

Problem Set #4

0.49 Evaluate $(f \circ g \circ h)(x)$ at $x=a$ for the following

(a) $f(x) = x^2 - 1$, $g(x) = 3x$, $h(x) = -x + 2$, for $a = 3$

$$\begin{aligned}(f \circ g \circ h)(a) &= f(g(h(3))) = f(g(-3+2)) \\ &= f(g(-1)) \\ &= f(-3) \\ &= (-3)^2 - 1 \\ &= 8\end{aligned}$$

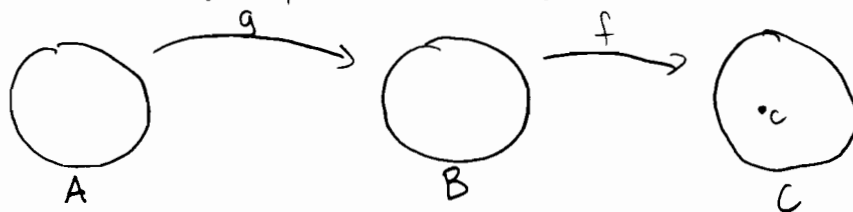
(b) $f(x) = x^2$, $g(x) = x - 3$, $h(x) = x - 3$, for $a = 1$

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(1))) = f(g(1-3)) \\ &= f(g(-2)) \\ &= f(-2-3) \\ &= f(-5) \\ &= (-5)^2 \\ &= 25\end{aligned}$$

(0.4.10) (a) Prove Proposition 0.4.16.

Let the functions $f: B \rightarrow C$ and $g: A \rightarrow B$ be onto. Then the composition $(f \circ g)$ is onto.

Let's spell out what we know and what we need to show



The function $f: B \rightarrow C$ is onto, so for every $c \in C$, there is a $b \in B$ such that $f(b) = c$. Note that I am not claiming that b is unique, merely that it must exist. Likewise, $g: A \rightarrow B$ is onto, so a similar statement applies.

We want to show that $f \circ g: A \rightarrow C$ is onto, so we must show that for any $c \in C$, there is an $a \in A$ such that $(f \circ g)(a) = c$.

Then let $c \in C$. Since f is onto, there is a $b \in B$ such that $f(b) = c$. For the b , the fact that g is onto means that there is an $a \in A$ such that $g(a) = b$. I claim that this is the a we want, since

$$(f \circ g)(a) = f(g(a)) = f(b) = c.$$

(b) Prove Proposition 0.4.17:

Let $f: B \rightarrow C$ and $g: A \rightarrow B$ be one to one. Then the composition $(f \circ g)$ is one to one.

Saying that $f: B \rightarrow C$ is one-to-one means that for every $c \in C$, there is at most one $b \in B$ such that $f(b) = c$. This is equivalent to saying that if $f(b) = f(b')$, then $b = b'$. We have a similar definition for g .

We want to show that $(f \circ g)$ is one-to-one, so I will show that if $(f \circ g)(a) = (f \circ g)(a')$, then $a = a'$.

Suppose that $(f \circ g)(a) = (f \circ g)(a')$ for some $a, a' \in A$. Then $f(g(a)) = f(g(a'))$, so $g(a) = g(a')$ since f is one to one. But then $g(a) = g(a')$ implies that $a = a'$ since g is one to one. Thus $(f \circ g)(a) = (f \circ g)(a')$ implies that $a = a'$, which is equivalent to the book's definition of $(f \circ g)$ being one-to-one.

(1.4.3) Normalize the following vectors.

