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MATH 180A. INTRODUCTION TO PROBABILITY LECTURE 4

David A. Meyer

Project in Geometry and Physics, Department of Mathematics
University of California/San Diego, La Jolla, CA 92093-0112
http://math.ucsd.edu/~dmeyer/; dmeyer@math.ucsd.edu

Probability distributions

Given an outcome space Ω, a probability distribution is a function $P : \mathcal{P}(\Omega) \to \mathbb{R}$, where $P(E)$ is the probability assigned to an event $E \subseteq \Omega$. In Lecture 1 [1] we learned de Finetti's definition of probability [2]:

DEFINITION. Suppose that for any stake $S \in \mathbb{R}$, an individual will pay pS to win S if E occurs. Then $P(E) = p$.

There are relations among events that we should expect to constrain the possible probability distributions. For example, if $A \subseteq B \subseteq \Omega$, we might expect $P(A) \leq P(B)$. These constraints follow from an assumption that individuals are rational [2]:

ASSUMPTION. An individual assigns probabilities to events *coherently*, *i.e.*, in such a way that the individual is never sure to lose.

This seemingly obvious assumption has surprisingly specific consequences.

PROPOSITION 1. *If P is a coherent probability distribution, $P(\Omega) = 1$.*

Proof. Suppose a stake S is paid if Ω occurs, and an individual will pay pS to play. Since Ω must occur—by definition it is the set of all possible outcomes—the individual "wins"

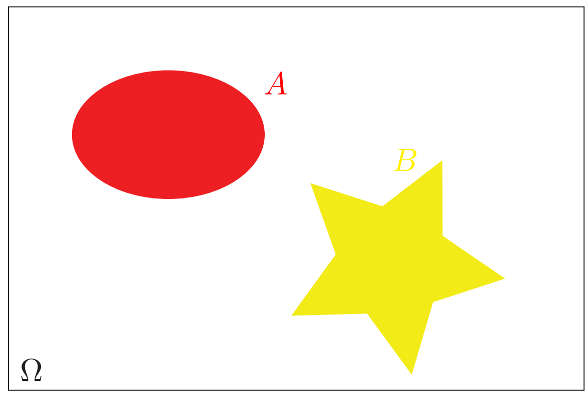
$$w = S - pS = (1 - p)S. \tag{1}$$

For any $w \in \mathbb{R}$, including $w < 0$, we can solve equation (1) for S, unless $1 - p = 0$. That is, unless $p = 1$, there is some S such that the individual is sure to lose money. Under the assumption that the individual assigns probabilities coherently, this means that $P(\Omega) = p = 1$. ∎

Notice that if $p < 1$ and $w < 0$, then the solution to equation (1) is some $S < 0$. We should think about the meaning of the wager in de Finetti's definition of probability when $S < 0$. In this case the individual is "paying" $-p|S|$ to "win" $-|S|$. "Paying" a negative amount means receiving a positive amount from the other party in the wager, while "winning" a negative amount means paying out the absolute value of that amount. Thus when $S < 0$ the parties to the wager change places: the first individual accepts a bet of $p|S|$ from the other party, and pays out $|S|$ if the event occurs. So the condition that the stake S in de Finetti's definition can be positive or negative means that the individual is willing to take either side of the wager with odds $p : 1 - p$ for E occurring.

PROPOSITION 2. *If P is a coherent probability distribution, $A, B \subseteq \Omega$, and $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.*

Proof. Let $p_A = P(A)$, $p_B = P(B)$, and $p = P(A \cup B)$. Suppose stakes S_A, S_B and S are wagered on the occurrence of A, B, and $A \cup B$, respectively.



As shown in the figure, $A\bar{B} = A \setminus B$, $\bar{A}B = B \setminus A$, and $\bar{A}\bar{B} = \overline{A \cup B}$ form a partition of Ω: these events have pairwise empty intersections and their union is Ω. Thus exactly one of these three events occurs, and we can calculate the individual's "winnings" in each case:

$$\begin{aligned}
A\bar{B}: &\qquad w_{A\bar{B}} = (1 - p_A)S_A - p_B S_B + (1 - p)S \\
\bar{A}B: &\qquad w_{A\bar{B}} = (1 - p_A)S_A - p_B S_B + (1 - p)S \\
\bar{A}\bar{B}: &\qquad w_{\bar{A}\bar{B}} = -p_A S_A - p_B S_B - pS
\end{aligned} \tag{2}$$

For any $w_{A\bar{B}}$, $w_{A\bar{B}}$, and $w_{\bar{A}\bar{B}}$, including negative values, we can solve the system of linear equations (2) for S_A, S_B, and S, unless the system is singular, *i.e.*, unless

$$0 = \begin{vmatrix} 1 - p_A & -p_B & 1 - p \\ -p_A & 1 - p_B & 1 - p \\ -p_A & -p_B & -p \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -p_A & -p_B & -p \end{vmatrix} = -p + p_B + p_A.$$

That is, unless $p = p_A + p_B$, there are some stakes S_A, S_B, and S such that the individual is sure to lose money. Under the assumption that the individual assigns probabilities coherently, this means that $P(A \cup B) = p = p_A + p_B = P(A) + P(B)$. ∎

The conclusions of Propositions 1 and 2, $P(\Omega) = 1$ and $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$, are the conditions required for P to be a probability distribution in Kolmogorov's formulation of probability [3]. We have derived them from de Finetti's assumption of coherence.

References

[1] http://math.ucsd.edu/~dmeyer/teaching/180Awinter08/notes01.pdf.
[2] B. de Finetti, "*La prévision: ses lois logiques, ses sources subjectives*", *Annales de l'Institute Henri Poincaré* **7** no. 1 (1937).
[3] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitrechnung* (1933); translation edited by N. Morrison, *Foundations of the Theory of Probability* (New York: Chelsea 1956).