Solving second order linear ODEs with constant coefficients —using differential operators and their inverses

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A general second order linear ODE with constant coefficients has the form

$$y''(t) + py'(t) + qy(t) = g(t). \tag{1}$$

We first consider *homogeneous* equations, which means $g(t) \equiv 0$. In this case, writing (1) in terms of the differential operator D introduced in our discussion of first order linear ODEs gives

$$(D^2 + pD + q)[y(t)] = 0, \tag{2}$$

where D^2 is interpreted as

$$D^2[y(t)] = D\big[D[y(t)]\big] = D[y'(t)] = y''(t).$$

If we knew the inverse operator $(D^2 + pD + q)^{-1}$, we could apply it to both sides of (2) and get

$$y(t) = (D^2 + pD + q)^{-1}[0],$$

which would be a solution to (2). So we need to figure out the inverse operator for $D^2 + pD + q$.

This *second order* differential operator should remind us of the quadratic polynomials we studied in high school algebra, like

$$x^2 + px + q.$$

Remember that such polynomials can be factored:

$$x^2 + px + q = (x - r_1)(x - r_2), \tag{3}$$

where r_1 and r_2 are the *roots* (solutions) of the quadratic equation

$$x^2 + px + q = 0. \tag{4}$$

This suggests that we might be able to factor the second order differential operator $D^2 + pD + q$ as the product of *first order* differential operators: $(D - r_1)(D - r_2)$, where r_1 and

r_2 are the roots of (4), which is called the *characteristic equation* for (2). To check if this is true, we can compute:

$$\begin{aligned}(D-r_1)(D-r_2)[y(t)] &= (D-r_1)\big[(D-r_2)[y(t)]\big] \\ &= (D-r_1)[y'(t)-r_2y(t)] \\ &= D[y'(t)-r_2y(t)]-r_1[y'(t)-r_2y(t)] \\ &= y''(t)-r_2y'(t)-r_1y'(t)+r_1r_2y(t) \\ &= y''(t)-(r_1+r_2)y'(t)+r_1r_2y(t).\end{aligned}$$

Since multiplying out the right side of (3) tells us that $-(r_1+r_2)=p$ and $r_1r_2=q$, we can conclude that this factorization of the second order differential operator is right:

$$D^2+pD+q=(D-r_1)(D-r_2). \tag{5}$$

Now we can use (5) to rewrite (2) as

$$(D-r_1)(D-r_2)[y(t)]=0. \tag{6}$$

Since

$$(D-r_1)(D-r_2)[y(t)]=(D-r_1)\big[(D-r_2)[y(t)]\big],$$

we can solve (6) be using what we already know about the inverses of first order differential operators:

$$\begin{aligned}(D-r_2)[y(t)] &= (D-r_1)^{-1}[0] \\ y(t) &= (D-r_2)^{-1}\big[(D-r_1)^{-1}[0]\big].\end{aligned} \tag{7}$$

That is, since

$$\begin{aligned}(D-r_1)^{-1}[0] &= e^{r_1t}\int^t e^{-r_1s}[0]\mathrm{d}s \\ &= e^{r_1t}c,\end{aligned}$$

(7) becomes

$$\begin{aligned}y(t) &= (D-r_2)^{-1}[ce^{r_1t}] \\ &= e^{r_2t}\int^t e^{-r_2s}ce^{r_1s}\mathrm{d}s \\ &= e^{r_2t}\int^t ce^{(r_1-r_2)s}\mathrm{d}s.\end{aligned} \tag{8}$$

Now there are two cases, depending on whether $r_1=r_2$ or not.

If $r_1\neq r_2$, we evaluate the integral in (8) to get:

$$\begin{aligned}y(t) &= e^{r_2t}\Big(\frac{c}{r_1-r_2}e^{(r_1-r_2)t}+c_2\Big) \\ &= c_1e^{r_1t}+c_2e^{r_2t},\end{aligned} \tag{9}$$

where we have renamed the constant $c/(r_1 - r_2) = c_1$.

If $r_1 = r_2 = r$, the integral in (8) is even simpler:

$$\begin{aligned} y(t) &= e^{rt} \int^t c \mathrm{d}s \\ &= e^{rt}(ct + c_2) \\ &= c_1 t e^{rt} + c_2 e^{rt}, \end{aligned} \tag{10}$$

where we have renamed the constant $c = c_1$.

Thus, to solve (2) we find the roots r_1 and r_2 of the characteristic equation (4). If they are equal the general solution is (10); if they are different, the general solution is (9).

Sometimes the roots of the characteristic equation (4) are complex numbers $a \pm bi$. In this case (9) becomes

$$\begin{aligned} y(t) &= c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} \\ &= e^{at}\big(c_1 e^{ibt} + c_2 e^{-ibt}\big). \end{aligned} \tag{11}$$

We can eliminate the imaginary exponents in (11) by using *Euler's formula*:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Then

$$\begin{aligned} y(t) &= e^{at}\Big(c_1\big(\cos(bt) + i\sin(bt)\big) + c_2\big(\cos(-bt) + i\sin(-bt)\big)\Big) \\ &= e^{at}\big((c_1 + c_2)\cos(bt) + i(c_1 - c_2)\sin(bt)\big) \\ &= e^{at}\big(d_1 \cos(bt) + d_2 \sin(bt)\big), \end{aligned} \tag{12}$$

where we have renamed the constant $c_1 + c_2 = d_1$ and the constant $i(c_1 - c_2) = d_2$. (12) is the solution to (2) when the roots of the characteristic equation (4) are complex.