(a) $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ $|\vec{v}| = \sqrt{0^2 + 1^2 + 4^2} = \sqrt{17}$ So normalized vector is $\frac{1}{\sqrt{17}} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ $|\vec{v}| = \sqrt{(-3)^2 + 7^2} = \sqrt{58}$ Normalized vector is $\frac{1}{\sqrt{58}} \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{-3}{\sqrt{58}} \\ \frac{7}{\sqrt{58}} \end{bmatrix}$

(c) $\vec{v} = \begin{bmatrix} \sqrt{2} \\ -2 \\ -5 \end{bmatrix}$ $|\vec{v}| = \sqrt{(\sqrt{2})^2 + (-2)^2 + (-5)^2} = \sqrt{31}$ Normalized vector is $\frac{1}{\sqrt{31}} \begin{bmatrix} \sqrt{2} \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{31}} \\ \frac{-2}{\sqrt{31}} \\ \frac{-5}{\sqrt{31}} \end{bmatrix}$

(1.4.5) Calculate the angles between the following pairs of vectors.

Recall that $\cos(\alpha) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|}$ where α is the angle between \vec{v} and \vec{w} with $0 \leq \alpha \leq \pi$.

(a) $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v} \cdot \vec{w} = 1 + 0 + 0 = 1$ so $\cos(\alpha) = \frac{1}{\sqrt{3}}$
 $|\vec{v}| = \sqrt{1^2 + 0^2 + 0^2} = 1$ $\alpha \approx 54.7^\circ$ or .955 radians
 $|\vec{w}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

(b) $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v} \cdot \vec{w} = 1 + 0 + (-1) + 0 = 0$ so $\cos(\alpha) = 0$
 $|\vec{v}| = \sqrt{1^2 + 0^2 + (-1)^2 + 0^2} = \sqrt{2}$ $\alpha = 90^\circ$ or $\frac{\pi}{2}$ radians
 $|\vec{w}| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$

(1.4.7) For each of the following matrices, compute its determinant and its inverse if it exists

Recall that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det A = ad - bc, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(provided $ad - bc \neq 0$)

(a) $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det A = 2 \cdot 0 - (-1)(1) = 1$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\det B = 1 \cdot 1 - 1 \cdot 1 = 0$$

B^{-1} does not exist

(c) $C = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

$$\det C = ad - 0b = ad$$

$$C^{-1} = \frac{1}{ad} \begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}, \quad \text{provided } ad \neq 0.$$

(d) $D = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$$\det D = (1)(-1) - (1)(-1) = 0$$

D^{-1} does not exist

(1.4.8) Compute the determinants of the matrices in the margin at left.

$$(a) A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1 \cdot \det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= 0 + (4 - 6) + 2(2 - 3) \\ &= -4 \end{aligned}$$

$$(b) B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

$$\begin{aligned} \det B &= a \cdot \det \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} - 0 \cdot \det \begin{bmatrix} b & c \\ 0 & f \end{bmatrix} + 0 \cdot \det \begin{bmatrix} b & c \\ d & e \end{bmatrix} \\ &= adf + 0 + 0 \\ &= adf \end{aligned}$$

$$(c) C = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{bmatrix}$$

$$\begin{aligned} \det C &= a \cdot \det \begin{bmatrix} d & 0 \\ f & g \end{bmatrix} - c \cdot \det \begin{bmatrix} b & 0 \\ f & g \end{bmatrix} + e \cdot \det \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} \\ &= a(dg - 0) - c(bg - 0) + e(0 - 0) \\ &= adg - cbg \end{aligned}$$

(1.4.10)

(a) Let A be a matrix, show that $|A^k| \leq |A|^k$. Compute $|A^3|$ and $|A|^3$ for $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

We know from Proposition 1.4.11 that $|BC| \leq |B||C|$ for any matrices B and C for which the product BC makes sense. Letting $B=A$ and $C=A$, this tells us that $|A^2| \leq |A|^2$. We will use this fact to prove the claim by induction.

Base case: $k=1$

This case is obvious since $|A^1| = |A| = |A|^1$.

Inductive step: Assume that $|A^k| \leq |A|^k$. We will show that $|A^{k+1}| \leq |A|^{k+1}$.

