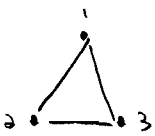
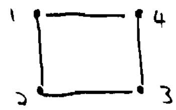
Homework #3

(1.2.17) (a) Compute the adjacency matrix A_T for a triangle and A_S for a square.

Note that the matrix you get depends on how you label the vertices (at least for the square).





$$A_T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(b) For each of these, compute the powers up to 5, and explain the meaning of the diagonal entries.

for A_T

$$\begin{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} & \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} & \begin{bmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{bmatrix} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ A_T & A_T^2 & A_T^3 & A_T^4 & A_T^5 \end{matrix}$$

$$\begin{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} & \begin{bmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} & \begin{bmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{bmatrix} & \begin{bmatrix} 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \\ 0 & 16 & 0 & 16 \\ 16 & 0 & 16 & 0 \end{bmatrix} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ A_S & A_S^2 & A_S^3 & A_S^4 & A_S^5 \end{matrix}$$

The diagonal entries of A^n give the number of ways that one can go from a point back to itself in exactly n steps.

(1.2.17 continued)

(c) For the triangle, observe that the diagonal terms differ by 1 from the off-diagonal terms. Can you prove that this will be true for all powers of A_T?

I claim that all powers of A_T look like $\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$ with $|a-b|=1$.

I will prove this by induction on the power of A_T. For the base case $n=1$ (that is, for A_T^1), we know from part (a) of the problem that A_T has this form. Now for the inductive step, assume that the claim holds for n, so that

$$A_T^n = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \qquad \text{with } |a-b|=1$$

Then A_T^{n+1} is the product of A^n and A:

$$\begin{array}{cc} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} & \begin{bmatrix} 2b & (a+b) & (a+b) \\ (a+b) & 2b & (a+b) \\ (a+b) & (a+b) & 2b \end{bmatrix} \end{array}$$

Note that $|2b-(a+b)| = |b-a| = |a-b| = 1$, so A_T^{n+1} has the desired form. Thus the inductive step works and the proof is complete.

(d) For the square, you should observe that half the terms are 0 for even powers, and the other half are 0 for odd powers. Can you prove that this will be true for all powers of A^S?

Again, we will prove this by induction on n. Specifically, we will show that for all $n \in \mathbb{N}$,

$$A_S^{2n+1} = \begin{bmatrix} 0 & a & 0 & a \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ a & 0 & a & 0 \end{bmatrix} \quad \text{and} \quad A_S^{2n+2} = \begin{bmatrix} b & 0 & b & 0 \\ 0 & b & 0 & b \\ b & 0 & b & 0 \\ 0 & b & 0 & b \end{bmatrix} \qquad a, b \in \mathbb{R} \setminus \{0\}$$

Again, this proof depends on my matrix for A_S, which may look slightly different from yours if you labeled the vertices differently. Also, I'm using the fact that $2n+2$ is always even and $2n+1$ is always odd for $n \in \mathbb{N}$.

For the base case, let $n=0$. Then A_S and A_S^2 have the desired forms by our work from part (b). For the inductive step, assume that the claim holds for n so

$$A_S^{2n+1} = \begin{bmatrix} 0 & a & 0 & a \\ a & 0 & a & 0 \\ 0 & a & 0 & a \\ a & 0 & a & 0 \end{bmatrix} \qquad A_S^{2n+2} = \begin{bmatrix} b & 0 & b & 0 \\ 0 & b & 0 & b \\ b & 0 & b & 0 \\ 0 & b & 0 & b \end{bmatrix}$$

(continues on next page) →

(1.2.17 (d) continued)

Then $A_5^{2(n+1)+1} = A_5^{2n+3} = A_5^{2n+2} \cdot A_5$, so it is given by the product

$$\begin{bmatrix} b & 0 & b & 0 \\ 0 & b & 0 & b \\ b & 0 & b & 0 \\ 0 & b & 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2b & 0 & 2b \\ 2b & 0 & 2b & 0 \\ 0 & 2b & 0 & 2b \\ 2b & 0 & 2b & 0 \end{bmatrix}}_{A_5^{2(n+1)+1}}$$

so $A_5^{2(n+1)+1}$ has the required form.

