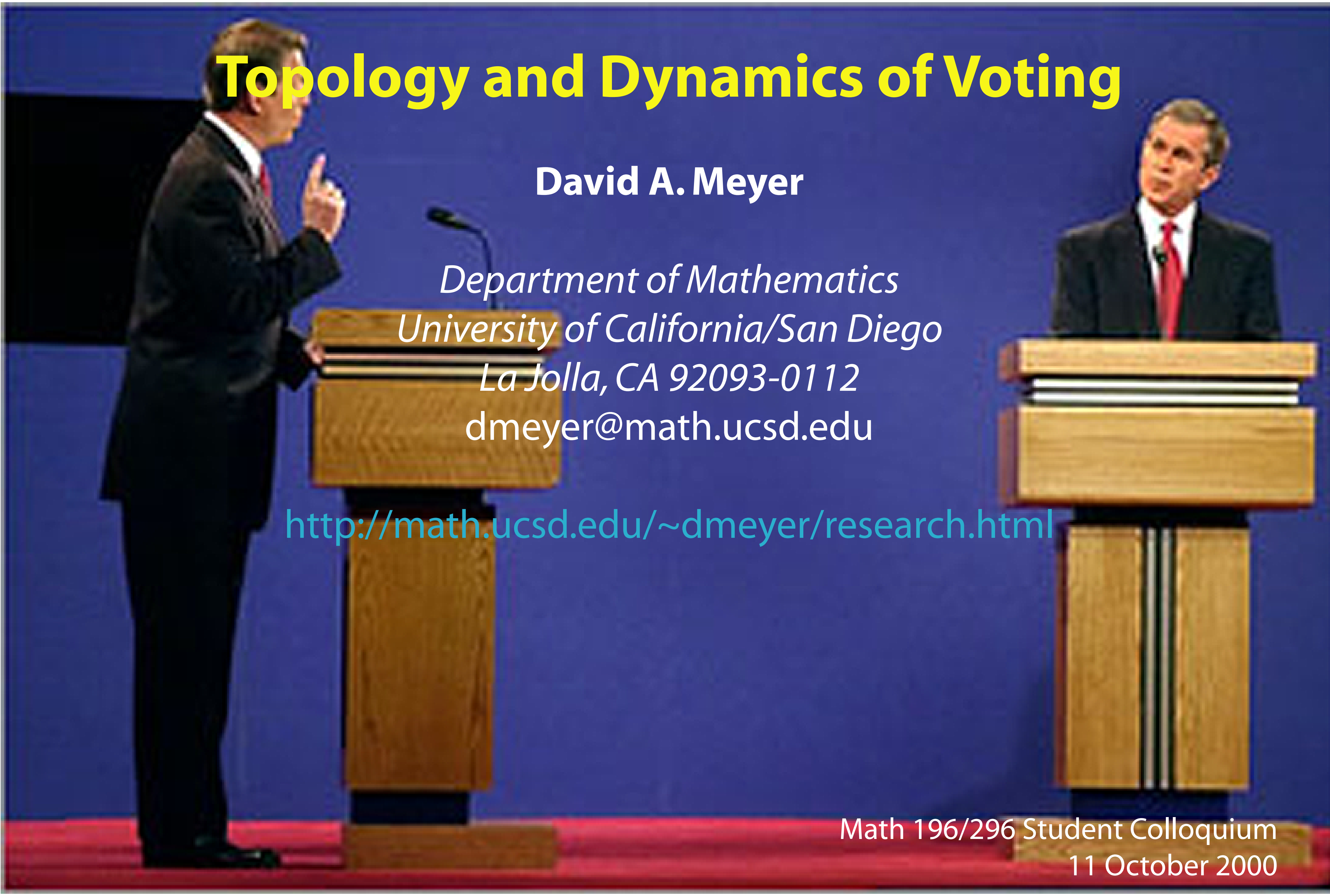

Topology and Dynamics of Voting

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Outline
voting "paradoxes"
Arrow's Theorem
chaotic dynamics
cycles *imply* chaos
open problems

The Marquis de Condorcet's example (1785)

60 voters, each with consistent preferences among 3 alternatives: A,B,C

	A	B	C
23 prefer A > B > C	23		
17 prefer B > C > A		17	
2 prefer B > A > C		2	
10 prefer C > A > B			10
8 prefer C > B > A			8
	23	19	18

A > B > C

in California 2000, say,

A = Gore

B = Bush

C = Nader

RATS

The Marquis de Condorcet's example (1785)

60 voters, each with consistent preferences among 3 alternatives: A,B,C

	A	B	C	A	B
23 prefer A > B > ~~C~~	23			23	
17 prefer B > ~~C~~ > A		17			17
2 prefer B > A > ~~C~~		2			2
10 prefer ~~C~~ > A > B			10	10	
8 prefer ~~C~~ > B > A			8		8
	23	19	18	33	27
	A > B > C			A > B	

The Marquis de Condorcet's example (1785)

60 voters, each with consistent preferences among 3 alternatives: A,B,C

	A	B	C	A	B	B	C
23 prefer ~~A~~ > B > C	23			23		23	
17 prefer B > C > ~~A~~		17			17	17	
2 prefer B > ~~A~~ > C		2			2	2	
10 prefer C > ~~A~~ > B			10	10			10
8 prefer C > B > ~~A~~			8		8		8
	23	19	18	33	27	42	18
	A > B > C			A > B		B > C	

The Marquis de Condorcet's example (1785)

