Problem Set 6

(2.2.3) (a) Confirm the solution for exercise 2.2.2(e) without using row reduction

Exercise 2.2.2(e) predicted that the system of equations

$$\begin{aligned} x+2y+z-4w+v&=0\\ x+2y-z+2w-v&=0\\ 2x+4y+z-5w+v&=0\\ x+2y+3z-10w+2v&=0 \end{aligned}$$

has infinitely many solutions.

I see two obvious ways to approach this problem without "row reduction". Either we can manipulate the equations by hand, effectively doing row reduction but without matrices, or we can use Theorem 2.2.1. I will do the latter.

Without actually doing the row reduction, note that the $\vec{b}$ vector in our matrix equation $A\vec{x}=\vec{b}$ consists only of zeros, so in echelon form, $\hat{b}$ cannot be a pivot column of $[\tilde{A}|\tilde{b}]$. Thus by the theorem, our system cannot have no solutions. Looking at $\tilde{A}$, note that there are a maximum of 4 pivots since we only have 4 rows. Hence not every column of $\tilde{A}$ has a pivot (there are 5 columns), so there is not a unique solution. Thus, there are infinitely many solutions.

(b) On how many parameters does the family of solutions for 2.2.2(e) depend?

We need the system in echelon form. Since it was specified not to use row operations, I will just add the equations to each other to reduce it that way.

$$\begin{aligned} x+2y+z-4w+v&=0\\ x+2y-z+2w-v&=0\\ 2x+4y+z-5w+v&=0\\ x+2y+3z-10w+2v&=0 \end{aligned} \quad\rightsquigarrow\quad \begin{aligned} x+2y+z-4w+v&=0\\ -2z+6w-2v&=0\\ -z+3w-v&=0\\ 2z-6w+v&=0 \end{aligned} \quad\rightsquigarrow\quad \begin{aligned} x+2y+z-4w+v&=0\\ z-3w+v&=0\\ -v&=0 \end{aligned}$$

$$\rightsquigarrow \begin{aligned} x+2y+z-4w+v&=0\\ z-3w+v&=0\\ v&=0 \end{aligned} \quad\rightsquigarrow\quad \begin{aligned} x+2y+z-4w&=0\\ z-3w&\\ v&=0 \end{aligned} \quad\rightsquigarrow\quad \begin{aligned} x+2y-w&=0\\ z-3w&=0\\ v&=0 \end{aligned}$$

There are two nonpivot columns, so by Theorem 2.2.1(b), the family of solutions depends on two parameters.

(2.2.5) (a) For what values of a does the system of equations

$$\begin{aligned} ax + y &= 2 \\ z + ay &= 3 \end{aligned}$$

have a solution?

We begin by translating the system of equations in to an augmented matrix

$$\left[\begin{array}{ccc|c} a & 1 & 0 & 2 \\ 0 & a & 1 & 3 \end{array}\right]$$

where the coefficients are for x, y, z in that order. Row reducing immediately splits in to cases. If $a = 0$, then

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

is already in echelon form. This system has solutions. In fact, it has infinitely many solutions (since x can take any value).

If $a \neq 0$, then we row reduce as follows:

$$\left[\begin{array}{ccc|c} a & 1 & 0 & 2 \\ 0 & a & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{a} & 0 & 2/a \\ 0 & a & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1/a & 0 & 2/a \\ 0 & 1 & 1/a & 3/a \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1/a^2 & 2/a - 3/a^2 \\ 0 & 1 & 1/a & 3/a \end{array}\right]$$

Writing this echelon form as $[\tilde{A} \mid \tilde{b}]$ as usual, note that $\tilde{b}$ does not have a pivot, so by Theorem 2.2.1, the system does have a solution. Thus the system has a solution for any value of a. Also, the system above has a nonpivot column in $\tilde{A}$, so there are infinitely many solutions.

(b) For what values of a does the system have a unique solution?

As explained in part (a), the system always has infinitely many solutions.

(2.2.7) Symbolically row reduce the system of equations in the matrix.

Basically they're asking us to row reduce the system even though some of the coefficients are not specified. We will split into cases as necessary.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & a & b \\ 2 & 0 & -b & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & (a-2) & (b-1) \\ 0 & -2 & (b-4) & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -2 & (a-2) & (b-1) \\ 0 & 0 & -a-b-2 & -b-1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -\frac{(a-2)}{2} & -\frac{(b-1)}{2} \\ 0 & 0 & a+b+2 & b+1 \end{array}\right]$$

(*)

At this point, the cases split. If $a+b+2=0$, then we cannot divide the third row by this value, if not, we proceed. I will formally continue the process. (This means I will go on as if no problems like dividing by zero arise).

