1.a. (15 points) Find the general solution to the equation $y' = 2y/x^3$. This equation is separable, *i.e.*,

$$\int \frac{\mathrm{d}y}{y} = \int \frac{2\mathrm{d}x}{x^3} \implies \ln|y| = -\frac{1}{x^2} + c \implies y = Ce^{-x^{-2}}.$$

b. (10 points) Find all solutions that satisfy the initial condition $y(1) = 1$.

$$1 = Ce^{-1^{-2}} \implies C = e \implies y = e^{1-x^{-2}}.$$

c. (10 points) Find all solutions that satisfy the initial condition $y(0) = 0$. If we define $y(0) = \lim_{x\to 0} y(x)$, then for all the solutions in 1.a, $y(0) = 0$. In fact, we can allow C to take different values for $t > 0$ and $t < 0$, so each element of the two parameter family of smooth functions:

$$y(t) = \begin{cases} C_1 e^{-x^{-2}} & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ C_2 e^{-x^{-2}} & \text{if } x < 0 \end{cases}$$

solves the equation and satisfies the initial condition.

2. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+y^2}{2xy}$.

   a. (5 points) What is the order of this equation? First.

   b. (5 points) Is this a linear differential equation? No.

   c. (20 points) Find the general solution to this equation. So we hope that it may be exact and write it as:

$$2x + y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Computing the partial derivatives:

$$\frac{\partial}{\partial y}(2x+y^2) = 2y = \frac{\partial}{\partial x}(2xy)$$

confirms that it is exact. Then

$$\frac{\partial \psi(x,y)}{\partial x} = 2x + y^2 \quad \Longrightarrow \quad \psi(x,y) = \int (2x+y^2)\mathrm{d}x = x^2 + xy^2 + c(y)$$

$$2xy = \frac{\partial \psi(x,y)}{\partial y} = 2xy + c'(y) \quad \Longrightarrow \quad c'(y) = 0 \quad \Longrightarrow \quad c(y) = c$$

$$\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + xy^2 + c) = 0 \quad \Longrightarrow \quad x^2 + xy^2 = k \quad \Longrightarrow \quad y = \pm\sqrt{x + k/x}.$$

3. Suppose $y_1(t) = e^{-t}\cos(3t)$ is a solution to a second order, linear, constant coefficient, homogeneous differential equation.

   a. (15 points) What is the general solution to this equation? We assume that the coefficients are real. In that case the roots of the characteristic equation are complex conjugates, $-1 \pm 3i$, and $y(t) = e^{-t}\big(c_1\cos(3t) + c_2\sin(3t)\big)$.

   b. (20 points) What is this equation? The characteristic equation must be $0 = \big(r - (-1+3i)\big)\big(r - (-1-3i)\big) = (r+1)^2 + 3^2 = r^2 + 2r + 10$. Thus the ODE must be $y'' + 2y' + 10y = 0$.

4. (Extra credit: 10 points) Let $y(x)$ be a differentiable function of a real variable x. Consider the two differential operators $D[\cdot]$ and $x[\cdot]$ defined by:

$$D[y] = y'$$
$$x[y] = xy.$$

Find the most general solution $y(x)$ to the equation:

$$(Dx - xD)[y] = x^3 e^x.$$

Hint: Remember that if $P[\cdot]$ and $Q[\cdot]$ are differential operators, $PQ[y] = P\big[Q[y]\big]$.

$(Dx - xD)[y] = D[xy] - xD[y] = xy' + y - xy' = y$, so $y = x^3 e^x$.