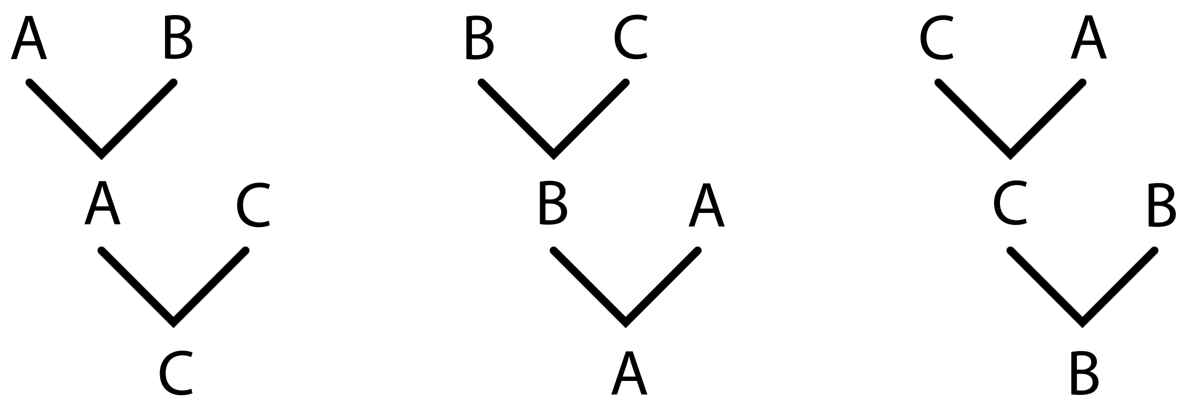
60 voters, each with consistent preferences among 3 alternatives: A,B,C

	A	B	C	A	B	B	C	A	C
23 prefer A > ~~B~~ > C	23			23		23		23	
17 prefer ~~B~~ > C > A		17			17	17			17
2 prefer ~~B~~ > A > C		2			2	2		2	
10 prefer C > A > ~~B~~			10	10			10		10
8 prefer C > ~~B~~ > A			8		8		8		8
	23	19	18	33	27	42	18	25	35
	A > B > C			A > B		B > C		C > A !	

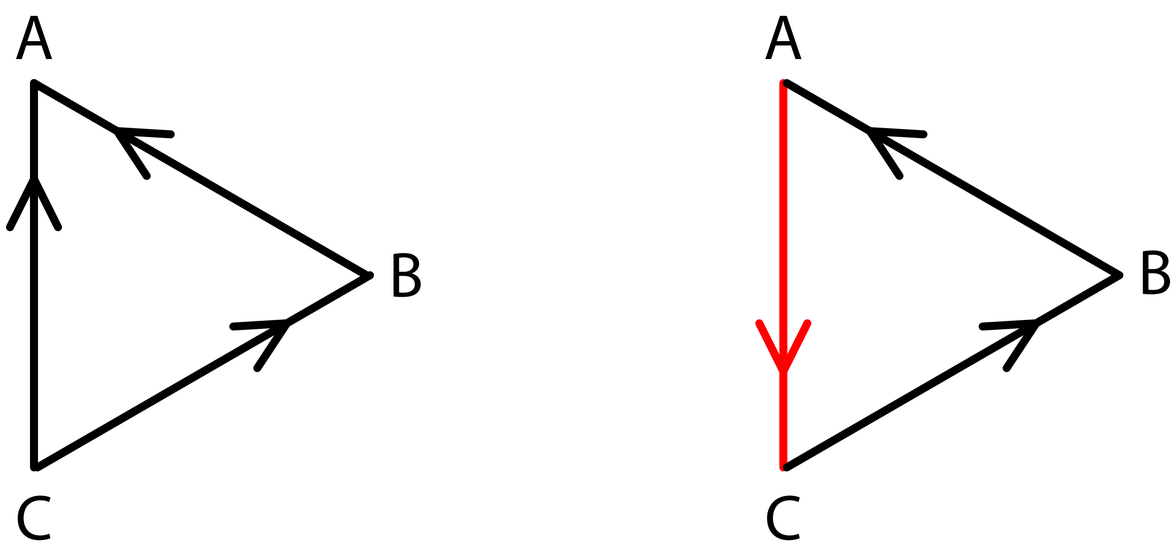
M. J. A. N. de Caritat, Marquis de Condorcet,
Essai sur l'application de l'analyse a la probabilite des decisions rendues a la pluralite des voix
(Paris: *l'Imprimerie Royale 1785*).

Consequences

The order of voting matters ("agenda control"):



Transitive preferences aggregate to cyclic preferences:



Arrow's Theorem (1951)

Let P be the set of preference orders on $n > 2$ alternatives.

A (strict) voting rule is a map from P^k to P.

If the map satisfies two reasonable conditions:

unanimity: if each voter orders $i > j$ then $i > j$

independence: the relation between i and j depends only on the voters' orders of i and j

then the rule is dictatorial, *i.e.*, a projection onto one factor.