Note that $A^{k+1} = (A^k) \cdot A$. Then

$$|A^{k+1}| = |A^k \cdot A| \leq |A^k| \cdot |A| \leq |A|^k \cdot |A| = |A|^{k+1}$$

by Prop. 1.4.11

by inductive hypothesis

This completes the proof.

For the second part,

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}$$

$A^2 \rightarrow$ $A^3 \rightarrow$

$$\begin{aligned} \text{thus } |A^3| &= \sqrt{5^2 + 7^2 + 10^2} \\ &= \sqrt{223} \quad (\text{slightly less than } 15) \end{aligned}$$

Meanwhile, $|A| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{7}$, so

$$|A|^3 = (\sqrt{7})^3 = \sqrt{7} \cdot \sqrt{7} \cdot \sqrt{7} = \sqrt{7 \cdot 7 \cdot 7} = \sqrt{343}$$

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$. Without computations, explain why the following are true or false:

(i) $|\vec{u} \cdot \vec{v}| < \|\vec{u}\| \|\vec{v}\|$

Schwarz's inequality says that $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ with equality if and only if one of \vec{u}, \vec{v} is a multiple of the other. Since $\vec{v} = -2\vec{u}$, this implies that $|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\|$ is true.

(ii) $|\vec{u} \cdot \vec{w}| = \|\vec{u}\| \|\vec{w}\|$

Note that \vec{u} and \vec{w} are not multiples of each other, so by Schwarz's inequality the claim is false. (continues) \rightarrow

(1.4.10 continued)

(c) Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be two vectors such that $v_1 w_2 < v_2 w_1$. Does \vec{w} lie clockwise or counterclockwise from \vec{v} ?

From Proposition 1.4.14(2) we know that $\det[\vec{a}, \vec{b}]$ is positive if and only if \vec{a} is clockwise from \vec{b} . It is negative if and only if \vec{a} lies counterclockwise from \vec{b} .

Note that $v_1 w_2 < v_2 w_1$ means $v_1 w_2 - v_2 w_1 < 0$. But $v_1 w_2 - v_2 w_1 = \det[\vec{v}, \vec{w}]$. Thus $\det[\vec{v}, \vec{w}]$ is negative, so \vec{v} lies counterclockwise from \vec{w} , so \vec{w} lies clockwise from \vec{v} .

(1.4.13) Show that the cross product of two vectors pointing in the same direction is zero.

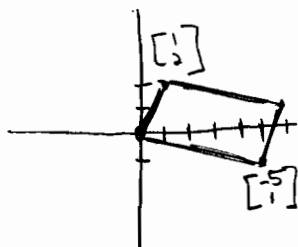
When we say that the cross product is zero, recall that the cross product is a vector, so we are really being asked to show that the cross product is $\vec{0}$.

Two vectors point in the same direction precisely when one is a multiple of the other. Therefore let our two vectors be

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{and} \quad \alpha \vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{bmatrix}$$

$$\text{Then } \vec{v} \times \alpha \vec{v} = \begin{bmatrix} v_2 \alpha v_3 - v_3 \alpha v_2 \\ -v_1 \alpha v_3 + v_3 \alpha v_1 \\ v_1 \alpha v_2 - v_2 \alpha v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \quad \text{So the proof is complete.}$$

(1.4.16) What is the area of the parallelogram with vertices at $(0, 0)$, $(1, 2)$, $(5, -1)$, $(6, 1)$?



This is the parallelogram spanned the vectors $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

By Proposition 1.4.14, the area of this parallelogram is equal to $|\det[\vec{a}, \vec{b}]|$

$$\text{Thus Area} = |\det[\vec{a}, \vec{b}]| = \left| \det \begin{bmatrix} 1 & -5 \\ 2 & 1 \end{bmatrix} \right| = |1 \cdot 1 - (-5)(2)| = |11| = 11.$$

(1.4.19) (a) What is the length of $\vec{v}_n = \vec{e}_1 + \dots + \vec{e}_n \in \mathbb{R}^n$.