Similarly, $A_5^{2(n+1)+2} = A_5^{2(n+1)+1} \cdot A_5$, so it is given by the product

$$\begin{bmatrix} 0 & 2b & 0 & 2b \\ 2b & 0 & 2b & 0 \\ 0 & 2b & 0 & 2b \\ 2b & 0 & 2b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 4b & 0 & 4b & 0 \\ 0 & 4b & 0 & 4b \\ 4b & 0 & 4b & 0 \\ 0 & 4b & 0 & 4b \end{bmatrix}}_{A_5^{2(n+1)+2}}$$

so $A_5^{2(n+1)+2}$ has the required form.

Thus the claim holds for $n+1$ as well, so the proof is complete.

(e) Show that you can color the vertices of a connected graph (one on which you can walk from any vertex to any other) in two colors, such that no two adjacent vertices have the same color, if and only if for all sufficiently high powers n of the adjacency matrix, those entries that are 0 for A^n are nonzero for A^{n+1}, and those that are nonzero for A^n are zero for A^{n+1}.

There are two parts to this proof. First I must do the ($\Longrightarrow$) direction, in which I assume that the graph can be colored in the desired way and demonstrate the alternating zero/nonzero behavior in successive powers of A. Then I must prove that if the alternating behavior holds, then I can color the graph in the desired way. This is the ($\Longleftarrow$) direction of the proof.

($\Longrightarrow$) I will break up the proof into several smaller claims.

<u>Claim 1</u>: I can travel from any vertex i to any vertex j in an even number of steps or an odd number, but not both.

(continued) →

(1.2.17 (e) continued)

Proof of Claim 1: We know we can color the graph using two colors so that no two vertices of the same color are adjacent. Then every vertex is adjacent only to vertices of the opposite color. Hence, each step must change the color you are standing at. Thus any path with an even number of steps will end on the color you started with, while a path with an odd number of steps will end on the opposite color. Hence, no vertex is reachable in both an even and an odd number of steps.

Claim 2: If you can go from vertex i to vertex j in k steps, then you can also do it in $k+2$ steps.

Proof of Claim 2: Start by traveling from i to j in k steps. Then simply step of vertex j along any edge, and then step back. This adds two steps, giving a path from i to j of length $k+2$.

Now to bring it all together. Let m be the number of vertices of our graph. For any two vertices i and j, there is a path from vertex i to vertex j within m steps since our graph is connected. Then by claims 1 and 2, there is a path from vertex i to vertex j of length n for every other sequential value of n. Said another way, for $n \geq m$, let a_{ij}^n be the number of paths of length n from i to j. Then a_{ij}^n is nonzero either of all odds greater than m or all evens greater than m. Hence for $n \geq m$ (this is the n being sufficiently large part), those entries of A^n which are 0 will be nonzero in A^{n+1}, and vice versa. This completes the ($\Rightarrow$) direction of the proof.

($\Leftarrow$) Now we assume that for sufficiently large n, those entries which are 0 for A^n will be nonzero for A^{n+1}, and those entries which are nonzero for A^n are 0 for A^{n+1}. Using my notation from before, this means that if $a_{ij}^n = 0$ then $a_{ij}^{n+1} \neq 0$ and if $a_{ij}^n \neq 0$ then $a_{ij}^{n+1} = 0$ for all vertices i and j. We must show that we can color our graph using two colors so that no two adjacent vertices share a color.

Choose a vertex V_i (I will show later that the following construction does not depend on which vertex you pick. Let n be large enough to have the zero/nonzero behavior explained above, and let n be even. Look at the row of A^n corresponding to vertex i (so the i^{th} row). Then for all vertices, if $a_{ij}^n \neq 0$, color vertex j blue, and if $a_{ij}^n = 0$ then color vertex j red. Thus all vertices (including V_i itself) which are reachable via a path of even length will be blue and all others will be red. Note that I'm using Claim 2 when I say "reachable in a path of even length" instead of "in a path of length precisely n". The proof of Claim 2 still holds, so this is legitimate.

(continues) →

(1.2.17(e) continued, still $\Leftarrow$ direction)

Similarly, the red vertices will only be reachable from vertex i via paths of odd length. Now we must show that given this coloring, no two points of the same color are adjacent. (This is me showing that my construction is independent of the choice of i.

