14 March 2008

MATH 180A. INTRODUCTION TO PROBABILITY
LECTURE 27

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§6.5 Solutions

Problems 1, 2 and 3 all require the same ideas, which are explained in the examples in this section. Since I didn't go over these examples in class, here are the solutions to the questions in problem 1. Problems 2 and 3 are similar.

1. Let X and Y be the normalized PSAT and SAT scores, respectively. Then $Y = \rho X + \sqrt{1-\rho^2}Z$, where Z is an $N(0,1)$ random variable, independent of X, and $\rho = 3/5$.

   a. When the PSAT score is 1000, $X = -2$, so $Y = -2\rho + \sqrt{1-\rho^2}Z$. An above average SAT score is $Y > 0$, so we need to compute $P(Y > 0) = P(-2\rho + \sqrt{1-\rho^2}Z > 0) = P(Z > 3/2) \approx 0.0668$, using the table in Appendix 5.

   b. $P(Y > 0 \mid X < 0) = P(Y > 0 \text{ and } X < 0)/P(X < 0) = 2P(Y > 0 \text{ and } X < 0) = 2P(\rho X + \sqrt{1-\rho^2}Z > 0 \text{ and } X < 0) = 2P\big(X < 0 \text{ and } Z > -(\rho/\sqrt{1-\rho^2})X\big)$. Since X and Z are independent standard normal random variables, their joint probability density function is circularly symmetric, so this probability is determined by the fraction of the whole circle spanned by the wedge $X < 0$ and $Z > -(\rho/\sqrt{1-\rho^2})X$. The angle between the lines $Z = -(3/4)X$ and $X = 0$ is $\arctan(3/4)$, so the answer is $2\arctan(3/4)/2\pi \approx 0.2048$.

c. Let U and V be the PSAT and SAT scores, respectively. Then

$$\begin{aligned}
P(V > U + 50) &= P\big(V - 1300 > (U - 1200) - 50\big) \\
&= P\Big(Y > \frac{10}{9}X - \frac{5}{9}\Big) \\
&= P\Big(\rho X + \sqrt{1-\rho^2}Z > \frac{10}{9}X - \frac{5}{9}\Big) \\
&= P\Big(Z > \frac{10/9 - \rho}{\sqrt{1-\rho^2}}X - \frac{5/9}{\sqrt{1-\rho^2}}\Big).
\end{aligned}$$

Using the circular symmetry of the joint probability density function of X and Z again, we observe that this probability is just $\Phi(d)$, where d is the distance from the origin to the line

$$Z = \frac{10/9 - \rho}{\sqrt{1-\rho^2}}X - \frac{5/9}{\sqrt{1-\rho^2}}.$$

A little geometry or trigonometry shows us that the distance from a line $z = mx + b$ to the origin is just $b/\sqrt{m^2+1}$, so plugging in the numbers gives $\Phi(0.5852) \approx 0.7207$.