1. Recall that the van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ can be rewritten as the system

$$\dot{x} = y - F(x)$$
$$\dot{y} = -x,$$

where $F(x) = \mu x(x^2/3 - 1)$ and $\mu > 0$.

a. [15 points] Let $R = \frac{1}{2}(x^2 + y^2)$. For which (x, y) is $\dot{R}$ positive? Negative? Is R a Lyapunov function? Why or why not?

$\dot{R} = x\dot{x}+y\dot{y} = x(y-F)+y(-x) = -xF = -\mu x^2(x^2/3-1)$. Thus $\dot{R} > 0$ for $|x| < \sqrt{3}$ and $\dot{R} < 0$ for $|x| > \sqrt{3}$. R is not a Lyapunov function since the textbook definition is that it should be weakly decreasing everywhere.

b. [15 points] Can you use what you learned about R in part (a), in conjunction with the Poincaré-Bendixson theorem, to prove that the van der Pol equation has a limit cycle? Explain your answer.

Let A be the annulus $R \leq 2$, for example, with a small disc (of radius less than $\sqrt{3}$) around the fixed point at $(x, y) = (0, 0)$ removed. Then along the part of the outer boundary of A on which $|x| > \sqrt{3}$, the flow is towards the interior of A, as it is also on the inner boundary. But this isn't quite enough to apply the Poincaré-Bendixson theorem, since part of the outer boundary of A has $|x| < \sqrt{3}$, and along that part of the boundary R is increasing. Can you find a different region which is trapping?

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2. Let $\ddot{x} + 2\epsilon\dot{x} + x = 0$, with $x(0) = 1$ and $\dot{x}(0) = 1$. Suppose we want to find a *three*-time approximation

$$x(t,\epsilon) = x_0(\tau, T, \mathcal{T}) + \epsilon x_1(\tau, T, \mathcal{T}) + \epsilon^2 x_2(\tau, T, \mathcal{T}) + O(\epsilon^3),$$

where $\tau = t$, $T = \epsilon t$, and $\mathcal{T} = \epsilon^2 t$.

a. [15 points] Compute the terms in the ϵ-expansions of $\dot{x}$ and $\ddot{x}$ up to and including $O(\epsilon^2)$.

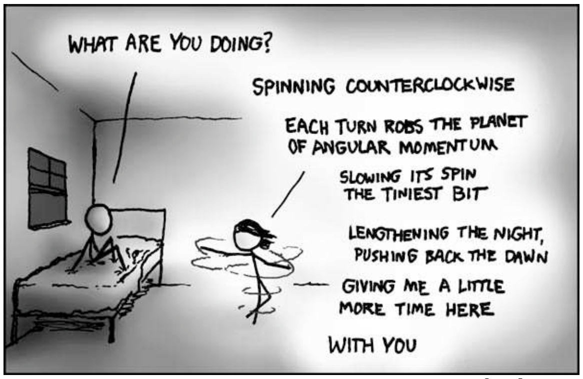
$$\begin{aligned}
\dot{x} &= \partial_\tau x_0 + \epsilon\partial_T x_0 + \epsilon^2\partial_{\mathcal{T}} x_0 + \epsilon(\partial_\tau x_1 + \epsilon\partial_T x_1) + \epsilon^2\partial_\tau x_2 + O(\epsilon^3) \\
&= \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + \epsilon^2(\partial_{\mathcal{T}} x_0 + \partial_T x_1 + \partial_\tau x_2) + O(\epsilon^3) \\
\ddot{x} &= \partial_{\tau\tau} x_0 + \epsilon\partial_{T\tau} x_0 + \epsilon^2\partial_{\mathcal{T}\tau} x_0 + \epsilon(\partial_{\tau T} x_0 + \epsilon\partial_{TT} x_0 + \partial_{\tau\tau} x_1 + \epsilon\partial_{T\tau} x_1) \\
&\qquad + \epsilon^2(\partial_{\tau\mathcal{T}} x_0 + \partial_{\tau T} x_1 + \partial_{\tau\tau} x_2) + O(\epsilon^3) \\
&= \partial_{\tau\tau} x_0 + \epsilon(2\partial_{\tau T} x_0 + \partial_{\tau\tau} x_1) \\
&\qquad + \epsilon^2(\partial_{TT} x_0 + 2\partial_{T\tau} x_1 + 2\partial_{\tau\mathcal{T}} x_0 + \partial_{\tau\tau} x_2) + O(\epsilon^3)
\end{aligned}$$

b. [15 points] Plugging the results of (a) into the differential equation, write down the equations implied by the ϵ^0, ϵ^1, and ϵ^2 terms.

$$\begin{aligned}
0 &= \partial_{\tau\tau} x_0 + \epsilon(2\partial_{\tau T} x_0 + \partial_{\tau\tau} x_1) + \epsilon^2(\partial_{TT} x_0 + 2\partial_{T\tau} x_1 + 2\partial_{\tau\mathcal{T}} x_0 + \partial_{\tau\tau} x_2) \\
&\qquad + 2\epsilon\big(\partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1)\big) + x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3) \\
(\epsilon^0) &\Rightarrow \partial_{\tau\tau} x_0 + x_0 = 0 \\
(\epsilon^1) &\Rightarrow \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - 2\partial_\tau x_0 \\
(\epsilon^2) &\Rightarrow \partial_{\tau\tau} x_2 + x_2 = -\partial_{TT} x_0 - 2\partial_{\tau\mathcal{T}} x_0 - 2\partial_T x_0 - 2\partial_{T\tau} x_1 - 2\partial_\tau x_1.
\end{aligned}$$

c. [Extra Credit, 20 points] Solve the first two of the equations you found in (b).

3. Consider the system of equations:



xkcd.com

$$\dot{u} = -\frac{1}{6}vw$$
$$\dot{v} = \frac{2}{3}wu$$
$$\dot{w} = -\frac{1}{2}uv.$$

a. [10 points] Show that $L = u^2 + v^2 + w^2$ is conserved in this system.
$\dot{L} = 2u\dot{u} + 2v\dot{v} + 2w\dot{w} = -\frac{1}{3}uvw + \frac{4}{3}uvw - uvw = 0.$

b. [15 points] Find an independent quadratic conserved quantity, $H(u, v, w)$.
Try $H = au^2 + bv^2 + cw^2$. Then $\dot{H} = \left(-\frac{1}{3}a + \frac{4}{3}b - c\right)uvw$, so we just need to pick a, b, and c to make $-\frac{1}{3}a + \frac{4}{3}b - c = 0$. An independent choice is $a = \frac{1}{2}$, $b = \frac{1}{4}$, and $c = \frac{1}{6}$.

c. [15 points] Use these two conserved quantities to reduce this system to a single nonlinear first order equation.

Since L and H are conserved, we have:

$$v^2 + w^2 = L - u^2$$
$$\frac{1}{2}v^2 + \frac{1}{3}w^2 = 2H - u^2.$$

Solving for v^2 and w^2 we get $w^2 = 3(L - 4H + u^2)$ and $v^2 = 2(6H - L - 2u^2)$. Substituting into the first ODE gives

$$\dot{u} = -\frac{1}{6}\sqrt{6(6H - L - 2u^2)(L - 4H + u^2)}.$$