HW #9 Solutions

6.1 #s 2, 3, 5, 9 , 6.2 #s 1, 2, 4, 7

6.3 #s 1, 2, 4

Sec 6.1 **#2**: Using the Laplace operator in spherical ~~cylindrical~~ coordinates, u must satisfy $u''(r) + \frac{2}{r}u'(r) = k^2 u(r) \Rightarrow r u''(r) + 2u'(r) = k^2 u(r) r$. Suppose u satisfies this equation and set $v = u \cdot r$. The $\frac{d}{dr} v = u' \cdot r + u$, and $\frac{d^2}{dr^2} v = u'' \cdot r + 2u' = k^2 u \cdot r = k^2 v$ using the equation for u. Solving $v'' - k^2 v = 0$ we find $v = Ae^{kr} + Be^{-kr}$, so that $u = \frac{v}{r} = \frac{Ae^{kr} + Be^{-kr}}{r}$.

#3: ~~Assume~~ We look for functions $u = u(r)$ that depend on r only. Using the polar coordinate form of Δ, we must solve $u_{rr} + \frac{1}{r}u_r = k^2 u$. Bessel's differential equation is

$$\frac{d^2}{dz^2}f(z) + \frac{1}{z}\frac{d}{dz}f(z) + \left(1 - \frac{s^2}{z^2}\right)f(z) = 0,$$

where s is a constant. Its solutions are the Bessel functions. Consider the equation with $s = 0$:

$$\frac{d^2}{dz^2}f(z) + \frac{1}{z}\frac{d}{dz}f(z) = -f(z). \qquad (*)$$

Suppose $f(z)$ is a solution of $(*)$ and set $u(r) = f(ikr)$. Then $\frac{d}{dr} u(r) = \frac{d}{dr} f(ikr) = f'(ikr) \cdot ik$, and

$$\frac{d^2}{dr^2} u(r) = f''(ikr)(ik)^2 = -k^2 f''(ikr).$$

So

$$\begin{aligned} u_{rr} + \frac{1}{r} u_r &= -k^2 f''(ikr) + \frac{ik}{r} f'(ikr) \\ &= -k^2 \left(f''(ikr) + \frac{1}{ikr} f'(ikr) \right) \\ &\overset{\text{by } (*)}{=} -k^2 \left(-f(ikr) \right) = k^2 u(r), \end{aligned}$$

So $u(r) = f(ikr)$ is a solution to the original problem $\Delta u = k^2 u$. According to section 10.5, the functions f that satisfy $(*)$ are just linear combinations of two Bessel functions I_0 and J_0. So we have solutions $u(r) = A I_0(ikr) + B J_0(ikr)$.

#5: Let's look for solutions u that depend only on r; the PDE becomes an ODE: $u_{rr} + \frac{1}{r} u_r = 1 \iff r u_{rr} + u_r = r$

i.e. $\frac{d}{dr}(r u_r) = r$. Integrating, we have $r u_r = \frac{1}{2} r^2 + c_1 \Rightarrow u_r = \frac{1}{2} r + \frac{c_1}{r}$.

$\Rightarrow u = \frac{1}{4} r^2 + c_1 \log r + c_2$. Since u must be defined for $r = 0$, we take $c_1 = 0$. The boundry condition gives

$0 = u(a) = \frac{a^2}{4} + c_2$, so

$u = \frac{r^2}{4} - \frac{a^2}{4}$.

#9 : a: Assuming the temperature T is a function of r only we have

$\Delta T = T_{rr} + \frac{2}{r} T_r = 0$; this is an ODE, in fact we have $\frac{d}{dr}(r^2 T_r) = 2r T_r + r^2 T_{rr} = r^2 (T_{rr} + \frac{2}{r} T_r) = 0$. Integrating gives $r^2 T_r = C_1 \Rightarrow T_r = C_1 r^{-2} \Rightarrow T = -C_1 r^{-1} + C_2$. Now we apply the boundry conditions:

$100 = T(1) = -C_1 + C_2 \Rightarrow C_2 = C_1 + 100$, and

$-\gamma = T_r(2) = C_1 2^{-2} = \frac{C_1}{4}$.

So $C_1 = -4\gamma$, $C_2 = -4\gamma + 100$, and

$T(r) = 4\gamma r^{-1} - 4\gamma + 100$.

b: The temperature decreases as r increases, so it's hottest at $r = 1$, w/ $T(1) = 100$, and coldest at $r = 2$, w/ $T(2) = 100 - 2\gamma$.

c: $\gamma = 40$

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Sec 6.2

#1 We take the hint and guess that the solution has form $u = Ax^2 + By^2 + Cxy + Dx + Ey + F$. Then $\Delta u = 2A + 2B$, so Laplace's Eqn. implies that $B = -A$. We apply the BC's: $u_x = 2Ax + Cy + D$ and $u_y = 2By + Cx + E = -2Ay + Cx + E$.

① $-a = u_x(0,y) = Cy + D \Rightarrow C = 0$ and $D = -a$.