Note that Claim 1 also still holds. Since either $a_{ij}^{n}=0$ or $a_{ij}^{n+1}=0$ for all j and for all n sufficiently large, no vertex can have paths to i both of even and of odd length.

Let vertices j and k be adjacent. Let n be sufficiently large to have the explained zero/nonzero behavior, and such that $a_{ij}^{n}\neq 0$. Then there is a path of lenth n from vertes i to vertex j. But this means there is a path of length $n+1$ from vertex i to vertex k, since we can do the first n steps to vertex j, and then one more step to vertex k. Hence $a_{ik}^{n+1}\neq 0$, so $a_{ik}^{n}=0$. But this means that vertices j and k are not the same color ($a_{ij}^{n}\neq 0$ and $a_{ik}^{n}=0$). Hence no two adjacent vertices share a color, so the proof is complete.

(1.2.20) Associate to $z = x+iy \in \mathbb{C}$ the matrix $M_z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$; let $z_1, z_2 \in \mathbb{C}$. Show that $M_{z_1+z_2} = M_{z_1} + M_{z_2}$, and $M_{z_1 z_2} = M_{z_1} M_{z_2}$.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, so that $z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$ and
$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Then $M_{z_1+z_2} = \begin{bmatrix} (x_1+x_2) & (y_1+y_2) \\ -(y_1+y_2) & (x_1+x_2) \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = M_{z_1} + M_{z_2}$.

Also, $M_{z_1 z_2} = \begin{bmatrix} (x_1x_2 - y_1y_2) & (x_1y_2 + x_2y_1) \\ -(x_1y_2 + x_2y_1) & (x_1x_2 - y_1y_2) \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}\begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = M_{z_1} \cdot M_{z_2}$.

(1.2.21) An oriented walk of length n on an oriented graph consists of a sequence of vertices $v_0, v_1, \cdots, v_n$ such that v_i, v_{i+1} are, respectively, the beginning and end of an oriented edge.

(a) Show that if A is the oriented adjacency matrix of an oriented graph, then the (i,j) entry of A^n is the number of oriented walks of length n going from vertex i to vertex j.

Unless I've made a mistake, the proof of Proposition 1.2.23 covers this as well, word for word. I won't rewrite that proof here.

(b) What does it mean for the oriented adjacency matrix of an oriented graph to be upper triangular? To be lower triangular? Diagonal?

If the matrix is upper triangular, then we can only travel from a vertex v_i to vertices with greater (or equal) subscripts. In particular, we can never travel to another vertex and get back to our starting point. Also, if the diagonal entries are zero, we will eventually get stuck in vertex with no paths out.

If the matrix is lower triangular, the situation is similar. We can only decrease (or keep the same) the subscript of the vertex we're on, so we can't go to a different vertex and get back, and if the diagonals are zero, we'll eventually get stuck somewhere.

If the matrix is diagonal, then our vertices have no paths between them. At most, we may have paths from vertices to themselves, so wherever you start, you're stuck there.

(*) Use the formulae

$$\frac{d}{dx}(x)=1 \quad \text{and} \quad \frac{d}{dx}(fg) = f\cdot\frac{d}{dx}g + g\cdot\frac{d}{dx}f$$

to prove by mathematical induction that for all $n \in \mathbb{N}$

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

We will prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $n \in \mathbb{N}$. Our base case is when $n=1$. This is just the fact that $\frac{d}{dx}(x^1) = 1\cdot x^0 = 1$, which is given to us.

For the inductive step, assume that the claim holds for n, that is, assume

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Then we must show that the claim holds for $n+1$, that is,

$$\frac{d}{dx}(x^{n+1}) = (n+1)(x^n)$$

Now note that $x^{n+1} = x^n \cdot x$, and think of $f = x^n$ and $g = x$. Then

$$\begin{aligned}
\frac{d}{dx}(x^{n+1}) &= \frac{d}{dx}(x^n \cdot x) \\
&= x^n \cdot \frac{d}{dx}(x) = x\cdot\frac{d}{dx}(x^n) \\
&= x^n \cdot 1 + x \cdot nx^{n-1} \\
&= (n+1)x^n
\end{aligned}$$

This completes the inductive step and the proof is complete.

