Name: **110 MIDTERM 2**

Some potentially useful formulae:

(*i*) $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

(*ii*) The Fourier coefficients of $f(x)$ for the eigenfunctions $\cos(n\pi x/\ell)$ and $\sin(n\pi x/\ell)$ on the interval $(-\ell, \ell)$ are

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\mathrm{d}x$$

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\mathrm{d}x.$$

In this case the Fourier series for $f(x)$ is $\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\big(A_n\cos\frac{n\pi x}{\ell} + B_n\sin\frac{n\pi x}{\ell}\big)$.

(*iii*) The Fourier coefficients of $f(x)$ for the eigenfunctions $e^{in\pi x/\ell}$ on the interval $(-\ell, \ell)$ are

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\mathrm{d}x.$$

In this case the Fourier series for $f(x)$ is $\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$.

	score
1	
2	
3	
total	

1. Let $\phi(x) = \sin x$.

   a. [10 points] Solve $u_t = u_{xx}$ for $0 < x < \pi$ and $0 < t < \infty$, with boundary conditions $u(0,t) = 0 = u(\pi,t)$ and initial condition $u(x,0) = \phi(x)$.
   Let $u(x,t) = X(x)T(t)$. Then $-X''(x) = \lambda X(x)$, with $X(0) = 0 = X(\pi)$, which has nontrivial solutions $X_n(x) = \sin(nx)$ with $\lambda_n = n^2$, $0 < n \in \mathbb{Z}$. Notice that $\phi(x) = \sin x = X_1(x)$, *so no other terms appear in the Fourier series expansion of* $\phi(x)$. Also $-T'(t) = \lambda T(t)$, which has solution $T(t) = Ae^{-\lambda t}$. Thus $u(x,t) = e^{-t}\sin x$.

   b. [15 points] Solve $u_t = iu_{xx}$ for $0 < x < \pi$ and $0 < t < \infty$, with the same boundary and initial conditions.
   Everything is the same as in part (a) up to the equation for $T(t)$: $-T'(t) = i\lambda T(t)$, which has solution $T(t) = Ae^{-i\lambda t}$. Thus $u(x,t) = e^{-it}\sin x$.

   c. [15 points] Let
   $$E(t) = \int_0^\pi \left(|u_x|^2 + |u_t|^2\right)\mathrm{d}x.$$
   Show that $E'(t) < 0$ for the solution in part (a), but $E'(t) = 0$ for the solution in part (b).
   For the solution to the diffusion equation in part (a),
   $$E(t) = \int_0^\pi \left(|e^{-t}\cos x|^2 + |-e^{-t}\sin x|^2\right)\mathrm{d}x = e^{-2t}\int_0^\pi (\cos^2 x + \sin^2 x)\mathrm{d}x = \pi e^{-2t},$$
   so $E'(t) = -2e^{-2t} < 0$. For the solution to the Schrödinger equation in part (b),
   $$E(t) = \int_0^\pi \left(|e^{-it}\cos x|^2 + |-e^{-it}\sin x|^2\right)\mathrm{d}x = |e^{-it}|^2\int_0^\pi (\cos^2 x + \sin^2 x)\mathrm{d}x = \pi,$$
   since $|e^{it}| = 1$ when $t \in \mathbb{R}$, so $E'(t) = 0$.

2. [30 points] Find the full Fourier series for $\cosh x$ on the interval $(-\ell, \ell)$.

Extra Credit. [10 points] Let $S_N(x)$ be the partial sum of the full Fourier series corresponding to eigenvalues $(n\pi/\ell)^2$ for $n \leq N$. Let $E_N(x) = \| \cosh x - S_N(x)\|^2$. How does the error $E_N(x)$ change as ℓ increases? If you double ℓ, how many terms in the new series should you use in order to keep the error the same? Why?

3. Give examples of functions whose Fourier cosine series on $(0, 1)$

a. [10 points] converges uniformly;
a continuous function with continuous first derivative, satisfying Neumann boundary conditions (the ones which lead to the Fourier cosine series) on $[0, 1]$, *e.g.*, $f(x) = 1$; or $f(x) = \cos(\pi x)$

b. [10 points] converges pointwise but not uniformly;
a piecewise continuous, but not continuous function on $[0, 1]$, or one that does not satisfy Neumann boundary conditions, *e.g.*, $f(x) = 0$ for $x \leq 1/2$ and $f(x) = 1$ for $x > 1/2$; or $f(x) = \sin(\pi x)$

c. [10 points] converges in mean-square but not pointwise.
a square-integrable function with a singularity worse than a jump discontinuity, *e.g.*, $f(x) = |x - 1/2|^{-1/4}$