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INTRODUCTION TO MATHEMATICAL MODELLING LECTURES 6-7: MODELLING WEALTH DISTRIBUTION

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Rank from probability density

In the previous lecture I introduced the Pareto distribution and showed that it has a heavy tail, *i.e.*, that its probability density function, $f(x)$, is asymptotically $x^{-\alpha-1}$ as $x \to \infty$ [1]. We can see that this leads to a linear log-log plot of random variable value as a function of rank as follows: Since $f(x) = \alpha k^\alpha x^{-\alpha-1}$, for $x \geq k$,

$$\text{(fraction of values at least } w) = \int_w^\infty \alpha k^\alpha x^{-\alpha-1} \mathrm{d}x,$$

so for a random sample of $N \gg 1$ independent values,

$$\text{(number of values at least } w) \approx \int_w^\infty N\alpha k^\alpha x^{-\alpha-1} \mathrm{d}x,$$

which implies that

$$\mathrm{rank}(w) \approx \int_w^\infty N\alpha k^\alpha x^{-\alpha-1} \mathrm{d}x = N k^\alpha w^{-\alpha},$$

where we have used the fact that $\alpha > 0$ so the integral converges. Taking the logarithm of both sides gives:

$$\log r \approx \log N k^\alpha - \alpha \log w,$$

which implies that

$$\log w \approx \frac{1}{\alpha} \log N k^\alpha - \frac{1}{\alpha} \log r.$$

Comparing this with eq. 5.2, we see that the *Forbes* 400 data is approximated by a Pareto distribution with $1/\alpha \approx 0.78502$, which implies that $\alpha \approx 1.27385$. We can also use the constant term to set $(\log N k^\alpha)/\alpha \approx 4.30542$; using $N \approx 2.9 \times 10^8$, this gives $k \approx 1.68527 \times 10^{-5}$.

A simple model for the distribution of wealth

In Lectures 3 and 4 we modelled the amount of money in the pocket of a gambler on the way out of a casino as a function of the initial amount and the outcomes of a sequence of random events (the coin flips) [2]. That model produced a binomial distribution of pocket money. To explain the *Forbes* 400 data, we must create a different model, that produces a different distribution. At first sight this might seem hopeless, since the economy in which individuals acquire wealth is so complicated. But just as with the flipping coin, perhaps we can avoid modelling many of the details, replacing them instead with some reasonable probabilistic description. The following simple model attempts this, although its reasonableness is debatable.

Let us suppose first that a model can be constructed that does not depend on population growth, *i.e.*, that we can postulate a constant number of people, N. Let us suppose second that the total wealth of the society is constant, *i.e.*,

$$\sum_{i=1}^{N} W_i(t) = W,$$

where W is constant, and $W_i(t)$ is the wealth of individual i at time t. Each of these assumptions is only an approximation that describes the real world over very short time scales. If the observed wealth distribution is a longer term effect, our model may not succeed; nevertheless, let us proceed.

Real economic interactions between people can transfer wealth between them. We make two very simple (and unrealistic) assumptions about this for our model: that pairs of people interact at random, and one (chosen at random) transfers a fixed amount of wealth, δ, to the other. We implement these events at a sequence of times, so for each time $t \in \mathbb{Z}_{>0}$, $A(t)$ and $B(t)$ are random variables with probability functions

$$\mathrm{prob}\big(A(t) = i\big) = \frac{1}{N} = \mathrm{prob}\big(B(t) = i\big), \tag{6.1}$$

for $i \in \{1, \ldots, N\}$, and

$$W_i(t) = \begin{cases} W_i(t-1) + \delta & \text{if } A(t) = i \text{ and } W_{B(t)}(t-1) > \delta; \\ W_i(t-1) - \delta & \text{if } B(t) = i \text{ and } W_{B(t)}(t-1) > \delta; \\ W_i(t-1) & \text{otherwise.} \end{cases} \tag{6.2}$$

The condition that $W_{B(t)}(t-1)$ be greater than δ prevents any individual's wealth from ever becoming zero (or negative).