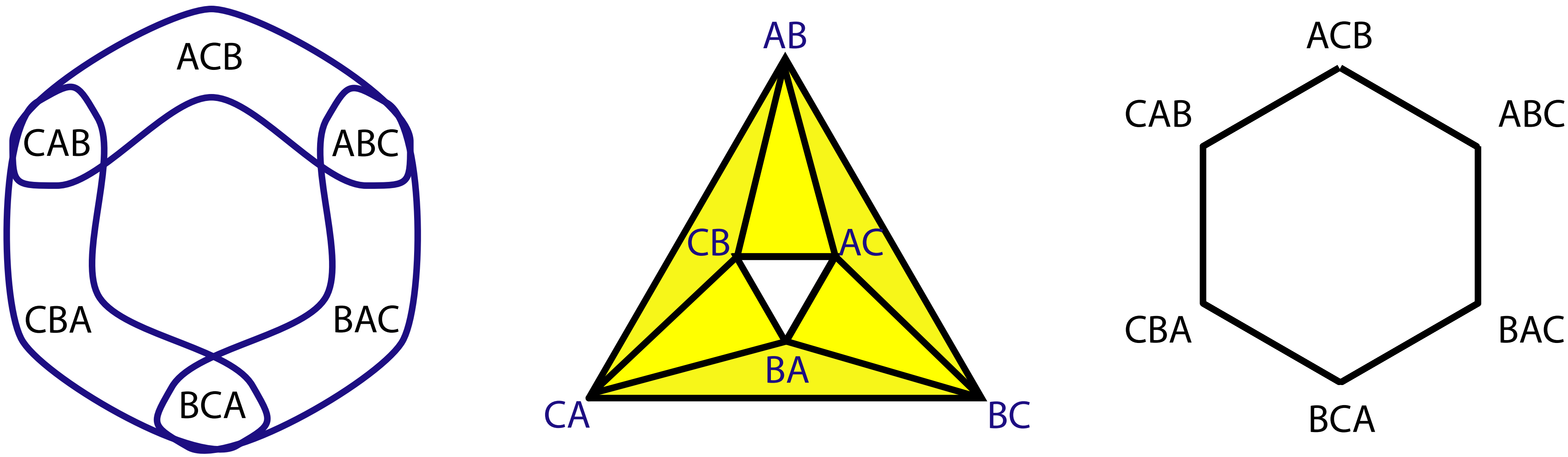
K. J. Arrow, *Social Choice and Individual Values* (New York: Wiley 1951).

Topological ingredients

Based on ideas of Chichilnisky and Baryshnikov.

G. Chichilnisky, "The topological equivalence of the Pareto condition and the existence of a dictator", *J. Math. Econ.* **9** (1982) 223.
Y. M. Baryshnikov, "Unifying impossibility theorems: a topological approach", *Advances in Appl. Math.* **14** (1993) 404.

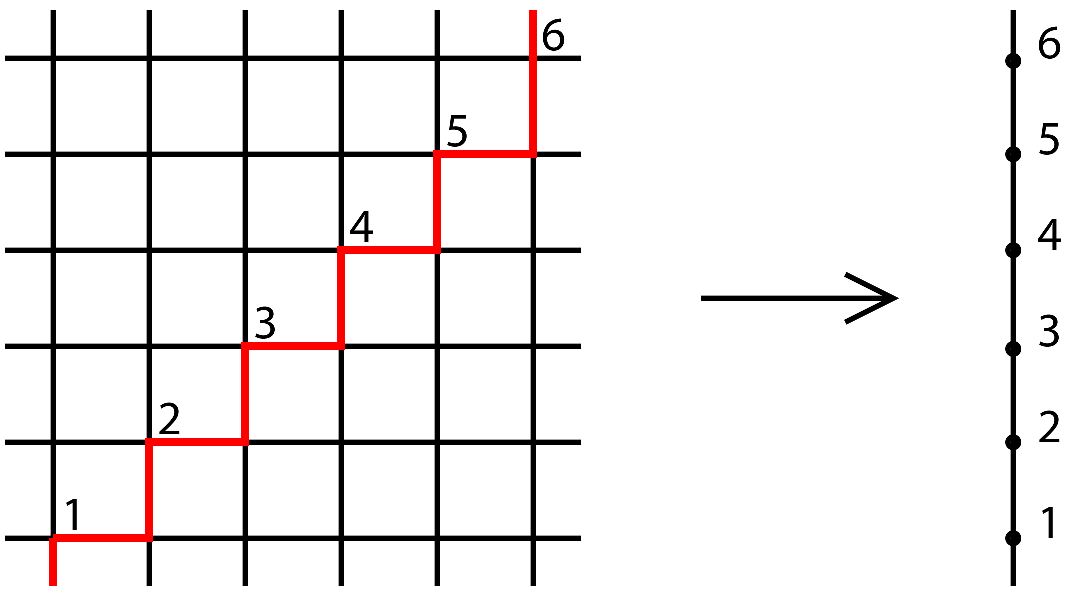
P is a finite topological space with open sets defined by $i > j$:



P nerve N S

Topological proof

Independence implies that the induced map from N^k to N is a simplicial map, and so is the induced map from S^k to S.



Unanimity implies that the diagonal map of S into S^k followed by the induced voting map from S^k to S is the identity.

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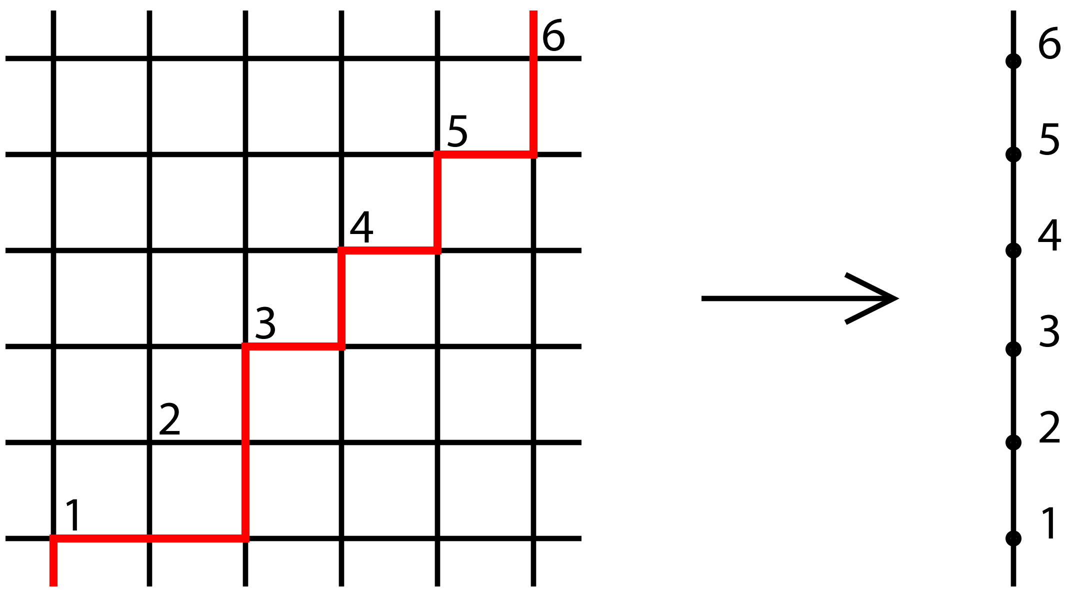

