Math 110 22 May 2009

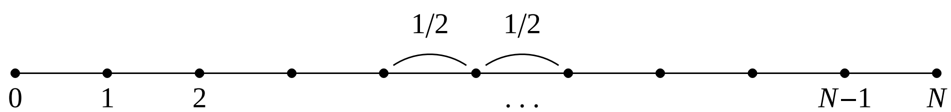
Extra Credit Problem 3

Our textbook gives the example of the function $u(x) = \text{prob}($a "randomly moving" particle starting at $x \in D$ ends at $C_1 \subset \partial D)$, where $D \subset \mathbb{R}^d$, as a harmonic function satisfying the boundary conditions

$$u(x) = \begin{cases} 0 & \text{if } x \in C_0; \\ 1 & \text{if } x \in C_1; \end{cases}$$

where $\partial D = C_0 \sqcup C_1$. To see why $u(x)$ satisfies Laplace's equation, consider a setting in which we can easily make precise what "randomly moving" means:

Let Γ be the graph (network) consisting of vertices labelled $0, 1, \ldots, N$, with an edge connecting each pair $\{x, x+1\}$ for $x \in \{0, \ldots, N-1\}$:



Now consider the motion of a particle on Γ, hopping repeatedly from vertex x to vertex $x-1$ with probability $1/2$ and to vertex $x+1$ with probability $1/2$, for $1 \le x \le N-1$. If the particle ever hops to vertex 0 or vertex N, it stops. Let $u(x) = \text{prob}($a particle that starts hopping at x will end at vertex $N)$. (So $C_0 = \{0\}$ and $C_1 = \{N\}$).

(a) Write $N+1$ equations for the $N+1$ unknowns $u(x)$. The first and last of these equations are $u(0) = 0$ and $u(N) = 1$.

(b) Write this set of equations in matrix form: $Au = f$, where u and f are $N+1$ dimensional vectors and A is an $(N+1) \times (N+1)$ matrix.

(c) Show that A is essentially the discretization of the Laplacian in 1 dimension, $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$.

(d) Solve the equations in (a) or (b) for $u(x)$.

(e) Do a similar analysis for a particle hopping randomly when Γ is a square lattice $\{(x, y) \mid 0 \le x, y \le N\}$ and the particle hops from an interior point to each of its left, right, up, and down neighboring points with probability $1/4$. Partition the boundary, for example, as:

$$C_0 = \{(x, y) \mid y = 0 \text{ or } (x = 0 \text{ and } y < N)\}$$
$$C_1 = \{(x, y) \mid y = N \text{ or } (x = N \text{ and } 0 < y)\}.$$