180A PRACTICE PROBLEMS FOR MIDTERM 2

Please simplify your answers to the extent reasonable without a calculator. Show your work. Explain your answers, concisely.

1. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and for $1 < s \in \mathbb{R}$ let X_s be a discrete random variable with

$$P(X_s = n) = \begin{cases} \frac{1}{\zeta(s)} \frac{1}{n^s} & \text{if } 0 < n \in \mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases} \qquad (*)$$

   a. Show that for $s > 1$, $(*)$ is a probability mass function.
   b. For which values of s is $\mathsf{E}[X_s] \in \mathbb{R}$? For these values of s, what is $\mathsf{E}[X_s]$?
   c. For which values of s is $\mathsf{Var}[X_s] \in \mathbb{R}$? For these values of s, what is $\mathsf{Var}[X_s]$?

2. Approximately 1/6 of 6 year old school children have learned at least as much as the average 7 year old school child.*
   a. Assuming learning rates, *e.g.*, 7 years of material/6 years = 7/6 years per year, are approximately normally distributed, what fraction of 6 year olds have learned at least as much as the average 8 year old?
   b. Assuming the learning rate of each student is approximately constant, what fraction of 12 year olds have learned at least as much as the average 14 year old?

3. After a big win at the slot machine, a gambler stuffs his pockets with quarters and walks down the Las Vegas strip. Unfortunately, there is a hole in one pocket and the coins drop out randomly, at a rate of 5 per 50 meters. If you follow the gambler for 100 meters, what is the probability mass function for the number of quarters you find on the sidewalk, head up?

4. Let $X \sim \text{Unif}[0,1]$. Find the moment generating function $M(t)$ of X.

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* See M. C. Makel, M. S. Matthews, S. J. Peters, K. Rambo-Hernandez and J. A. Plucker, "How can so many students be invisible? Large percentages of American students perform above grade level", Institute for Education Policy, Johns Hopkins School of Education (2016). I've simplified the numbers for the purposes of this problem.