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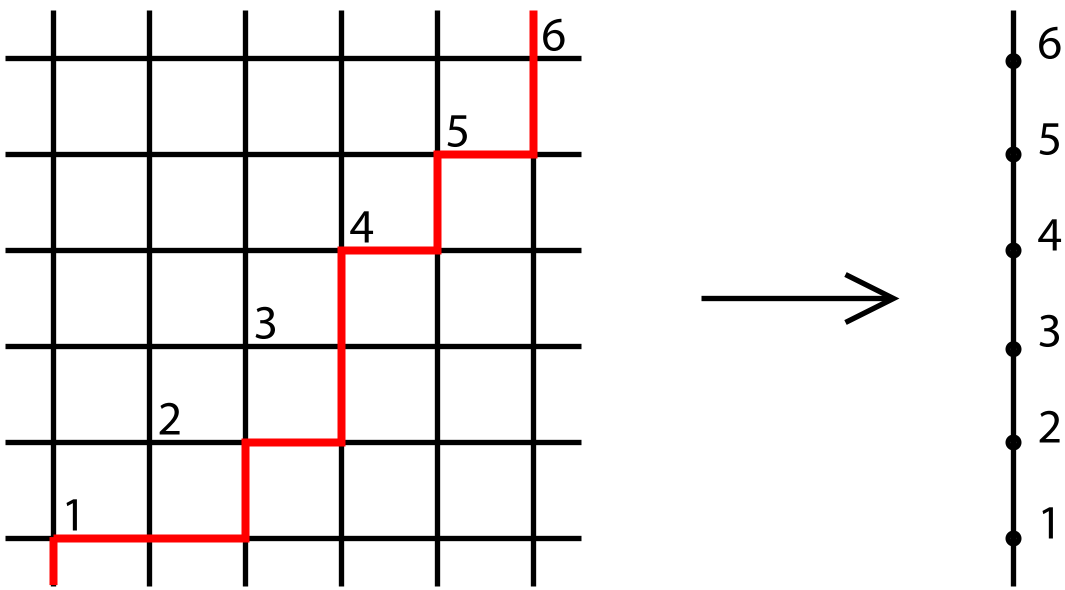

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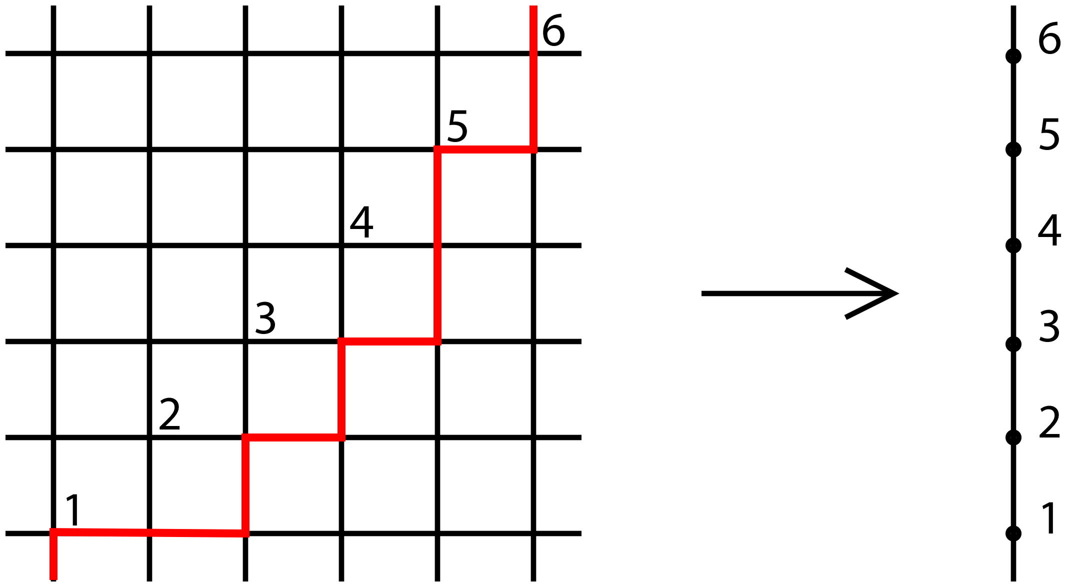

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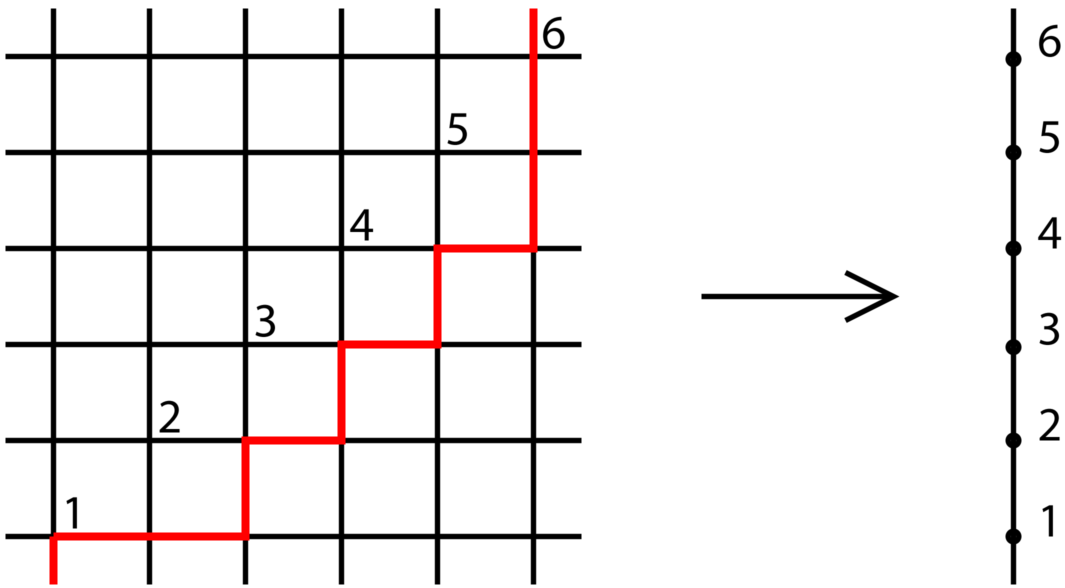