First recall that for any vector \vec{w} , the length of \vec{w} is $|\vec{w}| = \sqrt{\vec{w} \cdot \vec{w}}$. Next observe that

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\begin{aligned} \text{Thus } \vec{v}_n \cdot \vec{v}_n &= (\vec{e}_1 + \dots + \vec{e}_n) \cdot (\vec{e}_1 + \dots + \vec{e}_n) \\ &= \vec{e}_1 \cdot \vec{e}_1 + \dots + \vec{e}_n \cdot \vec{e}_n \quad (\text{other terms are } 0) \\ &= \underbrace{1 + \dots + 1}_{n \text{ times}} \\ &= n \end{aligned}$$

$$\text{Therefore } |\vec{v}_n| = \sqrt{n}$$

(b) What is the angle α_n between \vec{v}_n and \vec{e}_1 ? What is $\lim_{n \rightarrow \infty} \alpha_n$?

Let α_n be the angle between \vec{v}_n and \vec{e}_1 . Then by Definition 1.4.6,

$$\cos(\alpha_n) = \frac{\vec{v}_n \cdot \vec{e}_1}{|\vec{v}_n| |\vec{e}_1|}$$

We know from part (a) that $|\vec{v}_n| = \sqrt{n}$. Also,

$$\vec{v}_n \cdot \vec{e}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1 \quad \text{and } |\vec{e}_1| = \text{square root of } \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sqrt{1} = 1$$

$$\text{Thus } \cos(\alpha_n) = \frac{1}{\sqrt{n}}, \text{ so } \alpha_n = \arccos\left(\frac{1}{\sqrt{n}}\right)$$

Taking the limit as $n \rightarrow \infty$ of both sides, we get

$$\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \arccos\left(\frac{1}{\sqrt{n}}\right) = \arccos\left(\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}}\right) = \arccos(0) = 90^\circ \text{ or } \frac{\pi}{2} \text{ radians}$$

I need to know that \arccos is continuous, at least in the region $(0,1)$ of the domain, in order to pull the limit inside the arccosine.

(1.4.24) Let $\vec{v} \in \mathbb{R}^n$ be a nonzero vector, and denote by $\vec{v}^\perp \subset \mathbb{R}^n$ the set of vectors $\vec{w} \in \mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$, so $\vec{v}^\perp = \{ \vec{w} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \}$.

(a) Show that \vec{v}^\perp is a subspace of \mathbb{R}^n .

We must show that \vec{v}^\perp is closed under addition and scalar multiplication.

Let $\alpha \in \mathbb{R}$ and $\vec{w}_1, \vec{w}_2 \in \vec{v}^\perp$, so that $\vec{v} \cdot \vec{w}_1 = \vec{v} \cdot \vec{w}_2 = 0$. Then

$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2 = 0 + 0 = 0 \quad \text{so } (\vec{w}_1 + \vec{w}_2) \in \vec{v}^\perp$$

$$\vec{v} \cdot (\alpha \vec{w}_1) = \alpha (\vec{v} \cdot \vec{w}_1) = \alpha \cdot 0 = 0 \quad \text{so } \alpha \vec{w}_1 \in \vec{v}^\perp$$

Thus \vec{v}^\perp is closed under addition and scalar multiplication, so it is a subspace of \mathbb{R}^n .

(b) Given any vector $\vec{a} \in \mathbb{R}^n$, show that $\vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$ is an element of \vec{v}^\perp .

