HW #8 Solutions

Sec. 6.1 #1, 10-13

#1: Suppose $f(z) = f(x+iy) = \sum_{n=1}^{\infty} a_n (x+iy)^n$

Write f in its real and imaginary components:

$$u(x,y) + i v(x,y) = \sum_{n=0}^{\infty} a_n (x+iy)^n$$

Differentiating with respect to x gives

$$(1) \quad u_x + i v_x = \sum_{n=1} n \cdot a_n (x+iy)^{n-1}$$

Differentiating with respect to y gives

$$(2) \quad u_y + i v_y = i \sum_{n=1}^{\infty} n \cdot a_n (x+iy)^{n-1}$$

Multiply eqn (2) by $-i$ and equate the result with eqn (1):

$$-i u_y + v_y = \sum_{n=1}^{\infty} n \cdot a_n (x+iy)^{n-1} = u_x + i v_x$$

Now compare real and imaginary parts to obtain the Cauchy-Riemann eqns:

$$u_x = v_y, \quad u_y = -v_x$$

Differentiating again we obtain Laplace's Eqn. for both u and v. For example,

$$v_{yy} = (u_x)_y = (u_y)_x = -v_{xx}, \text{ so}$$

$v_{xx} + v_{yy} = 0$. Likewise, $u_{xx} + u_{yy} = 0$.

Hence $\Delta f = \Delta u + i\Delta v = 0 + 0$

#10: Suppose we have two solutions u and v to the given Dirichlet problem, i.e. $\Delta u = f = \Delta v$ in D and $u = g = v$ on ∂D. ~~Assu~~ Let $w = u - v$. Then $\Delta w = \Delta u - \Delta v = f - f = 0$ in D, which is Laplace's eqn., and $w = g - g = 0$ on ∂D. Multiplying by w, we have $w\Delta w = 0$ on D. The plan is to integrate this equation to show that $w = 0$. Note that

$$\begin{aligned}\nabla \cdot (w\nabla w) &= \nabla w \cdot \nabla w + w(\nabla \cdot \nabla w)\\ &= (w_x^2 + w_y^2 + w_z^2) + w\Delta w,\end{aligned}$$

so $w\Delta w = \nabla \cdot (w\nabla w) - (w_x^2 + w_y^2 + w_z^2)$.

Now we integrate:

$$\begin{aligned}0 = \iiint_D 0\, dx\,dy\,dz &= \iiint_D w\Delta w\, dx\,dy\,dz\\ &= \iiint_D [\nabla \cdot (w\nabla w) - (w_x^2 + w_y^2 + w_z^2)]\, dx\,dy\,dz\end{aligned}$$

We can integrate the first term using the Divergence Thm: $\iiint_D \nabla \cdot (w\nabla w)\, dx\,dy\,dz$ $= \iint_{\partial D} (w\nabla w) \cdot \vec{n}\, dS$. But $w = 0$ on ∂D, so $\iiint_D \nabla \cdot (w \nabla w)\, dx\,dy\,dz$

$= \iint_{\partial D} 0 \, dS = 0$. So we are left with $\iiint_D [w_x^2 + w_y^2 + w_z^2]\, dx\,dy\,dz = 0$

Since each of w_x^2, w_y^2, + w_z^2 is nonnegative and continuous, this implies that $w_x = w_y = w_z = 0$. Hence w = constant.

But we know $w = 0$ on ∂D, so $w = 0$ $\Rightarrow$ $u - v = 0$, i.e. $u = v$. Therefore the solution to the Dirichlet problem is unique.

