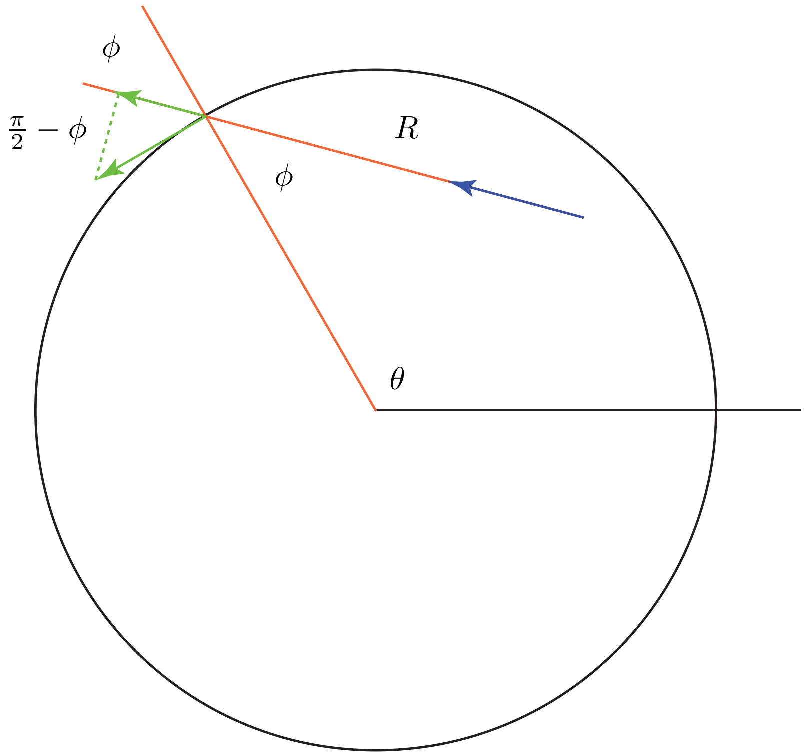
Problem 7.1.9

Let the vector from the center of the circle to the duck make an angle θ with the positive x-axis, let the vector from the dog to the duck make an angle ϕ with the vector from the duck to the center of the circle, and call the distance from the dog to the duck R, as shown below:



Let the speed of the dog be v (the length of the blue vector), and that of the duck be ω (the length of the green vector). Then R is decreasing by v and increasing by the projection of the duck's velocity vector in the direction of the dog-duck vector: $\omega\cos(\frac{\pi}{2} - \phi) = \omega \sin\phi$. Thus

$$\dot{R} = -v + \omega \sin\phi.$$

To understand how the angle ϕ is changing, extend the lines from the duck (k) to the center and the duck (k) to the dog (g) across the circle so that they become chords. Recall that the arc on the circle between the endpoints of these chords is 2ϕ. Consider the corresponding chords a moment later as the duck moves to a new position (k'), as shown in the next figure. The diameter is rotating at angular velocity ω, so the arc on the circle between the endpoints of the chords is decreasing by ω, but also increasing by the rate at which the chord through the dog is changing its endpoint (l). To see what this rate is, recall that the power of a point inside the circle is the product of the lengths into which it divides *any* chord. This means that the triangle $kk'g$ is similar to the triangle $ll'g$, and the ratio is $2\cos\phi - R$ (the length of the chord kl minus the length of the segment kg) to R (the length of the segment kg). Thus the endpoint l is moving at angular velocity

$\omega(2\cos\phi - R)/R$, since the endpoint k is moving at angular velocity ω, and hence

$$2\dot{\phi} = -\omega + \frac{2\cos\phi - R}{R}\omega \quad \Longrightarrow \quad \dot{\phi} = \frac{\cos\phi - R}{R}\omega.$$