Please simplify your answers to the extent reasonable without a calculator. Show your work. Explain your answers, concisely.

1. [25 points] Let A, B, and C be events in a probability space $(\Omega, \mathcal{F}, P)$. Suppose $P(A) = P(B) = P(C) = 1/2$. What is the smallest possible value for $P(A \cap B) + P(B \cap C) + P(C \cap A)$?

   Let $x = P(A \cap B) + P(B \cap C) + P(C \cap A)$. Using the Kolmogorov axioms and the inclusion-exclusion principle, we have:

$$1 = P(\Omega) \geq P(A \cup B \cup C) = P(A) + P(B) + P(C) - x + P(A \cap B \cap C)$$
$$= \frac{3}{2} - x + P(A \cap B \cap C)$$
$$\Rightarrow x \geq 1/2 + P(A \cap B \cap C).$$

   Since $P(A \cap B \cap C) \geq 0$, and can be 0, the smallest possible value for $x = P(A \cap B) + P(B \cap C) + P(C \cap A)$ is $1/2$.

2. A special *unfair* die has probabilities of rolling m and n whose ratio is m/n, for all $m, n \in \{1, 2, 3, 4, 5, 6\}$.

   a. [10 points] Find $P(n)$ for each $n \in \{1, 2, 3, 4, 5, 6\}$.

   $P(n)/P(1) = n/1$, so $P(n) = nP(1)$. Since $\Omega = \{1, 2, 3, 4, 5, 6\}$, by definition $1 = P(\{1, 2, 3, 4, 5, 6\}) = P(1) + \cdots + P(6) = P(1)(1 + 2 + 3 + 4 + 5 + 6) = 21P(1)$. Thus $P(n) = n/21$.

   b. [10 points] If you roll the die twice, what is the probability that the sum of your two rolls is 7?

$$P(\text{sum } 7) = 2\big(P(1)P(6) + P(2)P(5) + P(3)P(4)\big)$$
$$= \frac{2}{21^2}(1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4)$$
$$= \frac{2^2}{3^2 \cdot 7^2} 14 = \frac{2^3}{3^2 \cdot 7} = \frac{8}{63}.$$

   c. [5 points] Is your answer to (b) larger or smaller than what the probability would be if you were rolling a fair die?

   For a fair die, each probability $P(n) = 1/6$, so

$$P(\text{sum } 7) = 2 \cdot 3 \cdot \frac{1}{6^2} = \frac{1}{6} > \frac{8}{63}.$$

3. [25 points] You play the following game with a fair die: Roll the die. If it is n, you roll the die n more times. If you roll a second n, you win. What is the probability that you win?

$$\begin{aligned} P(\text{win}) &= 1 - P(\text{lose}) \\ &= 1 - \sum_{n=1}^{6} P(\text{lose} \mid n)P(n) \\ &= 1 - \frac{1}{6}\sum_{n=1}^{6}\left(\frac{5}{6}\right)^n \\ &= 1 - \frac{1}{6}\frac{5/6 - (5/6)^7}{1 - 5/6} \\ &= \frac{1}{6} + \left(\frac{5}{6}\right)^7. \end{aligned}$$

4. [25 points] Let $Z = (X, Y)$ be a point chosen uniformly at random in the unit square $[0,1]^2 = \{(x,y) : 0 \leq x, y \leq 1\}$. Find the cumulative distribution function for the random variable D = distance from Z to the closest point on the boundary of the square, and then find its probability density function.

For $0 \leq d \leq 1/2$,

$$\begin{aligned} P(D \leq d) &= P(Z \text{ is within } d \text{ of the boundary}) \\ &= P(Z \text{ is not in the square of side } 1 - 2d \text{ around the center of the square}) \\ &= 1 - (1 - 2d)^2. \end{aligned}$$

So

$$F(d) = \begin{cases} 0 & d \leq 0; \\ 1 - (1-2d)^2 & 0 \leq d \leq 1/2; \\ 1 & 1/2 \leq d. \end{cases}$$

Taking the derivative with respect to d gives the pdf:

$$f(d) = \begin{cases} 4(1-2d) & 0 < d < 1/2; \\ 0 & \text{otherwise.} \end{cases}$$