Section 1.3

(1.3.4):

(a) Let T be a linear transformation such that $T\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ v_2 \\ v_3 \end{bmatrix}$. What is its matrix?

We know from Theorem 1.3.4 that the i^{th} column of $[T]$ is $T(\vec{e}_i)$. Observe that

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This gives us the columns of $[T]$, so that

$$[T] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Let T be a linear transformation such that $T\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 + 2v_2 \\ v_3 + v_1 \end{bmatrix}$. What is its matrix?

As in part (a), observe that

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This gives us the columns of $[T]$, so that

$$[T] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(1.3.8)(a) Suppose you have a partner who is given the matrix of a linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^6$. He is not allowed to tell you what the matrix, but he can answer any question about what it does. In order to reconstitute the matrix, how many questions do you need answered? What are they?

First, let's figure out what the dimensions of this matrix are. The matrix sends vectors in $\mathbb{R}^5$ to vectors in $\mathbb{R}^6$, so the following matrix multiplication must make sense.

$$\underset{[T]}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}} \underset{\substack{5\times1 \\ \text{vector in } \mathbb{R}^5}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \underset{\substack{6\times1 \\ \text{vector in } \mathbb{R}^6}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}$$

Thus $[T]$ must be a 6×5 matrix in order for the multiplication to make sense. Now the question is a bit vague. If we are allowed to ask nonspecific questions, i.e. questions about where a general vector is sent, then simply asking where $[T]$ sends

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

we will be able to find $[T]$ with a single question. While I will take this as an answer, I believe that the question intended for us to only ask where <u>specific</u> vectors get sent. Then recall that the i^{th} column of $[T]$ is just $[T]\vec{e}_i$, that is, T applied to the i^{th} standard basis vector. Then by asking the five questions

$$T(\vec{e}_1) = ? \qquad T(\vec{e}_4) = ?$$
$$T(\vec{e}_2) = ? \qquad T(\vec{e}_5) = ?$$
$$T(\vec{e}_3) = ?$$

We will be able to recover $[T]$. The reason that five is the minimal number of specific questions we can ask to build $[T]$ requires the definition of a basis, something that we don't actually know yet, so don't worry about that. We'll revisit it later.

(b) Repeat for a linear transformation $T: \mathbb{R}^6 \times \mathbb{R}^5$.

By similar logic, $[T]$ will be a 5×6 matrix, so it will have six columns. Thus, using specific questions, we can build $[T]$ with the six questions

$$T(\vec{e}_1) = ? \qquad T(\vec{e}_4) = ?$$
$$T(\vec{e}_2) = ? \qquad T(\vec{e}_5) = ?$$
$$T(\vec{e}_3) = ? \qquad T(\vec{e}_6) = ?$$

(part c on next page) $\longrightarrow$

(1.3.8)(c) (problem continued)

In each case, is there only one right answer to the question "what are they"?

No, there are many sets of questions one could ask. I will only answer this for the 6×5 matrix in part (a). We said that we can build $[T]$ if we know

$$T(\vec{e}_1) \quad T(\vec{e}_4)$$
$$T(\vec{e}_2) \quad T(\vec{e}_5) \qquad (*)$$
$$T(\vec{e}_3)$$

Instead, we could ask

$$T(2\vec{e}_1) = ? \quad T(2\vec{e}_4) = ?$$
$$T(2\vec{e}_2) = ? \quad T(2\vec{e}_5) = ? \qquad (**)$$
$$T(2\vec{e}_3) = ?$$

Then by the linearity of T, note that $T(2\vec{e}_i) = 2 \cdot T(\vec{e}_i)$ for each i. Thus, we could figure out the information in $(*)$ from the answers given in $(**)$, but it would take a little work. This was just one alternate choice of questions; there are many, many other sets of 5 questions that would work.

(1.3.9) Let T be a transformation such that evaluated on the following five vectors it gives

$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\1\\1\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix},\quad T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix},\quad T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}4\\3\\3\end{bmatrix},\quad T\begin{bmatrix}2\\-1\\4\end{bmatrix}=\begin{bmatrix}7\\0\\4\end{bmatrix}$$

Is T linear? Justify your answer.

