HW #7 Solutions

Section 5.5 #'s 2, 4, 13, 14, 15

#2: We prove the Schwarz inequality:

$$|(f,g)|^2 \le \|f\|\,\|g\|.$$

Recall that $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$, and

$$\|f\| = \left[\int_a^b |f(x)|^2\,dx\right]^{1/2}.$$

We consider the quantity $\|f+tg\|^2$, where t is any complex number. By definition, $0 \le \|f+tg\|^2$. So

$$0 \le \|f+tg\|^2 = \int |f+tg|^2\,dx = \int (f+tg)\overline{(f+tg)}\,dx$$

$$= \int (f+tg)(\bar{f}+\bar{t}\bar{g})\,dx$$

$$= \int f\bar{f}\,dx + t\int \bar{f}g\,dx + \bar{t}\int f\bar{g}\,dx + t\bar{t}\int g\bar{g}\,dx$$

$$= \int |f|^2\,dx + t\,\overline{(f,g)} + \bar{t}\,(f,g) + |t|^2\int |g|^2\,dx$$

$$= \|f\|^2 + t\,\overline{(f,g)} + \bar{t}\,(f,g) + \|g\|^2|t|^2$$

The inequality holds for all t; in particular, let $t = -\dfrac{(f,g)}{\|g\|^2}$. Then

$$0 \le \|f\|^2 + \frac{-(f,g)\overline{(f,g)}}{\|g\|^2} + \frac{-\overline{(f,g)}}{\|g\|^2}(f,g) + \frac{|(f,g)|^2\|g\|^2}{\|g\|^4}$$

$$= \|f\|^2 - \frac{2|(f,g)|^2}{\|g\|^2} + \frac{|(f,g)|^2}{\|g\|^2}$$

$$= \|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2}.$$ Hence

$|(f,g)|^2 \le \|f\|^2\|g\|^2$, and the result follows by taking a square root. (Here we have used repeatedly that for a complex number z, we have $|z|^2 = z\bar{z}$).

#4: We reproduce the solution from the student solutions manual. Parts a & b are solved in exercise 12 of section 4.3, so we reproduce that solution as well.

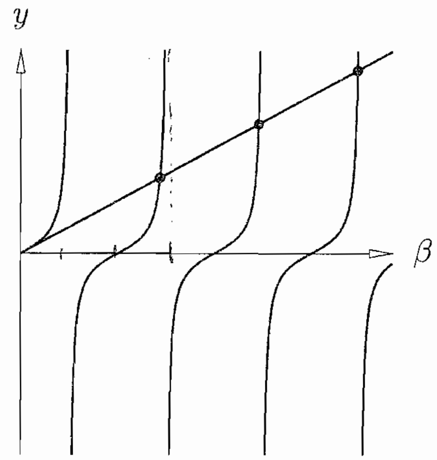

Figure 11: The graphs of $y = \tan\beta$ and $y = \beta$.

and the boundary conditions imply $C\cosh\gamma + D\sinh\gamma = 0$ and $C + D\gamma = 0$, so $D(\sinh\gamma - \gamma\cosh\gamma) = 0$, or $\tanh\gamma = \gamma$. Since $\tanh\gamma < \gamma$ for all positive γ, there are no non-zero solutions of this equation, and thus no negative eigenvalues.

4.3.12.

(a) If $\lambda = 0$, then $v(x) = C + Dx$ which implies $v'(x) = D$. The boundary condition $v'(0) = v'(l) = \frac{v(l)-v(0)}{l}$ therefore implies $D = D = \frac{[C+Dl]-C}{l}$, a condition which is satisfied for all C, D. Therefore, $v(x) = C + Dx$ is an eigenfunction for all constants C, D. In particular, $\lambda = 0$ has two linearly independent eigenfunctions $X_0(x) = 1$ and $Y_0(x) = x$.

