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MATH 180A. INTRODUCTION TO PROBABILITY LECTURE 5

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Conditional probability

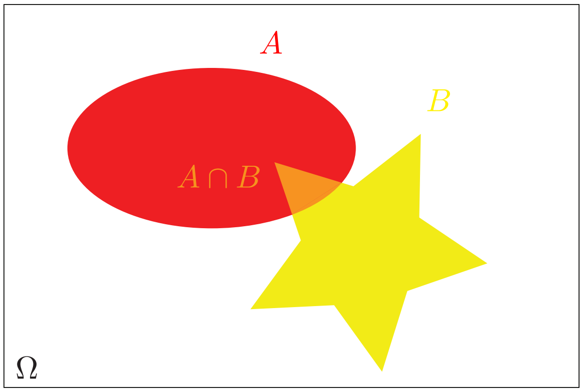
In Lecture 4 [1] we used de Finetti's assumption of coherence of probabilities [2] to derive basic properties of probability distributions. An argument similar to the ones we used there allows us to derive the basic formula relating conditional probability to probabilities of events. First we define conditional probability according to de Finetti [2]:

Definition. Let $A, B \subseteq \Omega$. Suppose that for any stake $S \in \mathbb{R}$, an individual will pay pS for a wager that:

s/he wins if B occurs and A occurs;
s/he loses if B occurs but A does not;
is cancelled if B does not occur.

That is, the change in the individual's wealth is $(1-p)S$, $-pS$, or 0 in these cases, respectively. Then the *conditional probability of A given B*, $P(A|B) = p$.

Proposition 3. *If P is a coherent probability distribution, $A, B \subseteq \Omega$, and $P(B) \neq 0$, then $P(A|B) = P(A \cap B)/P(B)$.*



Proof. Write $p_{AB} = P(A \cap B)$, $p_B = P(B)$, and $p = P(A|B)$. Suppose stakes S_{AB}, S_B, and S are wagered on the occurrence of $A \cap B$, B, and $A|B$, respectively.

As shown in the figure, $AB = A \cap B$, $\bar{A}B = B \setminus A$, and $\bar{B}$ form a partition of Ω. Thus exactly one of these three events occurs, and we can calculate the individual's "winnings" in each case:

$$\begin{aligned} AB: \quad & w_{AB} = (1-p_{AB})S_{AB} + (1-p_B)S_B + (1-p)S \\ \bar{A}B: \quad & w_{A\bar{B}} = -p_{AB}S_{AB} + (1-p_B)S_B - pS \\ \bar{B}: \quad & w_{\bar{B}} = -p_{AB}S_{AB} - p_B S_B + 0 \end{aligned} \tag{3}$$

For any w_{AB}, $w_{A\bar{B}}$, and $w_{\bar{B}}$, including negative values, we can solve the system of linear equations (3) for S_{AB}, S_B, and S, unless the system is singular, *i.e.*, unless

$$0 = \begin{vmatrix} 1-p_{AB} & 1-p_B & 1-p \\ -p_{AB} & 1-p_B & -p \\ -p_{AB} & -p_B & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1-p \\ 0 & 1 & -p \\ -p_{AB} & -p_B & 0 \end{vmatrix} = p_{AB} - p_B p.$$

That is, unless $p_{AB} = p_B p$, there are some stakes S_{AB}, S_B, and S such that the individual is sure to lose money. Under the assumption that the individual assigns probabilities coherently, this means that $P(A \cap B) = p_{AB} = pp_B = P(A|B)P(B)$. If $P(B) \neq 0$, dividing by $P(B)$ gives $P(A|B) = P(A \cap B)/P(B)$. ∎

EXAMPLE. Suppose there are two biased coins: C_1 with $P_1(H) = 1/10$ and C_2 with $P_2(H) = 3/4$. Consider a game in which we draw a card at random from a standard deck; if it is red, flip C_1, while if it is black, flip C_2.

1. Compute $P(H \cap \text{red})$:

$$P(H \cap \text{red}) = P(H|\text{red})P(\text{red}) = \frac{1}{10} \cdot \frac{1}{2} = \frac{1}{20},$$

   where the first equality follows from Proposition 3.

2. Compute $P(H \cap \text{black})$:

$$P(H \cap \text{black}) = P(H|\text{black})P(\text{black}) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8},$$

   where the first equality follows from Proposition 3.

3. Compute $P(H)$:

$$P(H) = P(H \cap \text{red}) + P(H \cap \text{black}) = \frac{1}{20} + \frac{3}{8} = \frac{17}{40},$$

   where the first equality follows from Proposition 2 [1].

4. Can we make betting on H a fair bet, *i.e.*, $P(H) = \frac{1}{2}$?

$$\begin{aligned} P(H) &= P(H \cap C_1) + P(H \cap C_2) \\ &= P(H|C_1)P(C_1) + P(H|C_2)P(C_2) \qquad (4) \\ &= \frac{1}{10}p + \frac{3}{4}(1-p), \qquad (5) \end{aligned}$$

where $p = P(C_1)$. Here the first equality follows from Proposition 2 [1], the second from Proposition 3 used twice, and the third from Propositions 1 and 2 [1]. Setting $P(H) = \frac{1}{2}$ and solving equation (5) gives $p = 5/13$. So we can make betting on H a fair bet by changing the game so that, for example, C_1 is flipped if the card drawn is in $\{A, K, Q, J, 10\}$, and C_2 is flipped otherwise.

Equation (4) is a special case of general, and very useful, formula which we should derive. First we prove a simple lemma:

LEMMA 4. *If* $\{C_1, \ldots, C_n\}$ *is a partition of* $A \subseteq \Omega$, *then*

$$P(A) = P(C_1) + \cdots + P(C_n).$$

Proof. This is a corollary of Proposition 2 [1]. Since $\{C_1, \ldots, C_n\}$ is a partition of A,

$$A = C_1 \cup C_2 \cup \cdots \cup C_n = C_1 \cup (C_2 \cup \cdots \cup C_n),$$

where $C_1 \cap (C_2 \cup \cdots \cup C_n) = \emptyset$, so by Proposition 2,

$$P(A) = P(C_1) + P(C_2 \cup \cdots \cup C_n).$$

Similarly,

$$C_2 \cup \cdots \cup C_n = C_2 \cup (C_3 \cup \cdots \cup C_n),$$

so using Proposition 2 again,

$$\begin{aligned} & P(C_2 \cup \cdots \cup C_n) = P(C_2) + P(C_3 \cup \cdots \cup C_n) \\ \Longrightarrow \quad & P(A) = P(C_1) + P(C_2) + P(C_3 \cup \cdots \cup C_n). \end{aligned}$$

Repeating this procedure gives the desired result. ∎

PROPOSITION 5. *If* $\{B_1, \ldots, B_n\}$ *is a partition of* Ω, *then for* $A \subseteq \Omega$,

$$P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n).$$

Proof. Since $\{B_1, \ldots, B_n\}$ is a partition of Ω, $\{A \cap B_1, \ldots, A \cap B_n\}$ is a partition of A. Then by Lemma 4,

$$P(A) = P(A \cap B_1) + \cdots + P(A \cap B_n).$$

Using Proposition 3 for each term on the right-hand side of this equation gives the desired result. ∎

Equation (4) in the example is an $n = 2$ version of this result.

References

[1] `http://math.ucsd.edu/~dmeyer/teaching/180Awinter08/notes04.pdf`.

[2] B. de Finetti, "*La prévision: ses lois logiques, ses sources subjectives*", *Annales de l'Institute Henri Poincaré* **7** no. 1 (1937).