Please simplify your answers to the extent reasonable without a calculator. Show your work. Explain your answers, concisely. In case you need them, here are the probability distributions we have learned ($0 \leq p \leq 1$, $0 < n \in \mathbb{Z}$, $0 < \lambda, \sigma \in \mathbb{R}$, $a < b \in \mathbb{R}$, $\mu \in \mathbb{R}$, $1 \leq r \in \mathbb{R}$, and any values not listed for the random variables have probability 0 or probability density 0):

$$B \sim \mathrm{Ber}(p) \qquad P(B = b) = \begin{cases} 1-p & \text{if } b = 0; \\ p & \text{if } b = 1. \end{cases}$$

$$K \sim \mathrm{Bin}(n,p) \qquad P(K = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k \in \{0, 1, \ldots, n\}.$$

$$K \sim \mathrm{Geom}(p) \qquad P(K = k) = (1-p)^{k-1} p, \quad 0 < k \in \mathbb{Z}.$$

$$K \sim \mathrm{Poisson}(\lambda) \qquad P(K = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad 0 \leq k \in \mathbb{Z}.$$

$$X \sim \mathrm{Unif}[a,b] \qquad f(x) = \frac{1}{b-a}, \quad x \in [a,b].$$

$$X \sim \mathcal{N}(\mu, \sigma^2) \qquad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \quad x \in \mathbb{R}.$$

$$T \sim \mathrm{Exp}(\lambda) \qquad f(t) = \lambda e^{-\lambda t}, \quad 0 \leq t \in \mathbb{R}.$$

$$T \sim \mathrm{Gamma}(r, \lambda) \qquad f(t) = \frac{\lambda^r t^{r-1}}{\Gamma(r)} e^{-\lambda t}, \quad 0 \leq t \in \mathbb{R}.$$

1. [25 points] For $1 < \alpha \in \mathbb{R}$, let X_α be a continuous random variable with probability density function:

$$f_\alpha(x) = \begin{cases} (\alpha - 1)x^{-\alpha} & \text{if } 1 \leq x \in \mathbb{R}; \\ 0 & \text{otherwise.} \end{cases}$$

a. [5 points] Show that f_α is a probability density function for $\alpha > 1$.

$f_\alpha(x) \geq 0$ and $\displaystyle\int_{-\infty}^{\infty} f_\alpha(x)\mathrm{d}x = \int_0^{\infty} (\alpha - 1)x^{-\alpha}\mathrm{d}x = -x^{-\alpha+1}\big|_0^{\infty} = 1.$

b. [10 points] For which values of α is $\mathsf{E}[X_\alpha] \in \mathbb{R}$? For these values of α, what is $\mathsf{E}[X_\alpha]$?

$\displaystyle\mathsf{E}[X_\alpha] = \int_0^{\infty} x(\alpha - 1)x^{-\alpha}\mathrm{d}x = \frac{\alpha - 1}{-\alpha + 2} x^{-\alpha+2}\big|_0^{\infty} = \frac{\alpha - 1}{\alpha - 2}$, provided $\alpha > 2$.

c. [10 points] For which values of α is $\mathsf{Var}[X_\alpha] \in \mathbb{R}$? For these values of α, what is $\mathsf{Var}[X_\alpha]$?

$\displaystyle\mathsf{Var}[X_\alpha] = \mathsf{E}[X_\alpha^2] - \mathsf{E}[X_\alpha]^2 = \int_0^{\infty} x^2(\alpha - 1)x^{-\alpha}\mathrm{d}x - \mathsf{E}[X_\alpha]^2 = \frac{\alpha - 1}{\alpha - 3} - \left(\frac{\alpha - 1}{\alpha - 2}\right)^2 = \frac{\alpha - 1}{(\alpha - 2)^2(\alpha - 3)}$, provided $\alpha > 3$.

2. Human heights have approximately normal distributions: American women with a mean of about 64 inches and a standard deviation of 2.5 inches; American men with a mean of about 69.5 inches and a standard deviation of 3 inches.*

   a. [8 points] Explain why a normal distribution can't be exactly right for human heights, but could still be a good approximation.

   Human heights can't be exactly distributed according to a normal distribution because any normal distribution gives positive probabilities to negative heights, and no human has a negative height. But a normal distribution can be a good approximation if it gives a very small probability to negative heights; notice that for American women this probability is $\Phi(-64/2.5) < \Phi(-25)$, which is tiny.

   b. [7 points] Mary is one standard deviation taller than the average American woman. Approximately what fraction of American women is she taller than?

   Recall that $\Phi(1) \approx 5/6$.

   c. [10 points] Approximately what fraction of American men is Mary taller than?

   Mary's height is $64 + 2.5 = 66.5$ inches. That is one standard deviation less than the average man's height, so she is taller than $\Phi(-1) \approx 1/6$ of American men.

3. [25 points] Recall that a Poisson process with intensity λ is defined to be a set of random points on $[0, \infty)$, satisfying three properties: the points are distinct; for a bounded interval $I \subset [0, \infty)$ the number of points $N(I) \sim \text{Poisson}(\lambda|I|)$; and for non-overlapping intervals $I_1, \ldots, I_n$, the random variables $N(I_1), \ldots, N(I_n)$ are mutually independent.

   a. [9 points] Argue that for any $a > 0$, the points of a Poisson process with intensity λ on $[0, \infty)$, that lie in the interval $[a, \infty)$, satisfy the same three properties.

   (1) If the points are distinct on $[0, \infty)$, the subset of points in $[a, \infty)$ also contains no duplicates. (2) Any bounded interval $I \subset [a, \infty)$ is also a bounded interval in $[0, \infty)$ so $N(I) \sim \text{Poisson}(\lambda|I|)$. (3) Non-overlapping intervals in $[a, \infty)$ are also non-overlapping intervals in $[0, \infty)$ so the corresponding random variables are mutually independent.

   b. [16 points] Let T_1 be the location of the first (smallest coordinate) point in a Poisson process with intensity $\lambda > 0$ on $[0, \infty)$. Let T_2 be the location of the second point. Use the result of part (a) to find the probability density function for $T_2 - T_1$.
   From part (a), the Poisson process restricted to the interval $[T_1, \infty)$ is still a Poisson process with intensity λ. Thus

$$P(T_2 - T_1 > t) = P\big(\text{no points in } (T_1, t)\big) = e^{-\lambda t},$$

   so $P(T_2 - T_1 \leq t) = 1 - e^{-\lambda t}$. Taking the derivative with respect to t, we find that the pdf for $T_2 - T_1$ is $\lambda e^{-\lambda t}$, so $T_2 - T_1 \sim \text{Exp}(\lambda)$.

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* See M. F. Schilling, A. E. Watkins and W. Watkins, "Is human height bimodal?", *The American Statistician* **56** (2012) 223–229.

4. [25 points] Let $Z \sim \mathcal{N}(0,1)$ be a standard normal random variable. Find the moment generating function $M(t)$ of Z.

$$M(t) = \mathsf{E}[e^{tZ}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2tx)/2} \mathrm{d}x$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2tx+t^2)/2} e^{t^2/2} \mathrm{d}x = e^{t^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} \mathrm{d}x$$
$$= e^{t^2/2},$$

where we completed the square in the exponent to get to the second line, and then recognized the last integrand as the pdf of an $\mathcal{N}(t,1)$ random variable, so it integrates to 1.