(b) If $\lambda = \beta^2 > 0$, then $v(x) = C\cos(\beta x) + D\sin(\beta x)$. The boundary condition $v'(0) = v'(l) = \frac{v(l)-v(0)}{l}$ implies

$$D\beta = -C\beta\sin(\beta l) + D\beta\cos(\beta l) = \frac{[C\cos(\beta l) + D\sin(\beta l)] - C}{l}.$$

The first equality implies that $D(1-\cos(\beta l)) = -C\sin(\beta l)$. The second equality implies that $D(\beta l - \sin(\beta l)) = C(\cos(\beta l) - 1)$. So to find a solution $(C, D) \neq (0,0)$ requires the equality

$$(1-\cos(\beta l))^2 = (\beta l - \sin(\beta l))\sin(\beta l).$$

Multiplying this out, it can be written as

$$2(1-\cos(\beta l)) = \beta l\sin(\beta l).$$

(c) Let $\gamma = \beta l/2$. Therefore,

$$\begin{aligned}
\beta\sin(\beta l) = \frac{2-2\cos(\beta l)}{l} &\implies \frac{\beta l}{2}\sin(\beta l) = 1 - \cos(\beta l) \\
&\implies \gamma\sin(2\gamma) = 1 - \cos(2\gamma) \\
&\implies 2\gamma\sin\gamma\cos\gamma = 1 - [1 - 2\sin^2\gamma] \\
&\implies \gamma\sin\gamma\cos\gamma = \sin^2\gamma.
\end{aligned}$$

(d) The equation $\gamma \sin\gamma \cos\gamma = \sin^2\gamma$ implies either $\gamma\cos\gamma = \sin\gamma$ or $\sin\gamma = 0$. If $\sin\gamma = 0$, then $\gamma = n\pi$. In this case, we conclude that we have eigenvalues $\lambda = \beta^2 = (2\gamma/l)^2 = (2n\pi/l)^2$.

In the other case, we have $\gamma = \tan\gamma$. Therefore, our other eigenvalues are given by $\lambda = \beta^2 = (2\gamma/l)^2$ where $\gamma = \tan\gamma$.

(e) For $\lambda = 0$, we have eigenfunctions $X_0(x) = 1$ and $Y_0(x) = x$. If $\lambda = \beta^2 = (2n\pi/l)^2$, then using the equality

$$D\beta = \frac{C\cos(\beta l) + D\sin(\beta l) - C}{l},$$

we see that $D(2n\pi/l) = 0$ which implies that $D = 0$. Therefore, the eigenfunctions corresponding to $\lambda_n = (2n\pi/l)^2$ are given by

$$X_n(x) = \cos\left(\frac{2n\pi}{l}x\right).$$

Our other eigenvalues are given by $\lambda = \beta^2 = (2\gamma/l)^2$ where $\gamma = \tan(\gamma)$. The condition $D(1 - \cos(\beta l)) = -C\sin(\beta l)$ implies the corresponding eigenfunctions are given by

$$Y_n(x) = \cos(\beta x) - \left(\frac{\sin(\beta l)}{1 - \cos(\beta l)}\right)\sin(\beta x) = \cos(\beta x) - \frac{2}{\beta l}\sin(\beta x)$$

by part (b).

(f) Now to solve the heat equation with the boundary conditions specified above, we use separation of variables to lead us to the eigenvalue problem stated above. To recap, our eigenvalues and eigenfunctions are given by

$$\begin{aligned}
\lambda_0 &= 0 \text{ with } X_0(x) = 1, Y_0(x) = x \\
\lambda_n &= \left(\frac{2n\pi}{l}\right)^2 \text{ with } X_n(x) = \cos\left(\frac{2n\pi}{l}x\right) \\
\alpha_n &= \beta_n^2 \text{ where } \beta_n = \frac{2\gamma_n}{l} \text{ for } \gamma_n = \tan(\gamma_n) \\
&\qquad \text{with } Y_n(x) = \cos(\beta_n x) - \left(\frac{\sin(\beta_n l)}{1 - \cos(\beta_n l)}\right)\sin(\beta_n x).
\end{aligned}$$

Our equation for T,

$$T_n' = -k\lambda_n T_n,$$

has solutions

$$T_n(t) = A_n e^{-k\lambda_n t}.$$

Therefore, the solution is

$$\begin{aligned}
u(x,t) = A_0 + B_0 x &+ \sum_{n=1}^{\infty} C_n \cos\left(\frac{2n\pi}{l}x\right) e^{-k\left(\frac{2n\pi}{l}\right)^2 t} \\
&+ \sum_{n=1}^{\infty} D_n \left[\cos(\beta_n x) - \frac{2}{\beta_n l}\sin(\beta_n x)\right] e^{-k\beta_n^2 t}
\end{aligned}$$

where the coefficients A_0, B_0, C_n, D_n are chosen such that $u(x,0) = \phi(x)$.