All that we need to show is that

$$\vec{v} \cdot \left(\vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \right) = 0$$

Note that $\frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2}$ is a constant and that $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ by definition. Then

$$\begin{aligned} \vec{v} \cdot \left(\vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \right) &= \vec{v} \cdot \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} (\vec{v} \cdot \vec{v}) \\ &= \vec{v} \cdot \vec{a} - \vec{a} \cdot \vec{v} \\ &= 0 \end{aligned}$$

(c) Define the projection of \vec{a} onto \vec{v}^\perp by the formula

$$P_{\vec{v}^\perp}(\vec{a}) = \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

Show that there is a unique number $t(\vec{a})$ such that $(\vec{a} + t(\vec{a})\vec{v}) \in \vec{v}^\perp$, and show that $\vec{a} + t(\vec{a})\vec{v} = P_{\vec{v}^\perp}(\vec{a})$.

We already know that $t(\vec{a}) = -\frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2}$ works from our work in part (b). Now to show that this is the unique such $t(\vec{a})$, suppose that $(\vec{a} + t(\vec{a})\vec{v}) \in \vec{v}^\perp$, so that $(\vec{a} + t(\vec{a})\vec{v}) \cdot \vec{v} = 0$. Then

$$\begin{aligned} \vec{a} \cdot \vec{v} + t(\vec{a}) \vec{v} \cdot \vec{v} &= 0 \\ t(\vec{a}) \vec{v} \cdot \vec{v} &= -\vec{a} \cdot \vec{v} \\ t(\vec{a}) &= \frac{-\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \end{aligned}$$

Thus this is the only solution. Plugging in for $t(\vec{a})$, we do indeed find that

$$\vec{a} + t(\vec{a})\vec{v} = \vec{a} - \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = P_{\vec{v}^\perp}(\vec{a}).$$

(1.4.26) Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

(a) What is the angle $\alpha(y)$ between $\begin{bmatrix} x \\ y \end{bmatrix}$ and $A\begin{bmatrix} x \\ y \end{bmatrix}$?

Note that $\begin{bmatrix} x \\ y \end{bmatrix}$ and $A\begin{bmatrix} x \\ y \end{bmatrix}$ are both vectors so we may ask for the angle between them, but this angle will depend on $\begin{bmatrix} x \\ y \end{bmatrix}$, hence the notation $\alpha\left[\begin{bmatrix} x \\ y \end{bmatrix}\right]$.

Explicitly, $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 3x+4y \end{bmatrix}$.

For general vectors \vec{v} and \vec{w} , $\alpha = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right)$, where α is the angle between them.

For us, let $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x-2y \\ 3x+4y \end{bmatrix}$. Then

$$\begin{aligned} |\vec{v}| &= \sqrt{x^2 + y^2} \\ |\vec{w}| &= \sqrt{(x-2y)^2 + (3x+4y)^2} = \sqrt{10x^2 + 20xy + 20y^2} \\ \vec{v} \cdot \vec{w} &= x^2 - 2xy + 3xy + 4y^2 = x^2 + xy + 4y^2 \end{aligned}$$

Thus $\alpha\left[\begin{bmatrix} x \\ y \end{bmatrix}\right] = \arccos\left(\frac{x^2 + xy + 4y^2}{\sqrt{x^2 + y^2} \cdot \sqrt{10x^2 + 20xy + 20y^2}}\right)$

I didn't find any way to simplify this nicely, so I'll leave it in this form.

(b) Is there any nonzero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ that is rotated by $\pi/2$?

If we plug in $\alpha\left[\begin{bmatrix} x \\ y \end{bmatrix}\right] = \pi/2$ and take the cosine of both sides, then since $\cos(\pi/2) = 0$, we get

$$\begin{aligned} 0 &= \frac{x^2 + xy + 4y^2}{\sqrt{x^2 + y^2} \cdot \sqrt{10x^2 + 20xy + 20y^2}} \\ \text{so that } 0 &= x^2 + xy + 4y^2 \end{aligned}$$

We can solve this in a number of ways. Using the quadratic formula (so solving for x in terms of y) we get

$$x = \frac{-y \pm \sqrt{y^2 - 16y^2}}{2}$$

The term under the square root is negative for all y , so there are no solutions. Hence, no vector is rotated by the angle $\pi/2$ when A is applied to it.