No, T is not linear. Observe that

$$T\begin{bmatrix}2\\-1\\4\end{bmatrix}=\begin{bmatrix}7\\0\\4\end{bmatrix}\neq\begin{bmatrix}7\\0\\5\end{bmatrix}=2\begin{bmatrix}2\\1\\1\end{bmatrix}-1\cdot\begin{bmatrix}1\\2\\1\end{bmatrix}+4\begin{bmatrix}1\\0\\1\end{bmatrix}=2\cdot T\begin{bmatrix}1\\0\\0\end{bmatrix}-T\begin{bmatrix}0\\1\\0\end{bmatrix}+4T\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

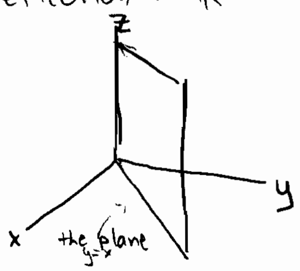
If T were linear, this would need to be an equal sign since then

$$T(a\vec{u}+b\vec{v}) = aT(\vec{u})+bT(\vec{v})$$

for all vectors $\vec{u},\vec{v}$ and scalars a,b.

(1.3.12)

(a) What is the matrix of the linear transformation $S:\mathbb{R}^3\to\mathbb{R}^3$ corresponding to reflection in the plane of equation $x=y$? What is the matrix corresponding to reflection $T:\mathbb{R}^3\to\mathbb{R}^3$ in the plane $y=z$? What is the matrix of $S\circ T$?



Reflecting across $y=x$ performs the following:

$$S\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad S\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad S\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

since it switches x and y and leaves z alone.

Thus $[S]$ is given by $[S]=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

By similar reasoning, $[T]$ is given by $[T]=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$.

The matrix of $S\circ T$ is just performing T first, then S, so $[S\circ T]=[S]\cdot[T]$ by Theorem 1.3.10. Hence

$$\begin{array}{cc} & \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\\ \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} & \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\end{array}$$

so $[S\circ T]=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$.

part (b) on next page. →

(1.3, 12 continued)

(b) What is the relationship between $[S\circ T]$ and $[T\circ S]$?

In part (a) we calculated that $[S\circ T] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Now we calculate $[T\circ S]$.

$$\begin{array}{cc} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

so $[T\circ S] = [T]\cdot[S] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

So now what is the relationship between $[T\circ S]$ and $[S\circ T]$? They're definitely different matrices, but if we take their product (in both orders) we'll get the identity. Thus $[S\circ T]$ and $[T\circ S]$ are inverses of each other.

$$\begin{array}{cc} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Do you see how we could have predicted this in advance?

(1.3.13) Let $A: \mathbb{R}^n \to \mathbb{R}^m$, $B: \mathbb{R}^m \to \mathbb{R}^k$, and $C: \mathbb{R}^k \to \mathbb{R}^n$ be linear transformations with k, m, n all different. Which of the following make sense? For those that make sense, give the domain and codomain of the composition.

In order for a composition $S \circ T$ to make sense, the codomain of the first linear transformation (the one of the right) must be the domain of the next transformation. Thus B can follow A, and C can follow B, and A can follow C. All other orderings are nonsense.

(a) $A \circ B$ nonsense (reasons given above)

(b) $C \circ B$ makes sense

$$\mathbb{R}^m \xrightarrow{B} \mathbb{R}^k \xrightarrow{C} \mathbb{R}^n$$

So the domain of $C \circ B$ is $\mathbb{R}^m$ and the codomain is $\mathbb{R}^n$.
I won't draw these pictures for the rest; the logic is the same.

(c) $A \circ C$ makes sense domain: $\mathbb{R}^k$
codomain: $\mathbb{R}^m$

(d) $B \circ A \circ C$ makes sense domain: $\mathbb{R}^k$
codomain $\mathbb{R}^k$

(e) $C \circ A$ nonsense

(f) $B \circ C$ nonsense

(g) $B \circ A$ makes sense domain: $\mathbb{R}^n$
codomain: $\mathbb{R}^k$

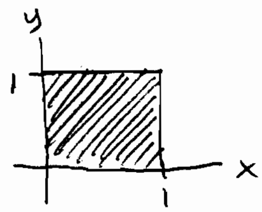
(h) $A \circ C \circ B$ makes sense domain: $\mathbb{R}^m$
codomain: $\mathbb{R}^m$

(i) $C \circ B \circ A$ makes sense domain: $\mathbb{R}^n$
codomain: $\mathbb{R}^n$

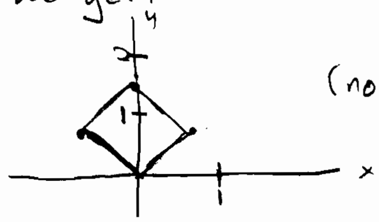
(j) $B \circ C \circ A$ nonsense

(1.3.18) Rotate the unit square by 45° counterclockwise, then stretch it using the linear transformation $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Sketch the result.