(g) From our solution in (f), we see that as $t \to +\infty$, all terms decay to zero, except for the term $A_0 + B_0 x$. Therefore,

$$\lim_{t\to+\infty} u(x,t) = A_0 + B_0 x.$$

4.3.15. Our eigenvalue problem can be rewritten as

$$\begin{cases} \kappa_1^2 X'' + \lambda\rho_1^2 X = 0 & 0 < x < a \\ \kappa_2^2 X'' + \lambda\rho_2^2 X = 0 & a < x < l \\ X(0) = 0 = X(l). \end{cases}$$

If $\lambda = \beta^2 > 0$, the solution of our equations is given by

$$X(x) = \begin{cases} A\cos\left(\frac{\beta\rho_1 x}{\kappa_1}\right) + B\sin\left(\frac{\beta\rho_1 x}{\kappa_1}\right) & 0 < x < a \\ C\cos\left(\frac{\beta\rho_2 x}{\kappa_2}\right) + D\sin\left(\frac{\beta\rho_2 x}{\kappa_2}\right) & a < x < l. \end{cases}$$

The boundary condition $X(0) = 0$ implies $A = 0$. The boundary condition $X(l) = 0$ implies $C\cos\left(\frac{\beta\rho_2 l}{\kappa_2}\right) + D\sin\left(\frac{\beta\rho_2 l}{\kappa_2}\right) = 0$. In order to guarantee that our eigenfunction is continuous at $x = a$, we require

$$B\sin\left(\frac{\beta\rho_1 a}{\kappa_1}\right) = C\cos\left(\frac{\beta\rho_2 a}{\kappa_2}\right) + D\sin\left(\frac{\beta\rho_2 a}{\kappa_2}\right).$$

Additionally, to guarantee our eigenfunction has a continuous derivative at $x = a$, we require

$$B\frac{\beta\rho_1}{\kappa_1}\cos\left(\frac{\beta\rho_1 a}{\kappa_1}\right) = -C\frac{\beta\rho_2}{\kappa_2}\sin\left(\frac{\beta\rho_2 a}{\kappa_2}\right) + D\frac{\beta\rho_2}{\kappa_2}\cos\left(\frac{\beta\rho_2 a}{\kappa_2}\right).$$

Eliminating B from the last two equations, we see that

$$\frac{\rho_1}{\kappa_1}\cot\left(\frac{\beta\rho_1 a}{\kappa_1}\right)\left[C\cos\left(\frac{\beta\rho_2 a}{\kappa_2}\right) + D\sin\left(\frac{\beta\rho_2 a}{\kappa_2}\right)\right] = -C\frac{\rho_2}{\kappa_2}\sin\left(\frac{\beta\rho_2 a}{\kappa_2}\right) + D\frac{\rho_2}{\kappa_2}\cos\left(\frac{\beta\rho_2 a}{\kappa_2}\right).$$

Now multiplying this equation by $\cos\left(\frac{\beta\rho_2 l}{\kappa_2}\right)$ and using the relation

$$C\cos\left(\frac{\beta\rho_2 l}{\kappa_2}\right) + D\sin\left(\frac{\beta\rho_2 l}{\kappa_2}\right) = 0,$$

we see that

$$\begin{aligned} &\frac{\rho_1}{\kappa_1}\cot\left(\frac{\beta\rho_1 a}{\kappa_1}\right)\left[-D\sin\left(\frac{\beta\rho_2 l}{\kappa_2}\right)\cos\left(\frac{\beta\rho_2 a}{\kappa_2}\right) + D\sin\left(\frac{\beta\rho_2 a}{\kappa_2}\right)\cos\left(\frac{\beta\rho_2 l}{\kappa_2}\right)\right] \\ &\quad = \left[D\frac{\rho_2}{\kappa_2}\sin\left(\frac{\beta\rho_2 l}{\kappa_2}\right)\sin\left(\frac{\beta\rho_2 a}{\kappa_2}\right) + D\frac{\rho_2}{\kappa_2}\cos\left(\frac{\beta\rho_2 a}{\kappa_2}\right)\cos\left(\frac{\beta\rho_2 l}{\kappa_2}\right)\right]. \end{aligned}$$

5.4.19

(a) Differentiating $-X'' = \lambda X$ with respect to λ gives $-X''_\lambda = \lambda X_\lambda + X$.