② $0 = u_x(a,y) = 2Aa + Cy + D$
$= 2Aa - a = a(2A - 1)$

$\Rightarrow A = \frac{1}{2}$

③ $b = u_y(x,0) = Cx + E \Rightarrow C = 0, \ E = b$

④ $0 = u_y(x,b) = 2Bb + Cx + E$
$= 2Bb + b = b(2B + 1)$

$\Rightarrow B = -\frac{1}{2}$, which agrees with our computation $B = -A$ above.

So $u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + F$, where F is arbitrary.

#2: We show that $\{\sin my \sin nz\}$ are orthogonal on $\{0<y<\pi,\ 0<z<\pi\}$, by integrating the product of two such functions over the square:

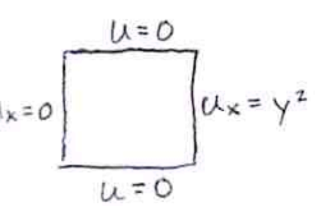
$$\int_0^\pi\int_0^\pi (\sin m_1 y \sin n_1 z)(\sin m_2 y \sin n_2 z)\,dy\,dz$$

$$= \left(\int_0^\pi \sin m_1 y \sin m_2 y\,dy\right)\left(\int_0^\pi \sin n_1 z \sin n_2 z\,dz\right)$$

$= 0$ unless $m_1 = m_2$ and $n_1 = n_2$ because the functions $\{\sin kx\}$ are orthogonal on $0<x<\pi$.

#4: There are two inhomogeneous boundry conditions; our strategy is to split the problem into two subproblems, each of which involves only one inhomogeneous boundry condition:

① Solve $\Delta u = 0$ on $D = \{0<x<1,\ 0<y<1\}$.



$u(x,0) = u(x,1) = u_x(0,y) = 0$, $u_x(1,y) = y^2$.

We separate variables: $u = X(x)\,Y(y)$

The PDE gives $\dfrac{X''}{X} = \dfrac{-Y''}{Y} = \lambda$.

First we solve $Y'' = -\lambda Y$, the BCs give $0 = Y(0) = Y(1)$. So $\lambda = (n\pi)^2$, $n>0$ with corresponding eigenfunctions $\cancel{X_n(x) = \sin n\pi x}$.

$Y_n(y) = \sin n\pi y$.

Next we solve $X'' = \lambda X$ with the single boundry condition $X'(0) = 0$. We have

$\Xi_n(x) = A\cosh n\pi x + B\sinh n\pi x$, and

$0 = \Xi'(0) = Bn\pi\cosh(0) = Bn\pi$

$\Rightarrow B = 0$, so $\Xi_n(x) = A\cosh n\pi x$. So

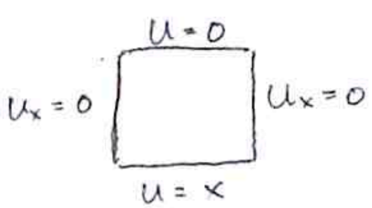
$u(x,y) = \sum_{n=1}^{\infty} A_n\cosh n\pi x\cdot\sin n\pi y$ satisfies the PDE and the three homogeneous BC's.

For the remaining side we require that

$$y^2 = u_x(1,y) = \sum_{n=1}^{\infty}(A_n\cdot n\pi\cdot\sinh n\pi)\sin n\pi y$$

Let $y^2 = \sum_{n=1}^{\infty} C_n\sin n\pi y$ be the Fourier sine series for y^2. Choosing $A_n = \frac{C_n}{n\pi\cdot\sinh n\pi} = \frac{1}{n\pi\cdot\sinh n\pi}\cdot\frac{2}{1}\int_0^1 y^2\sin n\pi y\,dy$, u has the required condition on the right side.

(2) Solve $\Delta u = 0$ on D with



$u_x(0,y) = u_x(1,y) = u(x,1) = 0$, $u(x,0) = x$.

Again, separate variables and obtain $-\frac{\Xi''}{\Xi} = \frac{Y''}{Y} = \lambda$.

First we solve $\Xi'' = -\lambda\Xi$; the BC's give $\Xi'(0) = \Xi'(1)$. These are Neumann conditions, so $\lambda = (n\pi)^2$ $n \geq 0$ with eigenfunctions $\Xi_0 = \text{constant}$, $\Xi_n = \cos n\pi x$ for $n > 0$.

Now we solve $Y'' = \lambda Y$ with the single boundary condition $Y(1) = 0$.

For $\lambda = 0$ we have $Y'' = 0 \Rightarrow$
$Y(y) = Ay + B$, and $0 = Y(1) = A + B$,
hence $B = -A$ and $Y_0(y) = A(y-1)$. For $n > 0$
$Y'' = \lambda Y$ has solutions $Y = A\cosh n\pi y + B\sinh n\pi y$
Then $0 = Y(1) = A\cosh n\pi + B\sinh n\pi$, so
$B = -A \operatorname{cotanh} n\pi$ and
$Y_n(y) = A(\cosh n\pi y - \operatorname{cotanh} n\pi y \cdot \sinh n\pi y)$

So $u(x,y) = A_0(y-1) + \sum_{n=1}^{\infty} A_n[\cosh n\pi y - \operatorname{cotanh} n\pi y \, \sinh n\pi y] \cdot \cos n\pi x$

satisfies the PDE and the three homogeneous BC's. For the remaining BC, we require that

$$x = u(x,0) = -A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x.$$

Writing the Fourier cosine series for x:

$x = \frac{1}{2}C_0 + \sum_{n=1}^{\infty} C_n \cos n\pi x$, we see that

we must have $A_0 = -\frac{1}{2}C_0$ and $A_n = C_n$ for $n > 0$, where $C_n = \frac{2}{1}\int_0^1 x \cos n\pi x \, dx$.