We start with the unit square:



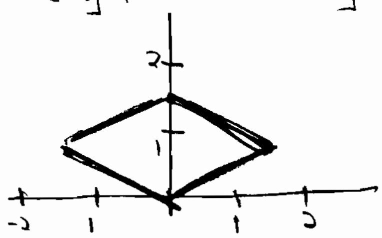
Rotating this by 45° counterclockwise, we get:



(note the rescaling)

Then we act by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. By Theorem 1.3.4, this matrix sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus our matrix stretches the x-coordinate by a factor of 2 and leaves the y-coordinate alone. Hence we get something like the flattened parallelogram:



(1.3.19) If A and B are $n \times n$ square matrices, their Jordan product is $\frac{AB+BA}{2}$. Show that this product is commutative but not associative.

I'm going to create my own notation here Let $J(A,B) = \frac{AB+BA}{2}$ be the Jordan product of A and B. For real numbers a and b (with multiplication), saying that a and b commute means saying that

$$a \cdot b = b \cdot a$$

So for the Jordan product, showing commutativity means we must show

$$J(A, B) = J(B, A)$$

Computing both sides we get

$$J(A,B) = \frac{AB+BA}{2} = \frac{BA+AB}{2} = J(B,A)$$

so the Jordan product is commutative.

For real numbers a, b, c, associativity means

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Thus, by parallel, we check whether

$$J(J(A,B), C) = J(A, J(B,C))$$

and show that it doesn't hold.

For the left side:

$$J(J(A,B),C) = J\left(\frac{AB+BA}{2}, C\right)$$

$$= \frac{ABC + BAC + CAB + CBA}{4}$$

not equal

The right hand side is:

$$J(A, J(B,C)) = J\left(A, \frac{BC+CB}{2}\right)$$

$$= \frac{ABC + ACB + BCA + CBA}{4}$$

These are not the same, so the Jordan product is not associative.

(1.3.20) Identify $\mathbb{R}^2$ to $\mathbb{C}$ by identifying $\binom{a}{b}$ to $z=a+ib$. Show that the following mappings $\mathbb{C} \to \mathbb{C}$ are linear transformations and give their matrices

(a) $Re: z \longmapsto Re(z)$ (the real part of z)

For linearity I will let $z=a+ib$ and $z'=c+id$ be two general complex numbers which I will freely associate with their vectors in $\mathbb{R}^2$, $z=\binom{a}{b}$, $z=\binom{c}{d}$. Let α be a scalar. (Note this means $\alpha \in \mathbb{R}$, since $\mathbb{R}^2$ is a real vector space). Note that $Re\binom{a}{b}=\binom{a}{0}$ since the real part of $a+ib$ is $a+i\cdot 0$.

Check linearity:

$$(1)\quad Re(z+z') = Re\left(\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right) = Re\left(\begin{pmatrix}a+c\\b+d\end{pmatrix}\right) = \begin{pmatrix}a+c\\0\end{pmatrix} = \begin{pmatrix}a\\0\end{pmatrix}+\begin{pmatrix}c\\0\end{pmatrix} = Re(z)+Re(z')$$

$$(2)\quad Re(\alpha\cdot z) = Re\left(\alpha\cdot\begin{pmatrix}a\\b\end{pmatrix}\right) = Re\left(\begin{pmatrix}\alpha a\\ \alpha b\end{pmatrix}\right) = \begin{pmatrix}\alpha a\\0\end{pmatrix} = \alpha\begin{pmatrix}a\\0\end{pmatrix} = \alpha\cdot Re\begin{pmatrix}a\\b\end{pmatrix} = \alpha\cdot Re(z)$$

Thus Re is linear.