(b) Green's second identity (5.3.3), with $X_1 = X_\lambda$ and $X_2 = X$ gives

$$\int_a^b -X''_\lambda X + X_\lambda X''\,dx = \Big[- X'_\lambda X + X_\lambda X' \Big]_a^b,$$

so using the fact that $X'' = -\lambda X$ and $-X''_\lambda = \lambda X_\lambda + X$, this becomes

$$\int_a^b X^2\,dx = \Big[- X'_\lambda X + X_\lambda X' \Big]_a^b.$$

(c) Let $X(x,\lambda) = \sin(\sqrt{\lambda}\,x)$. Then $X_\lambda = \frac{1}{2}\lambda^{-1/2} x\cos(\sqrt{\lambda}\,x)$ and $X' = \sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$, so part (b) applied at $\lambda = m^2\pi^2/l^2$ gives

$$\int_0^l \sin^2\frac{m\pi x}{l}\,dx = \Big[- X'_\lambda\left(x, \frac{m^2\pi^2}{l^2}\right)\sin\frac{m\pi x}{l} + \frac{1}{2}x\cos^2\frac{m\pi x}{l}\Big]_0^l = \frac{l}{2}.$$

Section 5.5

5.5.2 If g is the zero function, then the inequality holds since both sides are zero. Now suppose g is not the zero function, and let $H(t) = \|f + tg\|^2$. Then $H(t) \geq 0$ for all t. Since

$$H(t) = (f + tg, f + tg) = (f, f) + 2(f, g)t + (g, g)t^2$$
$$H'(t) = 2(f, g) + 2(g, g)t,$$

the only critical point of H is $t = -(f, g)/(g, g)$. Thus

$$0 \leq H(-(f,g)/(g,g)) = (f, f) - 2\frac{(f,g)^2}{(g,g)} + \frac{(f,g)^2}{(g,g)} = (f,f) - \frac{(f,g)^2}{(g,g)}.$$

Multiplying both sides by (g, g), this yields $(f, f)(g, g) - (f, g)^2 \geq 0$, so $(f, g)^2 \leq \|f\|^2\|g\|^2$. Taking the square root of both sides gives $|(f, g)| \leq \|f\| \cdot \|g\|$.

5.5.4 Parts (a) and (b) follow from Exercise 4.3.12.

(c) Let v be an eigenfunction with eigenvalue λ. That is $-v_{xx} = \lambda v$ and $v_x(0) = v_x(l) = \frac{v(l)-v(0)}{l}$. Applying Green's first identity (see Exercise 5.3.12), we have

$$-\int_0^l v_{xx} v\,dx = \int_0^l v_x^2\,dx - v_x v\Big|_0^l$$

Using the properties of v, this becomes

$$\lambda\int_0^l v^2\,dx = \int_0^l v_x^2\,dx - [v(l) - v(0)]^2/l.$$

By Exercise 5.5.3, the right hand side is nonnegative, from which it follows that $\lambda \geq 0$.

(d) Note that
$$\phi(x) = A + Bx + (\text{orthogonal terms}),$$
where the orthogonal terms are orthogonal to both 1 and x on $(0, l)$. Let
$$\alpha = \int_0^l \phi(x)\,dx, \qquad \text{and} \qquad \beta = \int_0^l x\phi(x)\,dx.$$
Then
$$\alpha = \int_0^l A + Bx\,dx = Al + \frac{1}{2}Bl^2$$
and
$$\beta = \int_0^l x(A + Bx)\,dx = \frac{1}{2}Al^2 + \frac{1}{3}Bl^3.$$
Then we can solve these two equations for A and B. We get
$$A = \frac{4l\alpha - 6\beta}{l^2} \qquad \text{and} \qquad B = \frac{12\beta - 6l\alpha}{l^3}.$$

5.5.6 The solution of the diffusion equation is given by the series
$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt/l^2}$$
where
$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx.$$
Since ϕ is continuous on $[0, l]$, $|\phi(x)|$ is bounded by some constant M on $[0, l]$. Thus
$$|A_n| \le \frac{2}{l}\int_0^l M\,dx = 2M$$
for all n. Now let
$$f_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt/l^2}.$$
Then
$$\frac{\partial f_n}{\partial x} = A_n \frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt/l^2}.$$
Note that
$$\frac{n\pi}{l} e^{-n^2\pi^2 kt/l^2} \le Ce^{-n\pi^2 kt/l^2} = C\left[e^{-\pi^2 kt/l^2}\right]^n$$
for some constant C. Fix $t > 0$ and let $r = e^{-\pi^2 kt/l^2}$. Then $0 < r < 1$ and we have
$$\left|\frac{\partial f_n}{\partial x}\right| \le |A_n|\, e^{-n^2\pi^2 kt/l^2} \le 2MC(e^{-\pi^2 kt/l^2})^n = 2MCr^n$$