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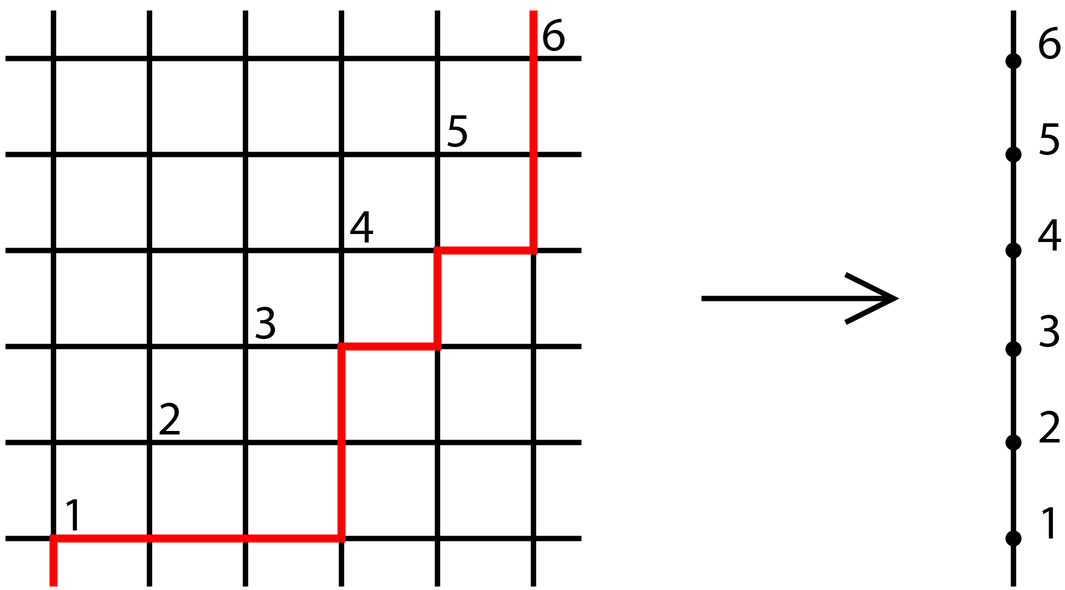

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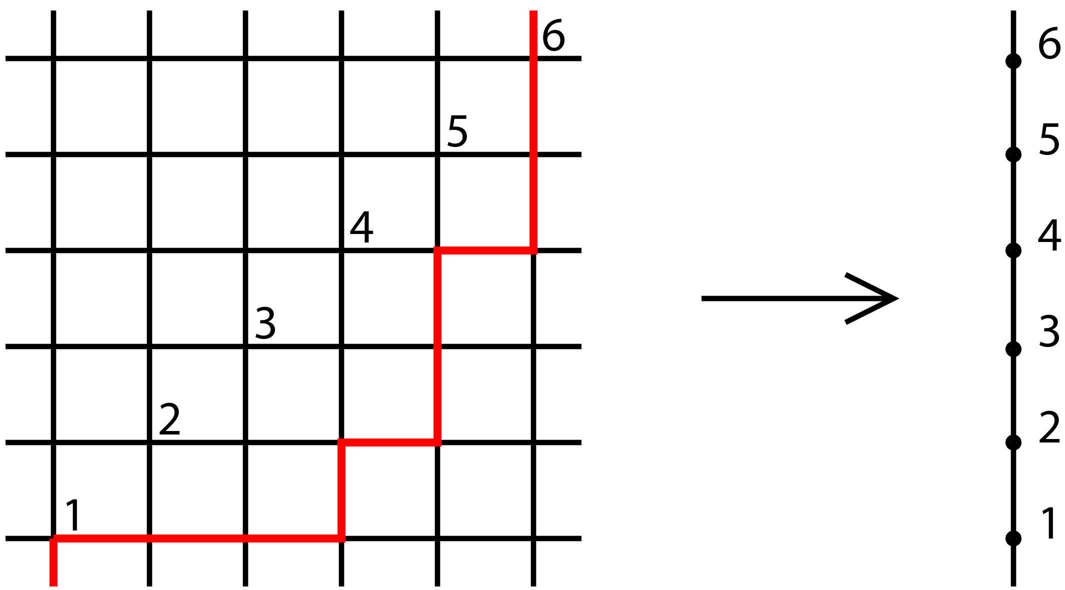