To find the matrix of Re, we recall Theorem 1.3.4, which says that the first column will be $Re(\vec{e}_1) = Re\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ and the second column will be $Re(\vec{e}_2) = Re\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Thus $[Re] = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

I will not show this much detail on the later parts.

(b) $\mathcal{I}: z \longmapsto Im(z)$ (the imaginary part of z)

This map sends $\binom{a}{b} \longmapsto \binom{b}{0}$ since the imaginary part of $a+bi$ is $b+i0$.

Check linearity:

$$(1)\quad \mathcal{I}(z+z') = \mathcal{I}\left(\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right) = \begin{pmatrix}b+d\\0\end{pmatrix} = \begin{pmatrix}b\\0\end{pmatrix}+\begin{pmatrix}d\\0\end{pmatrix} = \mathcal{I}(z)+\mathcal{I}(z')$$

$$(2)\quad \mathcal{I}(\alpha z) = \mathcal{I}\left(\alpha\begin{pmatrix}a\\b\end{pmatrix}\right) = \mathcal{I}\left(\begin{pmatrix}\alpha a\\ \alpha b\end{pmatrix}\right) = \begin{pmatrix}\alpha b\\0\end{pmatrix} = \alpha\begin{pmatrix}b\\0\end{pmatrix} = \alpha\,\mathcal{I}(z)$$

Then as in part (a), we construct the matrix by action on standard basis vectors to get

$$[\mathcal{I}] = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

(continues on next page) →

(1.3.20 continued) ...

(c) $\mathcal{C}: z \mapsto \bar{z}$ (the complex conjugate of z, i.e. $\bar{z} = a - ib$ if $z = a + ib$)

Thus $\mathcal{C}$ will send $\begin{pmatrix} a \\ b \end{pmatrix}$ to $\begin{pmatrix} a \\ -b \end{pmatrix}$ as explained above. (I changed the look of my map to $\mathcal{C}$ to prevent confusion).

Check linearity:

(1) $\mathcal{C}(z+z') = \mathcal{C}\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = \mathcal{C}\left(\begin{pmatrix} a+c \\ b+d \end{pmatrix}\right) = \begin{pmatrix} a+c \\ -b-d \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} + \begin{pmatrix} c \\ -d \end{pmatrix} = \mathcal{C}(z) + \mathcal{C}(z')$

(2) $\mathcal{C}(\alpha z) = \mathcal{C}\left(\alpha \begin{pmatrix} a \\ b \end{pmatrix}\right) = \mathcal{C}\left(\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}\right) = \begin{pmatrix} \alpha a \\ -\alpha b \end{pmatrix} = \alpha \begin{pmatrix} a \\ -b \end{pmatrix} = \alpha \mathcal{C}(z).$

And we construct the matrix using Theorem 1.3.4 as before:

$$[\mathcal{C}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) $m_w: z \longmapsto wz$ where $w = u + iv$ is a fixed complex number

Let's see where this sends $z = \begin{pmatrix} a \\ b \end{pmatrix}$. Since $z = a + ib$, then

$$\begin{aligned} wz &= (u+iv)(a+ib) \\ &= (ua - vb) + i(va + ub) \end{aligned}$$

Thus $m_w(z) = \begin{pmatrix} ua - vb \\ va + ub \end{pmatrix}$

Check linearity:

(1) $m_w(z+z') = m_w\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right) = m_w\left(\begin{pmatrix} a+c \\ b+d \end{pmatrix}\right) = \begin{pmatrix} u(a+c) - v(b+d) \\ v(a+c) + u(b+d) \end{pmatrix}$

✓

$$= \begin{pmatrix} ua - vb \\ va + ub \end{pmatrix} + \begin{pmatrix} uc - vd \\ vc + ud \end{pmatrix}$$

$$= m_w(z) + m_w(z')$$

(2) $m_w(\alpha z) = m_w\left(\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}\right) = \begin{pmatrix} u\alpha a - v\alpha b \\ v\alpha a + u\alpha b \end{pmatrix} = \alpha \cdot \begin{pmatrix} ua - vb \\ va + ub \end{pmatrix} = \alpha \cdot m_w(z)$

Now construct the matrix using Theorem 1.3.4 to get

$$[m_w] = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}$$