#12: If f is C^1 in $[-\pi, \pi]$ that satisfies periodic BC's and if $\int_{-\pi}^{\pi} f(x)\,dx = 0$, then $\int_{-\pi}^{\pi} |f|^2 dx \leq \int_{-\pi}^{\pi} |f'|^2 dx$

Pf: Since f is C^1, f and f' are continuous on $[-\pi, \pi]$, hence $\|f\|$ and the $\|f'\|$ are finite. By Thm 3 of 5.4, the fourier series for each function converges to the function in the L^2 sense. By Thm 6, we get Parseval's Equality for each function:

Let $\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$ and

$\frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n \cos nx + D_n \sin nx$ be

the Fourier series of f and f', respectively.

By Parsevals Equality we have

$$\cancel{|A_0|^2 \cdot 2\pi + \sum_{n=1}^{\infty} |A_n|^2}$$

$$|A_0|^2 \cdot \int_{-\pi}^{\pi} |1|^2 dx + \sum_{n=1}^{\infty} |A_n|^2 \int_{-\pi}^{\pi} |\cos nx|^2 dx$$
$$+ \sum_{n=1}^{\infty} |B_n|^2 \int_{-\pi}^{\pi} |\sin nx|^2 dx = \int_{-\pi}^{\pi} |f|^2 dx$$

and
$$|C_0|^2 \cdot \int_{-\pi}^{\pi} |1|^2 dx + \sum_{n=1}^{\infty} |C_n|^2 \int_{-\pi}^{\pi} |\cos nx|^2 dx$$
$$+ \sum_{n=1}^{\infty} |D_n|^2 \int_{-\pi}^{\pi} |\sin nx|^2 dx = \int_{-\pi}^{\pi} |f'|^2 dx.$$

We have $\int_{-\pi}^{\pi} |1|^2 dx = 2\pi$ and

$$\int_{-\pi}^{\pi} \cos^2 nx\,dx = \int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi,$$

so it suffices to prove that

$$|A_0|^2 \cdot 2\pi + \pi \sum_{n=1}^{\infty} |A_n|^2 + |B_n|^2 \leq |C_0|^2 \cdot 2\pi + \pi \sum_{n=1}^{\infty} |C_n|^2 + |D_n|^2$$

Note that $A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$, so

$0 = |A_0|^2 \cdot 2\pi \leq |C_0|^2 \cdot 2\pi$. For $n > 0$

we have

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi}\left[f(x) \cdot \frac{-1}{n} \cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} f'(x) \cos nx\,dx \right]$$

$$= \frac{1}{n} \cdot \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \cos nx\,dx = \frac{1}{n} C_n,$$

so $|C_n|^2 = n^2 |B_n|^2$. A similar calculation shows that $|D_n|^2 = n^2 |A_n|^2$.

To summarize, we have $|A_0|^2 = 0 \leq |C_0|^2$, $|B_n|^2 \leq |C_n|^2$, and $|A_n|^2 \leq |D_n|^2$,

so we certainly must have

$$|A_0|^2 \cdot 2\pi + \pi \sum_{n=1}^{\infty} |A_n|^2 + |B_n|^2 \leq |C_0|^2 \cdot 2\pi + \pi \sum_{n=1}^{\infty} |C_n|^2 + |D_n|^2,$$

as desired.

#13: Let f be C^1 (continuous on the whole real line) with period 2π. The goal is to show that the Fourier series of f converges to f pointwise.

a: In this part, we justify two simplifying assumptions on f.

① If $f(0) \neq 0$, let $g(x) = f(x) - f(0)$. Let C_n, D_n be the Fourier coefficients for f, g.