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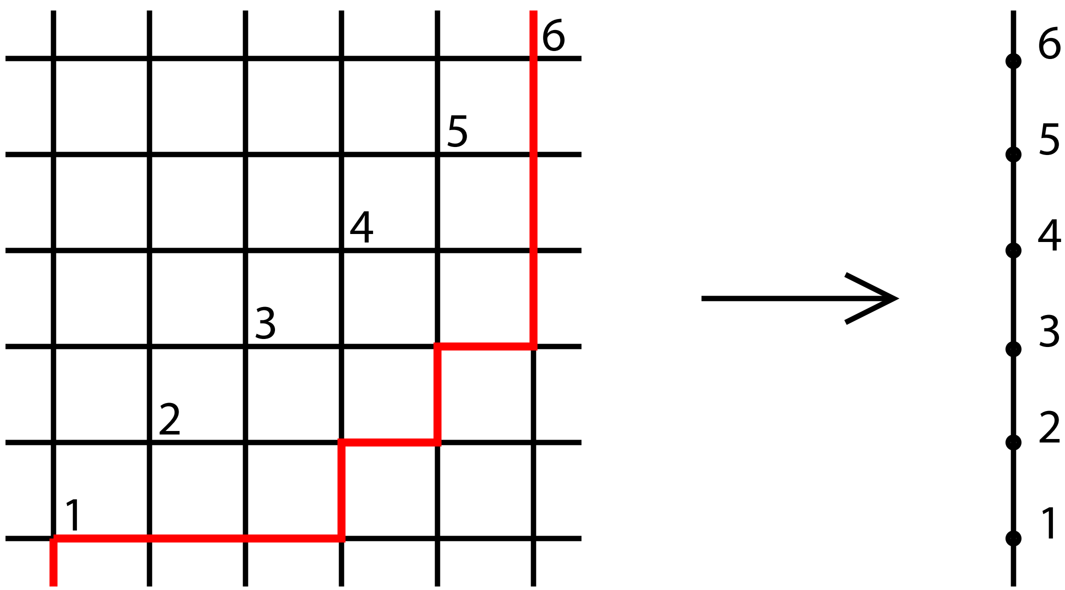

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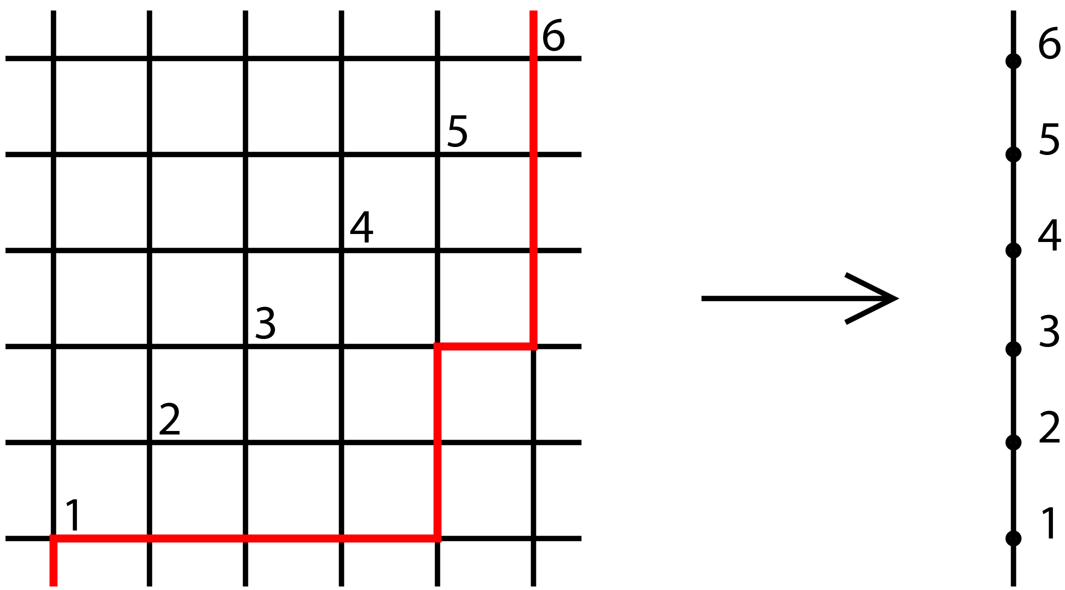

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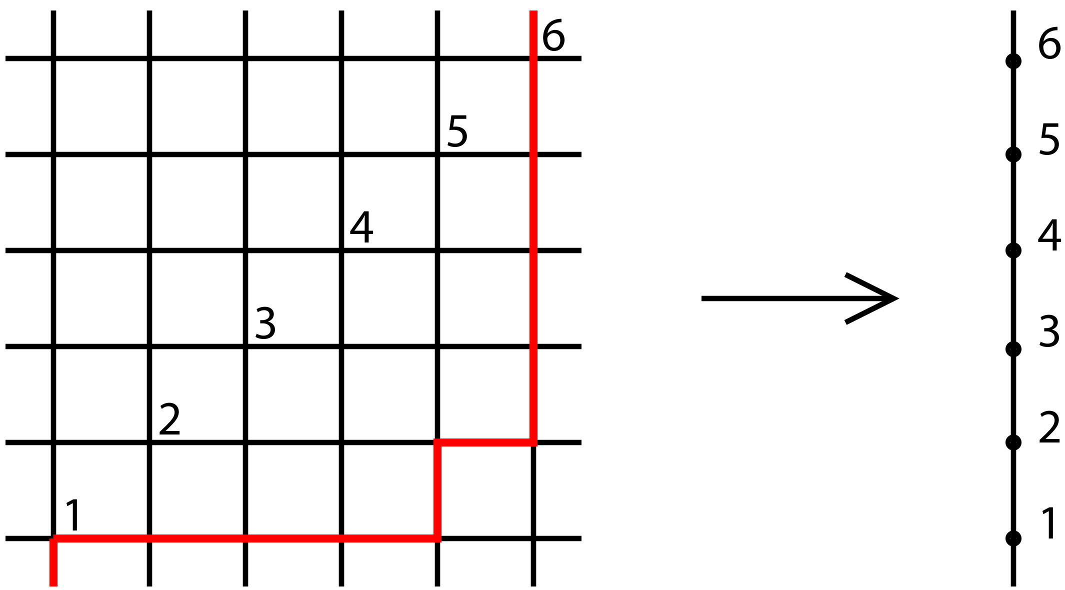