③ Let u_1 be the solution found in ①, and u_2 the solution found in ②. Let $u = u_1 + u_2$. Since u_1, u_2 harmonic, so is u. Moreover, 4

We get $Y(y) = A\cosh n\pi y + \cancel{B\sinh}$ $B\sinh n\pi y$. The boundry condition gives $0 = Y(1) = A\cosh n\pi + B\sinh n\pi$ so $B = -A\operatorname{cotanh} n\pi$, and

$$Y(y) = A(\cosh n\pi y - \operatorname{cotanh} n\pi \cdot \sinh n\pi y)$$

for $n > 0$.

~~So~~ $\cancel{u(x,y) = \sum_{n=1}^{\infty} \cos n\pi x}$

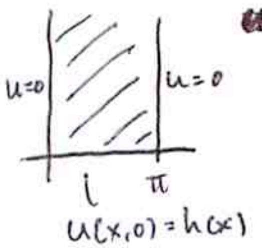
So $u(x,y) = \sum_{n=1}^{\infty} A_n[\cosh n\pi y - \operatorname{cotanh} n\pi \cdot \sinh n\pi y]\cos n\pi x$ satisfies the PDE and the three homogeneous boundry conditions. For the remaining BC, we require

$$x = u(x,0) = \sum_{n=1}^{\infty} A_n \cos$$

For $n = 0$

the sum satisfies the two homogeneous and two inhomogeneous BC's that we are looking for, so u is our solution.

#7: a: We solve $\Delta u = 0$ on the strip $\{0 < x < \pi,\ y > 0\}$ with boundry conditions $u = 0$ for $x = 0$ and $x = 1$, and $u(x,0) = h(x)$, and $\lim_{y\to\infty} u(x,y) = 0$.



Separating variables, the PDE gives

$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda$. The BC's imply that

$X(0) = X(\pi) = 0$. The eigenvalue problem for X has solutions $\lambda = n^2$ for $n \geq 1$ w/ eigenfunctions $X_n(x) = \sin nx$. For Y we solve $Y'' = \lambda Y$. The solutions have form $Y_n(y) = Ae^{ny} + Be^{-ny}$. The condition at ∞ implies that $Y_n \to 0$ as $y \to \infty$ so we must have $A = 0$. Then the function

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-ny} \cdot \sin nx$$

satisfies $\Delta u = 0$ and three of the four BC's. For the fourth BC we must have

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx,$$

so the B_n's must be the Fourier sine coefficients for $h(x)$: $B_n = \frac{2}{\pi}\int_0^{\pi} h(x) \sin nx \, dx$.

b: If we omit the condition at ∞, the problem is solved in the same way except we no longer throw out the term e^{ny} in the expressio for Y_n. Thus,

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{ny} + B_n e^{-ny}\right) \sin nx$$

satisfies the PDE and the homogeneous BC's on the left + right sides. For the last BC, we ~~regr~~ require that

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} (A_n + B_n) \sin nx,$$

so that $A_n + B_n$ must be the n^{th} Fourier sine coefficient for h. The point here is that there are many ways to choose A_n + B_n so that this is the case, therefore the solution u is not unique. This does not violate the proof for uniqueness in Sec. 6.1 since that proof assumes that the region D is bounded.

HW #7 Solutions

Section 5.5 #s 2, 4, 12, 13, 14.

#2: We prove the Schwar~~tz~~z in-

$\boxed{\text{Sec } 6.3}$:

#1: a: By the Maximum Principle,

$$\max_{\text{on } \bar{D}} u = \max_{\text{on } \partial D} u = \max(3\sin 2\theta + 1) = 4.$$

b: By the mean value property,

$u(0,0)$ = average value of u on the circle of radius 2 $= \frac{1}{2\pi}\int_0^{2\pi}(3\sin 2\theta + 1)\,d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} 3\sin 2\theta\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta$$

$$= 0 + \underline{1}$$

#2: According to Sec. 6.3, $\Delta u = 0$ on the disk has solution

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta).$$

For the boundry condition, we require that

$$1 + 3\sin\theta = u(a,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n A_n\cos n\theta + a^n B_n\sin n\theta$$

Comparing coefficients, we see that $A_0 = 2$, ~~A Bn~~ $B_1 = \frac{3}{a}$, and all other coefficients are zero.

#4: We must show that

$P(r,\theta) = \dfrac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2}$ is harmonic. In polar coordinates, the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$ so we must compute P_{rr}, $\frac{1}{r} P_r$, and $\frac{1}{r^2} P_{\theta\theta}$ and show that they sum to zero. Good luck!