This is the final from the last time I taught this class. We have not covered Chapter 9, so you do not need to be able to do problems 1, 2 and 6.b.

There are 6 problems worth a total of 100 points, and 1 extra credit problem worth 15 points. Make sure that you look on the back of this page.

1. [16 points] Suppose $\langle x_n \rangle \to x$ and for all $n \in \mathbb{N}$, $x_n \neq x$. Prove that
$$\langle \frac{x^2 - x_n^2}{x - x_n} \rangle \to 2x.$$

2. [16 points] Use the Least-Upper-Bound Axiom to prove that if $S \subset \mathbb{R}$ has a lower bound then it has a greatest lower bound.

3. For $n \in \mathbb{N}$, let $[n]$ denote the equivalence class of n modulo p, where $p > 1$ is a prime number. Explain your answer to each of the following questions:

   a. [2 points] What is the cardinality of $\mathbb{Z}_p = \{[n] \mid n \in \mathbb{N}\}$?

   b. [4 points] Define $f : \mathbb{N} \to \mathbb{Z}_p$ by $f(n) = [n]$. Is f onto? Is f one-to-one?

   c. [6 points] Define $g : \mathbb{N} \to \mathbb{Z}_p$ by $g(n) = [n^{p-1}]$. Is g onto? Is g one-to-one?

   d. [6 points] Let $q > 1$ be a prime number not equal to p. Define $h : \mathbb{Z}_p \to \mathbb{Z}_p$ by $h([n]) = [qn]$. Is h onto? Is h one-to-one?

4. a. [6 points] Solve $x^2 - x \equiv 6 \pmod 7$.

   b. [12 points] Find all solutions to the system of congruences:
$$x^2 - x \equiv 6 \pmod 7$$
$$x \equiv 1 \pmod 5.$$

(over)

5. Let $n \in \mathbb{N}$.

   a. [5 points] How many solutions are there to the equation $x_1 + x_2 = n$ for x_1 and x_2 nonnegative integers? Prove that your answer is right.

   b. [10 points] Prove that there are $(n+1)(n+2)/2$ solutions to the equation $x_1 + x_2 + x_3 = n$ for x_1, x_2 and x_3 nonnegative integers.

6. For $n \in \mathbb{N}$, let $U_n = \left\{ x \in \mathbb{R} \;\middle|\; 1 - \frac{1}{n} < x < 1 + \frac{1}{n} \right\}$.

   a. [5 points] Find $\cap_{n \in \mathbb{N}} U_n$.

   b. [10 points] Let $\langle a_n \rangle$ be a sequence with the property that for all $n \in \mathbb{N}$, $a_n \in U_n$. Prove $\langle a_n \rangle \to 1$.

Extra Credit. [15 points] Prove that the following propositional expression is a contradiction:

$$\begin{aligned}
&(\neg R_1 \vee \neg R_2) \wedge (\neg B_1 \vee \neg B_2) \wedge \\
&(\neg R_2 \vee \neg R_3) \wedge (\neg B_2 \vee \neg B_3) \wedge \\
&(\neg R_3 \vee \neg R_1) \wedge (\neg B_3 \vee \neg B_1) \wedge \\
&(R_1 \vee B_1) \wedge (R_2 \vee B_2) \wedge (R_3 \vee B_3)
\end{aligned}$$

Your proof does not have to include a complete truth table.