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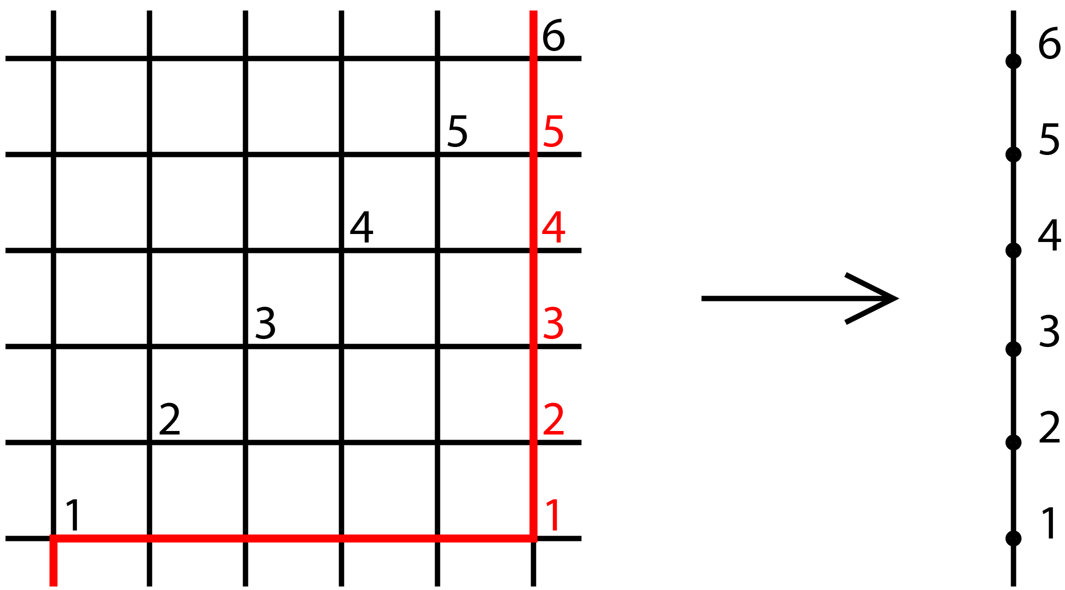
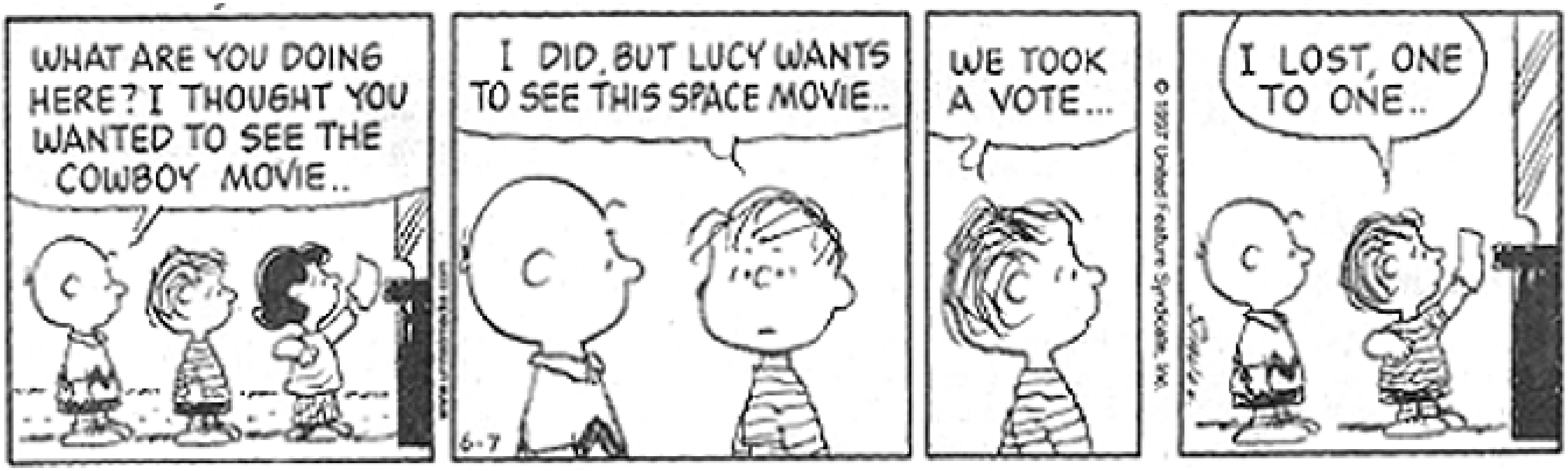


Unanimity implies that the diagonal map of S into S^k followed by the induced voting map from S^k to S is the identity. ………. which implies either a horizontal or a vertical cycle, but not both, maps onto S. Say vertical.

But then every vertical cycle maps onto S.

Consequences

Either accept that the only "reasonable" voting rule is dictatorship ...



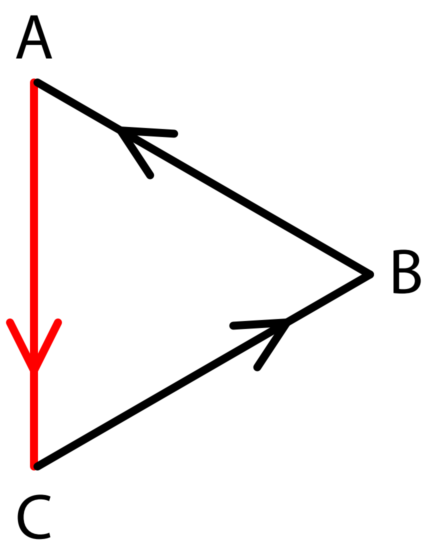
... or allow voting rules which do not map into P, like majority rule.