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & \frac{-a+2}{2} & \frac{-b+1}{2} \\ 0 & 0 & 1 & \frac{b+1}{a+b+2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 - \frac{2(b+1)}{(a+b+2)} \\ 0 & 1 & 0 & \left(\frac{a-2}{2}\right)\left(\frac{b+1}{a+b+2}\right) + \left(\frac{-b+1}{2}\right) \\ 0 & 0 & 1 & \frac{b-1}{a-b-2} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 - \frac{2(b+1)}{(a+b+2)} - \left(\frac{a-2}{2}\right)\left(\frac{b+1}{a+b+2}\right) - \frac{(-b+1)}{2} \\ 0 & 1 & 0 & \left(\frac{a-2}{2}\right)\left(\frac{b+1}{a+b+2}\right) + \left(\frac{-b+1}{2}\right) \\ 0 & 0 & 1 & \frac{b+1}{a+b+2} \end{array}\right]$$

This last is the echelon form we get if $a+b+2 \neq 0$, so note that it is not always applicable.

(a) For what values of a, b does the system have a unique solution? Infinitely many solutions? No solutions?

As we showed above, if $a+b+2 \neq 0$, then the matrix row reduces to the identity, so our system has a unique solution. If $a+b+2=0$, then go back and look at step (*) in the row reduction.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -\frac{(a-2)}{2} & -\frac{b-1}{2} \\ 0 & 0 & 0 & b+1 \end{array}\right]$$

If $b+1=0$, then the system will have infinitely many solutions (since the last column will not be a pivot), If $b+1=0$, then $b=-1$, and since we're already supposing $a+b+2=0$, this means $a=-1$. Hence $a=-1, b=-1$ is the only choice yielding infinitely many solutions. If $b \neq -1$ (but $a+b+2=0$ still holds), then the augmented column will be a pivot, so there will be no solutions.

(b) I was told that Professor Meyer said to skip part (b).

(2.2.11) I am not repeating the full prompt here. $R(n)$ counts the number of steps needed to row reduce an $n \times n$ matrix by usual row reduction in a worst case scenario, clearing each column left to right. $Q(n)$ counts the number of steps to do partial row reduction (clearing below the pivots) and then back substituting.

First we will show that the number of operations needed to row reduce an $n \times n$ matrix is $R(n) = n^3 + n^2/2 - n/2$. I think the book is in error when it says this formula gives the number of operations to row reduce an augmented matrix $[A|\vec{b}]$. This formula actually gives the number of operations to reduce A itself.

(a) Compute $R(1)$ and $R(2)$ and show that the formula is correct for $n=1, n=2$.

A general 1×1 matrix looks like $[a]$. It only takes 1 operation to reduce this to $[1]$, namely dividing by a. Plugging 1 into $R(n)$ gives $R(1) = 1 + \frac{1}{2} - \frac{1}{2} = 1$, which agrees with our analysis.

A general 2×2 matrix can be row reduced as follows. I will count the operations later. Note that I may fill in entries with asterisks $*$ when I really don't care what the values are.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{①} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \xrightarrow{②} \begin{bmatrix} 1 & b/a \\ 0 & * \end{bmatrix} \xrightarrow{③} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \xrightarrow{④} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In step ①, we divide the first row by a. This takes two operations, one for each division.

In step ②, we add a multiple of row one to row two. This has two phases. First we multiply row one by $-c$ (which takes two operations), then we add these values to the second row (two more operations) for a total of four operations. This is generally how it works. Multiplying a row by a constant takes operations equal to the number of nonzero entries in the row, and adding one row to another takes twice as many (for multiplication and addition).

In step ③, the first column is effectively done. Multiplying the second row by a constant only takes one operation since the first entry is a 0.

In step ④, we add a multiple of row two to row one. Again, the first column is done, so this only takes two operations: one to multiply by an appropriate constant, and one more to add. This gives a total of $4+2+1+2=9$ operations, which agrees with $R(2) = 2^3 + 2^2/2 - 2/2 = 8+2-1 = 9$.

Q.2.11 continued)

(b) Suppose that columns $1, \dots, k-1$ each contain a pivotal 1, and that all other entries in those columns are 0. Show that you will require another $(2n-1)(n-k+1)$ operations for the same to be true of column k.

The first $k-1$ columns are done, so any row operations only affects the remaining $n-k+1$ columns. Row reduction of the next column would happen as follows:

$$\begin{array}{c} \begin{bmatrix} 1 & & 0 & * & \\ & \ddots & & \vdots & \\ 0 & & 1 & * & \\ & & & \boxed{*} & \cdots \\ & & & * & \\ & & & * & \end{bmatrix} \\ 1 \cdots k-1 \;\; k \cdots n \end{array} \xrightarrow{①} \begin{array}{c} \begin{bmatrix} 1 & & 0 & * & \\ & \ddots & & \vdots & \\ 0 & & 1 & * & \\ & & & \boxed{1} & \cdots \\ & & & * & \\ & & & * & \end{bmatrix} \\ 1 \cdots k-1 \;\; k \cdots n \end{array} \xrightarrow{②} \begin{array}{c} \begin{bmatrix} 1 & & 0 & 0 & \\ & \ddots & & \vdots & \\ 0 & & 1 & 0 & \\ & & & \boxed{1} & \cdots \\ & & & 0 & \\ & & & \vdots & \\ & & & 0 & \end{bmatrix} \\ 1 \cdots k-1 \;\; k \cdots n \end{array}$$

In step ①, we multiply the k^{th} row of our matrix by a constant to get the boxed value to be equal to 1. This takes $n-k+1$ operations by the logic we developed in part (a), namely one operation for each non-zero numerical multiplication we conduct.

