Name: **110 MIDTERM 1**

1. Let $-xu_x + u_t = 0$ and define $I(t) = \int_{-\infty}^{\infty} u(x,t)\mathrm{d}x$.

   a. [20 points] Find $u(x,t)$ solving this PDE, subject to the initial condition $u(x,0) = e^{-x^2}$.

   $0 = -xu_x + u_t = (-x, 1) \cdot \nabla u$, so the characteristics satisfy

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = -x \Longrightarrow x = Ce^{-t}.$$

   This means that $u(x,t) = f(xe^t)$. The initial condition tells us that

   $$u(x,0) = e^{-x^2} = f(x) \Longrightarrow u(x,t) = e^{-(xe^t)^2}.$$

   b. [20 points] Show that for any initial condition like this one, namely $u(x,0) = \phi(x) \geq 0$ and $\lim_{|x|\to\infty} x\phi(x) = 0$, $I(t)$ is decreasing, *i.e.*, $\frac{\mathrm{d}I}{\mathrm{d}t} \leq 0$.

   $$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{-\infty}^{\infty} u_t \mathrm{d}x = \int_{-\infty}^{\infty} xu_x \mathrm{d}x = xu\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u\mathrm{d}x \leq 0,$$

   where the first equality is true assuming uniform convergence of the integrals, the second equality is obtained by applying the PDE, and the third equality follows from integration by parts. Since u is constant along the characteristics, if $U(x,0) = \phi(x) \geq 0$, then $u(x,t) \geq 0$ for all x and t, so the last integral is non-negative. Also, since $x\phi(x) \to 0$ as $|x| \to \infty$, $x\phi(C) = xf(C) = xu(x,t) \to 0$ as $|x| \to \infty$, so the boundary term vanishes. The last two statements justify the final inequality.

	score
1	
2	
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2. [30 points] Consider the wave equation $u_{tt} = c^2 u_{xx}$, for $0 \leq x \leq \pi$, subject to the boundary conditions $u(0,t) = 0 = u(\pi,t)$. Notice that $u(x,t) = f(t)\sin kx$ satisfies these boundary conditions if k is an integer. In this case, find the most general $f(t)$ such that $u(x,t)$ is also a solution to the wave equation.

   $u_{tt} = f''(t)\sin kx$ and $u_{xx} = -k^2 f(t)\sin kx$; plugging these into the wave equation gives $f'' + c^2k^2 f = 0$. The general solution to this second order constant coefficient ODE is $f(t) = A\cos ckt + B\sin ckt$.

3. [30 points] Suppose that u_1 and u_2 both solve the diffusion equation $u_t = ku_{xx}$, with $k > 0$, on the interval $0 \leq x \leq \ell$, and suppose $u_1(x,t) \geq u_2(x,t)$ if $t = 0$ or $x = 0$ or $x = \ell$. Prove that $u_1(x,t) \geq u_2(x,t)$ for all $0 \leq x \leq \ell$ and $0 \leq t < \infty$.

   Let $w(x,t) = u_1(x,t) - u_2(x,t)$. Then w satisfies the diffusion equation since it is a linear combination of solutions, and $w(x,t) \geq 0$ if $t = 0$ or $x = 0$ or $x = \ell$. By the Minimum Principle, this means that $w(x,t) \geq 0$ for all $0 \leq x \leq \ell$ and $0 \leq t < \infty$. Thus in this domain, $u_1(x,t) - u_2(x,t) \geq 0$, or equivalently, $u_1(x,t) \geq u_2(x,t)$.