Then $D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-inx}\,dx = C_n - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(0) e^{-inx}\,dx$

$$= \begin{cases} C_n & n \neq 0 \\ C_0 - f(0) & n = 0 \end{cases}.$$

If it is known that the Fourier series of g converges to ~~zero at~~ $g(0) = 0$ at $x=0$, then we have

$$0 = g(0) = \sum_n D_n e^{in(0)} = \left(\sum_n C_n e^{in(0)}\right) - f(0)$$

so that $\sum_n C_n e^{in(0)} = f(0)$, i.e. the Fourier series of f converges to $f(0)$ at $x=0$. So we may as well assume that $f(0) = 0$.

② Let a be fixed and set $g(x) = f(x+a)$. Then, with notation as above,

$$D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+a) e^{-inx}\,dx$$

$$= e^{ina} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+a)\, e^{-in(x+a)}\, dx$$

$$= e^{ina} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad \left(\text{since } f(x)e^{-inx} \text{ is periodic}\right)$$

$$= e^{ina} \cdot C_n .$$

If it is known that the Fourier series of g converges to $g(0)$ at $x = 0$ then

$$f(a) = g(0) = \sum_n D_n e^{in(0)} = \sum_n C_n e^{ina},$$

i.e. the Fourier series of f converges to $f(a)$ at $x = a$. So it suffices to show convergence at $x = 0$.

Note: In light of ① + ②, from now on we assume that $f(0) = 0$, and we need only show that the Fourier series of f converges pointwise to zero. at $x = 0$.

b: Let $g(x) = \dfrac{f(x)}{e^{ix}-1}$. We want to show that g is continuous. This is clear for points where $e^{ix} - 1 \neq 0$, i.e. for $x \neq 2\pi n$ for any integer n. Since g is periodic, we only need to show continuity at $x = 0$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{e^{ix}-1} = \lim_{x\to 0} \frac{f(x)-0}{x-0} \cdot \frac{x-0}{e^{ix}-1}$$

$$= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \cdot \lim_{x\to 0} \frac{x}{e^{ix}-1},$$

provided these limits exist. The first limit is just $f'(0)$. The second can be computed directly or, assuming L'hopital's rule is valid: $\lim_{x\to 0} \frac{x}{e^{ix}-1} = \lim_{x\to 0} \frac{1}{ie^{ix}} = \frac{1}{i}$.

(Technically we have not defined g at $x=0$, but since $\lim_{x\to 0} g(x)$ exists, we simply define $g(0) := \lim_{x\to 0} g(x)$, so that g is defined and continuous everywhere).

c: Since g is continuous,

$\int_{-\ell}^{\ell} |g|^2$ is finite, so we have Bessel's inequality: $\sum_n D_n^2 \cdot \int_{-\pi}^{\pi} |e^{inx}|^2 dx$

$$= \sum_n D_n^2 \cdot 2\pi \leq \int_{-\pi}^{\pi} |g|^2 dx < \infty.$$

Since this sum converges, we must have $D_n \to 0$ as $|n| \to \infty$.

d: ~~and~~ Since $f(x) = g(x)(e^{ix}-1)$, we have

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{ix}-1) g(x) e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-i(n-1)x}\, dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-inx}\, dx$$

$= D_{n-1} - D_n$. So the series $\sum c_n$ is telescoping: e.g

$$\cancel{\sum_{-n}^{n} c_n = c_{-n} + c_{-(n-1)} + \dots + c_{n-1} + c_n}$$

$$\sum_{k=m}^{n} c_k = c_m + c_{m+1} + \dots + c_{n-1} + c_n$$

$$= (D_{m-1} - D_m) + (D_m - D_{m+1}) + \dots + (D_{n-2} - D_{n-1}) + (D_{n-1} - D_n)$$

$$= D_{m-1} - D_m.$$

e: The Fourier series of f at $x = 0$ is $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{-in(0)} = \sum_{n=-\infty}^{\infty} c_n = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n$

By part d, this partial sum is just $D_{-N-1} - D_N$. By part c, D_{-N-1} and $D_N \to 0$ as $N \to \infty$, so this limit converges to zero, i.e. $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{-in(0)} = 0$.

#14: Consider the problem $u_t = k u_{xx}$ on $(0, l)$ with BC's $u_x(0,t) = u_x(l,t) = 0$ and $u(x,0) = \phi(x)$ is continuous with piecewise continuous derivative. According to section 4.2, the solution is $u(x,t) = \frac{1}{2}A_0 + \displaystyle\sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{l^2}kt} \cos\frac{n\pi x}{l}$.

where the A_n's are the Fourier cosine coefficients of ϕ, i.e.