<u>#11</u>: Suppose u satisfies $\Delta u = f$ in D a subset of $\mathbb{R}^3$, and $\frac{\partial u}{\partial n} = g$ on ∂D. We must show that $\iiint_D f\, dx\,dy\,dz = \iint_{\partial D} g\, dS$. Recall that the directional derivative $\frac{\partial u}{\partial n}$ of u in the direction of the normal vector n is given by the dot product of the gradient of u with n: $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$. We have

$$\iiint_D f\, dx\,dy\,dz = \iiint_D \Delta u\, dx\,dy\,dz$$

$$= \iiint_D \nabla \cdot \nabla u\, dx\,dy\,dz = \iint_{\partial D} \nabla u \cdot n\, dS$$

$$= \iint_{\partial D} \frac{\partial u}{\partial n}\, dS = \iint_{\partial D} g\, dS,$$

where we have used the Divergence Thm. The two dimensional case is almost identical except we use Green's Thm instead of the Divergence Thm (see A.3 in Appendix). We have

$$\Delta u = f(x,y) \text{ in } D \quad \text{and} \quad \frac{\partial u}{\partial n} = g \text{ on } \partial D$$

where D is a subset of $\mathbb{R}^2$. We must show that $\iint_D f\, dx\, dy = \int_C g\, ds$ where C is the curve that travels around the boundry of D and the right hand side is a path integral. We have

$$\iint_D f\, dx\, dy = \iint_D \Delta u\, dx\, dy = \iint_D \nabla \cdot \nabla u\, dx\, dy$$

$$\overset{\text{Green's Thm}}{=} \int_C \nabla u \cdot n\, ds = \int_C \frac{\partial u}{\partial n}\, ds = \int_C g\, ds.$$

For the one-dimensional case D is just an interval (a,b) and ∂D is just the two points a and b. So specifying boundry conditions means we specify $\frac{\partial u}{\partial n}$ at a and b. At the point b, the outward normal n just points in the positive direction, so $\frac{\partial u}{\partial n}(b) = u'(b)$. At a, n points in the negative x direction, so $\frac{\partial u}{\partial n}(a) = -u'(a)$. So the

problem is $u_{xx} = f(x)$ on (a,b) and $-u'(a) = y_1$, $u'(b) = y_2$, where y_1, y_2 are arbitrary constants, and we must show that $\int_a^b f(x)\,dx = y_1 + y_2$. We have

$$\int_a^b f(x)\,dx = \int_a^b u_{xx}(x)\,dx = u_x \Big|_a^b$$
$$= u'(b) - u'(a) = y_1 + y_2 .$$

#12: Let $u(x,y) = \dfrac{1-(x^2+y^2)}{(1-x)^2+y^2}$, $D = \{x^2+y^2<1\}$

Note that u is continuous except where $(1-x)^2 + y^2 = 0$, i.e. except at the point $(1,0)$. If (x,y) is a boundry point then $x^2+y^2 = 1$; for $(x,y) \neq (1,0)$ we have $u(x,y) = 0$. So $u = 0$ on the boundry except at $(1,0)$. At an interior point, $x^2+y^2 < 1$ so that $u(x,y) > 0$. So u does not take its max on the boundry, and the maximum principle is not valid. On the other hand if we allow u to take on the value ∞, we can define

$u(1,0) = \infty$. In this sense the max principle does hold since u is finite on D but infinite at $(1,0)$.

#13: Let $u(\vec{x})$ be subharmonic, i.e. $\Delta u \geq 0$ on D. We will prove that u takes its maximum on ∂D. One way is just to follow the proof for harmonic functions given on page 155. In fact, we must only make one change: Let $\varepsilon > 0$ and set

$v(\vec{x}) = u(\vec{x}) + \varepsilon |\vec{x}|^2$. Then (in two dimensions)

$\Delta v = \Delta u + \varepsilon \Delta(x^2+y^2) \geq 0 + 4\varepsilon > 0$ in D.

The rest of the proof is identical.

Note that the minimum principle does not hold for subharmonic functions. For example let $u(x,y) = x^2+y^2$ on $\bar{D} = \{x^2+y^2 \leq 1\}$.

Then $\Delta u = 2+2 > 0$ so u is subharmonic, but u takes its minimum at $(0,0)$: $u(0,0) = 0$, whereas $u = 1$ on the boundry.