Solving first order linear ODEs
or
Where do integrating factors come from?

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A general first order linear ODE has the form

$$y'(t) = a(t)y(t) + b(t). \tag{1}$$

When a and b are constants, we can solve (1) by direct integration:

$$\begin{aligned} y'(t) = a\big(y(t) + b/a\big) \implies \frac{y'(t)}{y(t)+b/a} &= a \\ \implies \int \frac{y'(t)}{y(t)+b/a}\mathrm{d}t &= \int a\mathrm{d}t. \end{aligned} \tag{2}$$

The integrand on the left side of (2) can be recognized as

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln|y(t) + b/a| \tag{3}$$

by the Chain Rule. Then, using the Fundamental Theorem of Calculus, we can integrate both sides of (2) to get

$$\begin{aligned} \ln|y(t) + b/a| = at + c \implies y(t) + b/a &= Ce^{at} \\ \implies y(t) &= -b/a + Ce^{at}. \end{aligned} \tag{4}$$

When b/a depends on t, however, the Chain Rule does not give the integrand on the left side of (2) when we evaluate (3), so we need a different solution method.

We begin by rewriting (1) as

$$y'(t) - ay(t) = b. \tag{5}$$

The left side of (5) is a function that we can think of as the result of starting with the function $y(t)$, differentiating it, and subtracting a times it. That is, there is a *differential operator* $D - a$ that changes the function $y(t)$ into the function $y'(t) - ay(t)$. "D", of course, stands for "derivative" or "differentiate". With this notation we can rewrite (5) as

$$(D - a)[y(t)] = b. \tag{6}$$

Now, suppose we had another operator on functions that changed $y'(t) - ay(t)$ back into $y(t)$, *i.e.*, that undid, or *inverted*, the operator $D - a$. We could call this operator $(D-a)^{-1}$ by analogy with multiplication and division: For the simple operator that acts on the function $y(t)$ to change it into the function $dy(t)$, for some $d \in \mathbb{R}$,

$$d[y(t)] = dy(t),$$

the inverse operator is d^{-1}, since

$$d^{-1}\big[d[y(t)]\big] = d^{-1}[dy(t)] = d^{-1}dy(t) = y(t).$$

If we knew $(D-a)^{-1}$, we could apply it to both sides of (6):

$$(D-a)^{-1}\big[(D-a)[y(t)]\big] = (D-a)^{-1}[b]. \tag{7}$$

Since $(D-a)^{-1}$ is the inverse of $D-a$, the left hand side of (7) is just $y(t)$, so

$$y(t) = (D-a)^{-1}[b], \tag{8}$$

would give a solution to (1), *if we knew* $(D-a)^{-1}$.

To start figuring out what $(D-a)^{-1}$ is, notice that we already know what the inverse of D, namely D^{-1}, is:

$$D[y(t)] = y'(t),$$

and to change $y'(t)$ back into $y(t)$, the Fundamental Theorem of Calculus tells us simply to integrate:

$$\int y'(t)\mathrm{d}t = y(t).$$

Thus the *integral operator*,

$$\int [\cdot]\mathrm{d}t$$

that takes functions of t and integrates them, is D^{-1}:

$$D^{-1}\big[D[y(t)]\big] = \int D[y(t)]\mathrm{d}t = \int y'(t)\mathrm{d}t = y(t).$$

The inverse of $D-a$ is also an integral operator. To figure out what it is, notice that if d is our simple multiplication operator, then $d-a$ is also a multiplication operator: $(d-a)[y(t)] = (d-a)y(t)$. So

$$(d-a)^{-1} = \frac{1}{d-a} = \frac{1}{d}\cdot\frac{1}{1-a/d} = d^{-1}\cdot\frac{1}{1-a/d}.$$

That is, when we subtract a from d, the inverse operator is d^{-1}, multiplied by the factor $1/(1-a/d)$. So let's guess that the inverse of $D-a$ has the form

$$\int \mu(t)[\cdot]\mathrm{d}t, \tag{9}$$

just like D^{-1}, but with a multiplicative factor of $\mu(t)$ inside the integral, for some function $\mu(t)$ that we must determine. Let's see what happens if we apply the integral operator (9) to $(D-a)[y(t)]$:

$$\int \mu(t)(D-a)[y(t)]\mathrm{d}t = \int \mu(t)\big(y'(t) - ay(t)\big)\mathrm{d}t. \tag{10}$$

We would like to find some function $\mu(t)$ so that when we evaluate this integral we get $y(t)$. The only way we know to evaluate general integrals is to use the Fundamental Theorem of Calculus (again), so we need to recognize the integrand in (10) as the derivative with respect to t of something. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mu(t)y(t)\big) = \mu(t)y'(t) + \mu'(t)y(t),$$

which would be the same as the integrand on the right side of (10) if

$$\mu'(t)y(t) = -\mu(t)ay(t).$$

This simplifies to a condition on $\mu(t)$:

$$\mu'(t) = -a\mu(t), \tag{11}$$

which is easy to solve. Dividing by $\mu(t)$ and integrating, we have:

$$\int \frac{\mu'(t)}{\mu(t)}\mathrm{d}t = -\int a\mathrm{d}t. \tag{12}$$

Just as in (2) we use the Chain Rule to recognize the integral on the left side of (12) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln|\mu(t)|,$$

and then integrate both sides of (12) to get

$$\ln|\mu(t)| = -at \implies \mu(t) = e^{-at}. \tag{13}$$

This choice for the function $\mu(t)$ makes the integrand on the right side of (10) a total derivative, so (10) becomes

$$\int e^{-at}(D-a)[y(t)]\mathrm{d}t = \int \frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-at}y(t)\big)\mathrm{d}t = e^{-at}y(t).$$

This means that our guess (9) for the inverse of $D-a$ is close—with $\mu(t)=e^{-at}$, applied to $(D-a)[y(t)]$ it gives e^{-at} times $y(t)$, rather than just $y(t)$, which is what the real inverse gives. But that means we just need to divide (9) by e^{-at} to get

$$(D-a)^{-1}[\cdot]=e^{at}\int^{t}\mu(s)[\cdot]\mathrm{d}s. \tag{14}$$

Now we can finally use this inverse of $D-a$ to find $y(t)$, using (8):

$$\begin{aligned}
y(t)&=(D-a)^{-1}[b]\\
&=e^{at}\int e^{-at}b\,\mathrm{d}t\\
&=e^{at}\big(-\frac{b}{a}e^{-at}+C\big)\\
&=-\frac{b}{a}+Ce^{at},
\end{aligned}$$

which is the same as (4). But this same method works even when a and b depend on t.

When a is not a constant, we can still integrate (12) to get

$$\ln|\mu(t)|=-\int^{t}a(s)\mathrm{d}s \implies \mu(t)=e^{-\int^{t}a(s)\mathrm{d}s}, \tag{15}$$

which we plug into the general form of (14):

$$(D-a)^{-1}[\cdot]=\frac{1}{\mu(t)}\int^{t}\mu(s)[\cdot]\mathrm{d}s, \tag{16}$$

to find the inverse of $D-a(t)$. This gives the general solution to (1):

$$y(t)=(D-a)^{-1}[b(t)]=\frac{1}{\mu(t)}\int^{t}\mu(s)b(s)\mathrm{d}s,$$

with $\mu(t)$ given by (15).