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.$$

We will prove that this really is the solution.

Let ϕ_{ext} be the $2l$-periodic even extension of ϕ. Then the solution to $u_t = k u_{xx}$ on the whole real line with initial condition $u(x,0) = \phi_{ext}(x)$ is

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi_{ext}(y)\, dy.$$

Since $\phi_{ext} = \phi$ on $(0,l)$, this solution satisfies the correct initial condition. It also satisfies the right boundry conditions:

$$u_x(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \frac{-2(x-y)}{4kt} e^{-\frac{(x-y)^2}{4kt}} \phi_{ext}(y)\, dy$$

$$\Rightarrow u_x(0,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \frac{2y}{4kt} e^{-\frac{y^2}{4kt}} \phi_{ext}(y)\, dy.$$

Here the integrand is odd, so $u_x(0,t) = 0$.

Likewise,

$$u_x(l,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \frac{-2(l-y)}{4kt} e^{-\frac{(l-y)^2}{4kt}} \phi_{ext}(y)\, dy.$$

Set $z = y - l$ and set $\psi(z) = \phi_{ext}(z + l)$.

Then
$$u_x(l,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \frac{2z}{4kt} e^{-\frac{z^2}{4kt}} \psi(z)\, dz.$$

Since $\psi(z)$ is even, we get $u_x(l,t) = 0$ as above.

So the restriction of u to the interval $(0, \ell)$ is the solution to the boundry value problem. We show that u is equal to the Fourier series given above. Since ϕ_{ext} is just the even periodic extension of ϕ, these functions have the same Fourier series. So

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \left(\tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi y}{\ell} \right) dy$$

$$= \frac{\frac{1}{2}A_0}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy + \sum_{n=1}^{\infty} A_n \cdot \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cos \tfrac{n\pi y}{\ell} \, dy$$

Claim: $\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy = 1$, and

$$\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cos \tfrac{n\pi y}{\ell} \, dy = e^{-\frac{n^2\pi^2}{\ell^2}kt} \cos \tfrac{n\pi x}{\ell}.$$

From this claim it follows that

$$u(x,t) = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2}{\ell^2}kt} \cos \tfrac{n\pi x}{\ell},$$

as desired. The first part of the claim is proved by making the substitution $p = \frac{y-x}{\sqrt{4kt}}$ and using the fact that $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$.

For the second part we have

$$\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cos \tfrac{n\pi y}{\ell} \, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \left(\frac{e^{i\frac{n\pi y}{\ell}} + e^{-i\frac{n\pi y}{\ell}}}{2} \right) dy.$$

Distributing the sum, the first integral is

$\frac{1}{2\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt} + \frac{in\pi y}{\ell}} dy$. We complete the square in the exponent: (details left to the reader).

$$-\frac{(x-y)^2}{4kt} + \frac{in\pi y}{\ell} = -\frac{\left(y - x - \frac{in\pi(2kt)}{\ell}\right)^2}{4kt} - \frac{n^2\pi^2}{\ell^2}kt - \frac{in\pi x}{\ell}.$$

Letting $p = \frac{y - x - \frac{in\pi(2kt)}{\ell}}{\sqrt{4kt}}$, the integral becomes $\frac{1}{2\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-p^2 - \frac{n^2\pi^2}{\ell^2}kt - \frac{in\pi x}{\ell}} \cdot \sqrt{4kt}\, dp$

$$= e^{-\frac{n^2\pi^2}{\ell^2}kt} \cdot \frac{e^{-\frac{in\pi x}{\ell}}}{2} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp$$

$$= e^{-\frac{n^2\pi^2}{\ell^2}kt} \cdot \frac{e^{-\frac{in\pi x}{\ell}}}{2}.$$

Likewise, the second integral is equal to $e^{-\frac{n^2\pi^2}{\ell^2}kt} \cdot \frac{e^{\frac{in\pi x}{\ell}}}{2}$, so the sum is just

$$e^{-\frac{n^2\pi^2}{\ell^2}kt}\left(\frac{e^{\frac{in\pi x}{\ell}} + e^{-\frac{in\pi x}{\ell}}}{2}\right) = e^{-\frac{n^2\pi^2}{\ell^2}kt} \cos\frac{n\pi x}{\ell}$$

as claimed.