This model is simple enough that we could compute $\mathrm{prob}(W_i(t) = w)$ as a function of w, t, and the model parameters N, W and δ. Rather than doing so, Figure 6.1 shows a movie of 10^5 timesteps for $N = 400$, $W = 2.4N$, $\delta = 0.6$, starting from the initial condition $W_i(0) = 2.4$. The vertical axis is $\log W_i(t)$ and the horizontal axis is log(rank), just as in

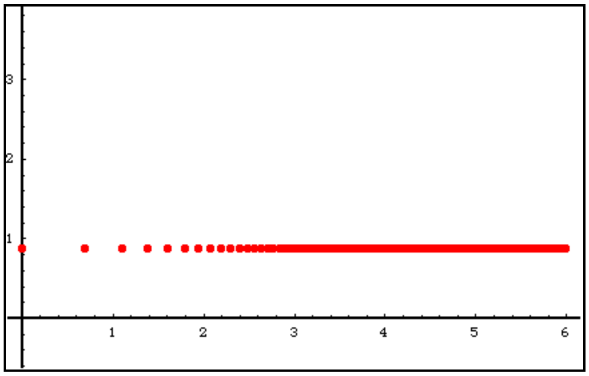

Figure 6.1. QuickTime movie of 10^5 timesteps of the simple model for wealth distribution. Each frame is 100 timesteps.

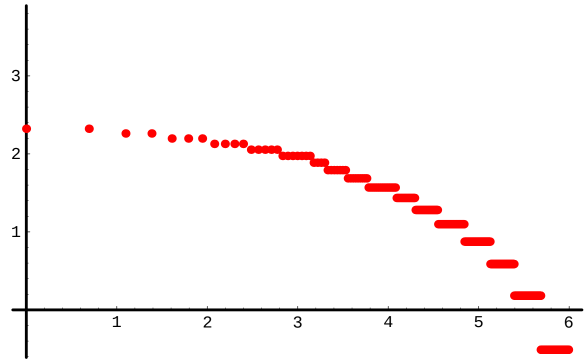

Figure 6.2. The distribution of the W_i after $t = 10^5$ timesteps. $\log W_i$ is plotted as a function of log(rank).

Figure 5.8 [1]. Figure 6.2 shows the same plot at $t = 10^5$. Since the plot is not linear, this model has not produced a Pareto distribution for the W_i.

We can imagine modifying this simple model in many ways: the amount of wealth transfered need not be constant, the interactions need not be random, the direction of the transfer need not be random, wealth might grow over time, there could be taxes, *etc.*

Homework: Modify the model for wealth distribution described by equations 6.1–6.2 in some ways that seem realistic to you. Do any of them lead to a Pareto distribution?

A particularly simple version of this model, in which an interaction between two individuals ends with them sharing their combined wealth evenly, was interpreted by Melzak as describing a 'socialist' economy [3]. Other versions have been investigated by Mainieri [4], and Ispolatov, Krapivsky and Redner [5]. Two natural modifications are to transfer a constant fraction of the wealth of the individual who is losing wealth during a transaction, and to include some rate of return on wealth. That is, the model is described by eq. 6.1, and with

$$W_i(t) = \big(1 + R_i(t)\big) \begin{cases} W_i(t-1) + fW_{B(t)}(t-1) & \text{if } A(t) = i; \\ W_i(t-1) - fW_{B(t)}(t-1) & \text{if } B(t) = i; \\ W_i(t-1) & \text{otherwise,} \end{cases} \tag{6.3}$$

replacing eq. 6.2. Here $0 \leq f \leq 1$ and $R_i(t)$ is a new random variable that represents the rate of return on wealth during the time period from $t-1$ to t. This model is more complicated than the previous one, but it is still easy to simulate numerically. Figure 6.3 shows a movie of the wealths of 400 individuals for 10^5 timesteps, using $W_i(0) = 0.6$ (the minimum wealth on the *Forbes* 400 list), $f = 0.1$, and