By chaos, we mean complete disorder or the lack of predictability of the system. The choice of the word "chaos" is selected to reflect both this generic usage of the word as well as to provoke comparisons with its technical usage coming from dynamical systems. In the dynamical systems literature, chaos is used to mean the existence of a subsystem of a deterministic dynamical system which is highly "random". ... The weights and voters' preferences create a deterministic system. Yet, for a random selection of outcomes, a subsystem can be found (subset of voters' preferences) for which this system realizes the specified outcome.

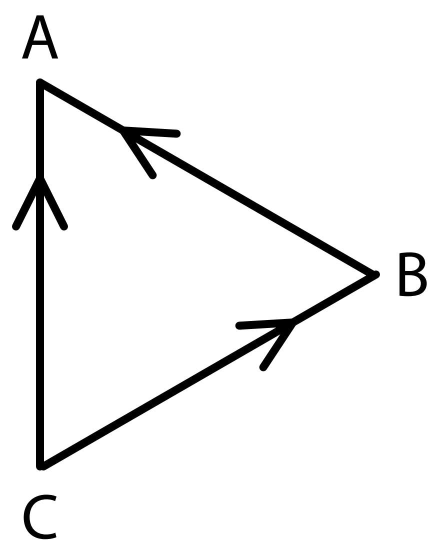
D. G. Saari, "The ultimate of chaos resulting from weighted voting systems", *Advances in Appl. Math.* **5** (1984) 286.

Voting dynamics

A voting rule applied to a specific set of preferences can be represented by a directed graph:



CBBACBAAACC ...
BBACBAAACC ...
BACBAAACC ...



CBBAAAAA ...
BBAAAAA ...
BAAAAA ...

are admissible sequences.

D. A. Meyer and T. A. Brown, "Statistical mechanics of voting", *Phys. Rev. Lett.* **81** (1998) 1718.

Dynamical systems

Let X be a set of admissible sequences $(x_0, x_1, x_2, \ldots)$ defined by a finite directed graph, with a metric defined by:

$$d(x,y) = 2^{-t(x,y)} \qquad \text{where} \qquad t(x,y) := \min\{\, t : x_t \neq y_t \,\}.$$

Let T acting on X be the shift map which takes $(x_0, x_1, x_2, \ldots)$ to $(x_1, x_2, \ldots)$.

The iterated action of T on X defines a kind of dynamical system, a subshift of finite type.

W. H. Gottshalk and G. A. Hedlund, *Topological Dynamics*, AMS Colloquium Publications **36** (Providence, RI: AMS 1955).

Chaotic dynamics

A dynamical system (X,T) is chaotic if

T has sensitive dependence on initial conditions

T is topologically transitive

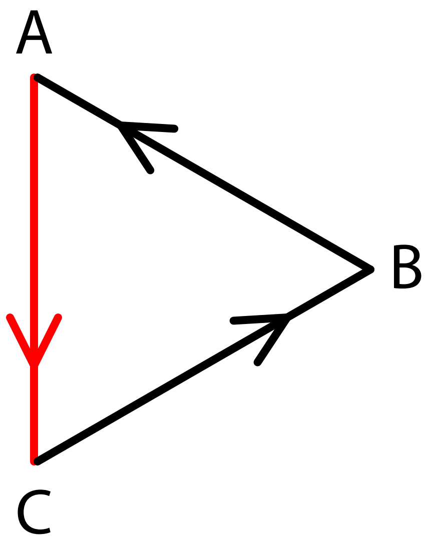
periodic points are dense in X

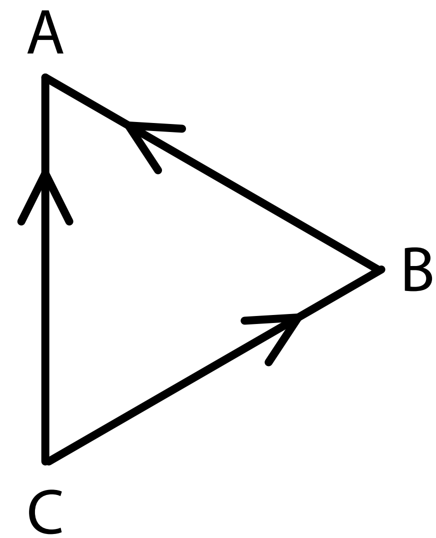
the topological entropy is positive

T. Li and J. Yorke, "Period three impies chaos", *Amer. Math. Monthly* **82** (1975) 985.
M. Misiurewicz, "Horseshoes for continuous mappings of an interval", in
C. Marchioro, ed., *Dynamical Systems* (Napoli, Italy: *Liguori Editore* 1980) 125.

The transition matrix

For a directed graph, define the transition matrix: $T_{ab} = \begin{cases} 1 & \text{if } a \leftarrow b; \\ 0 & \text{otherwise.} \end{cases}$





$$T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$S(T_1) = 1$

$S(T_2) = 0$

Topological entropy

The topological entropy is

$$S(T) := \lim \frac{1}{n} \log (\#\text{ admissible sequences of length } n)$$

$$= \lim \frac{1}{n} \log \operatorname{Tr} T^n$$

$$= \log (\text{ largest eigenvalue of } T)$$

$$> 0 \quad \text{iff} \quad \text{there is a preference cycle}$$

So "paradoxes" imply chaos, and the topological entropy quantifies the degree of chaos, given a voting rule and a set of voter preferences.

Some (partially) open questions

Are there topological proofs for other results in social choice/decsion theory?

What is the average entropy for specific voting rules over some distribution of voter preferences?

Is there a best voting rule by a minimum entropy criterion?

How does the difficulty of choosing an agenda leading to a desired outcome depend on the entropy?

What happens when the space of alternatives is continuous?

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What happens when the space of alternatives is continuous?

How does this apply to real political situations?