$$\text{prob}\big(R_i(t) = \pm 0.125\big) = \frac{1}{2}$$

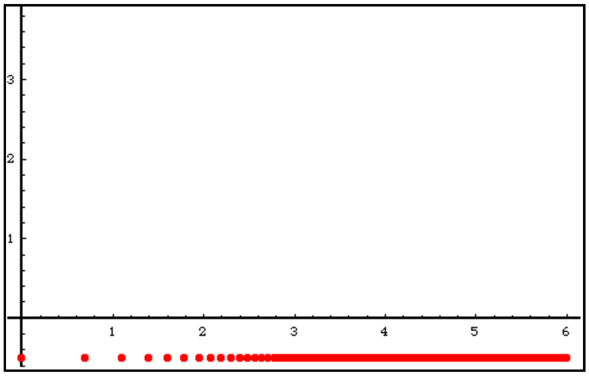

Figure 6.3. QuickTime movie of 10^5 timesteps of the more complicated model for wealth distribution. Each frame is 100 timesteps.

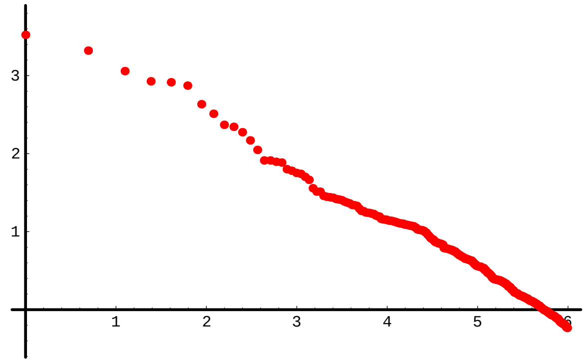

Figure 6.4. The distribution of the largest 400 W_i, after $t = 10^5$ timesteps. $\log W_i$ is plotted as a function of log(rank).

when $t \equiv 0 \bmod 100$ and $R_i(t) = 0$ otherwise. (That is, each individual only collects interest on his/her wealth every 100 timesteps.) As in Figures 6.1 and 6.2, the vertical axis is $\log W_i(t)$ and the horizontal axis is log(rank). Figure 6.4 shows the same plot at $t = 10^5$. This plot is very similar to the plot of the *Forbes* 400 data shown in Figure 5.8, so it appears that we have constructed a reasonably good model.

Figure 6.5, however, shows the results for the whole model. The simulation shown in Figure 6.3 actually has $N = 2000$, with only the 400 largest wealths plotted at each timestep. When all 2000 wealths are plotted, the distribution looks quite different; there is a sharp bend in the log-log plot, and there is a substantial number of very poor individuals. Since we have not collected data on the true distribution of wealth in the U.S. beyond the wealthiest 400 people, we cannot know if this model is consistent with reality. But we do know that it does not produce a Pareto distribution, since the log-log plot is not linear throughout its domain.

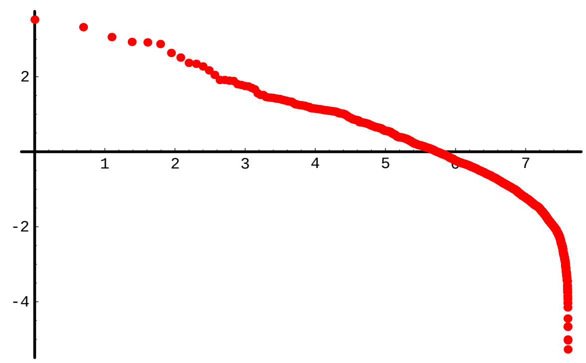

Figure 6.5. The distribution of all 2000 W_i, after $t = 10^5$ timesteps. $\log W_i$ is plotted as a function of log(rank).

References

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[3] Z. A. Melzak, *Mathematical Ideas, Modeling and Applications*, Vol. II of *Companion to Concrete Mathematics* (New York: Wiley 1976) 279–280.

[4] R. Mainieri, lectures at Santa Fe Institute Complex Systems Summer School.

[5] S. Ispolatov, P. L. Krapivsky and S. Redner "Wealth distributions in asset exchange models", http://arXiv.org/abs/cond-mat/9708018; *European Phys. J. B* **2** (1998) 267–276.