In step ②, we add appropriate multiples of row k to all of the other rows to zero out the other entries of column k. For each row, this will take $2(n-k+1)$ operations. Specifically, it will take $n-k+1$ operations to multiply the k^{th} row by an appropriate value (the nonzero entries that is), and $n-k+1$ to add to another row. This gives $2(n-k+1)$ operations per row, times the $n-1$ rows we must do this to. Hence step ② takes $2(n-1)(n-k+1)$ operations in all.

Thus the total cost to row reduce column k is

$$(n-k+1) + 2(n-1)(n-k+1) = (2n-1)(n-k+1) \text{ operations.}$$

(c) Show that $\displaystyle\sum_{k=1}^{n}(2n-1)(n-k+1) = n^3 + \frac{n^2}{2} - \frac{n}{2}$.

In the previous part, we showed that the k^{th} row takes $(2n-1)(n-k+1)$ operations to row reduce. Adding up these operations for each k is the sum on the left, so in this step we're finding the total number of steps to row reduce an $n \times n$ matrix. That is, we're deriving the formula for $R(n)$.

There are several approaches here. We could prove the formula as one massive induction, or break it up into smaller, manageable pieces. I choose the latter.

(continued →)

(2.2.11(c) continued)

$$\sum_{k=1}^{n}(2n-1)(n-k+1) = \sum_{k=1}^{n}(2n^2-2nk+n+k-1)$$
$$= \sum_{k=1}^{n}2n^2 - \sum_{k=1}^{n}2nk + \sum_{k=1}^{n}n + \sum_{k=1}^{n}k - \sum_{k=1}^{n}1$$

I will do each of these sums separately.

$$\sum_{k=1}^{n}2n^2 = \underbrace{2n^2+2n^2+\cdots+2n^2}_{n\text{ times}} = 2n^3$$

$$\sum_{k=1}^{n}2nk = 2n\sum_{k=1}^{n}k$$

The sum here is over k, so $2n$ is a "constant". Think of this as pulling a constant past an integral sign in integration.

I now need to know what $\sum_{k=1}^{n}k$ is. I claim that $\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$, and I will prove it by induction. As a base case, let $n=1$. Then

$\sum_{k=1}^{1}k = 1$ and $\frac{1(1+1)}{2}=1$, so the case $n=1$ is proven.

Now for the inductive step, assume that $\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$. We must show that

$$\sum_{k=1}^{n+1}k = \frac{(n+1)(n+2)}{2}$$

Starting with the left side:

$$\sum_{k=1}^{n+1}k = \left(\sum_{k=1}^{n}k\right)+(n+1) = \frac{n(n+1)}{2}+(n+1) = \frac{(n+1)(n+2)}{2}.$$

Thus the inductive step holds and $\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$ for all positive integers n.

Therefore:

$$\sum_{k=1}^{n}2nk = 2n\sum_{k=1}^{n}k = \frac{2n(n)(n+1)}{2} = n^3+n^2$$

$$\sum_{k=1}^{n}n = n^2$$

$$\sum_{k=1}^{n}k = \frac{n(n+1)}{2} \qquad \text{(we proved this above)}$$

$$\sum_{k=1}^{n}1 = n$$

Adding up all of these pieces, we get

$$\sum_{k=1}^{n}(2n-1)(n-k+1) = 2n^3-n^3-n^2+n^2+\frac{n^2}{2}+\frac{n}{2}-n$$
$$= n^3+\frac{n^2}{2}-\frac{n}{2}$$

Thus we have the formula for $R(n)$

(2.2.11 continued)

(d) Compute $Q(1), Q(2), Q(3)$. Show that $Q(n) < R(n)$ when $n \geq 3$.

Note that $Q(n)$ corresponds to the number of operations to do partial row reduction, in which we first clear only below each pivot and then clear above by back substitution. After our practice in part (a), I will only show the steps and count the number of operations. You may need to convince yourselves that these counts are correct.

<u>$Q(1)$</u> $[a] \to [1]$ takes one operation

<u>$Q(2)$</u>
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{2 operations}]{} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \xrightarrow[\text{4 operations}]{} \begin{bmatrix} 1 & b/a \\ 0 & * \end{bmatrix} \xrightarrow[\text{1 operation}]{} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{2 operations}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>$Q(3)$</u>
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & b/a & c/a \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{12} \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & * & * \end{bmatrix}$$
$$\xrightarrow{4} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The number over each arrow gives the number of operations in that step, so the total number of operations is $Q(3) = 28$.

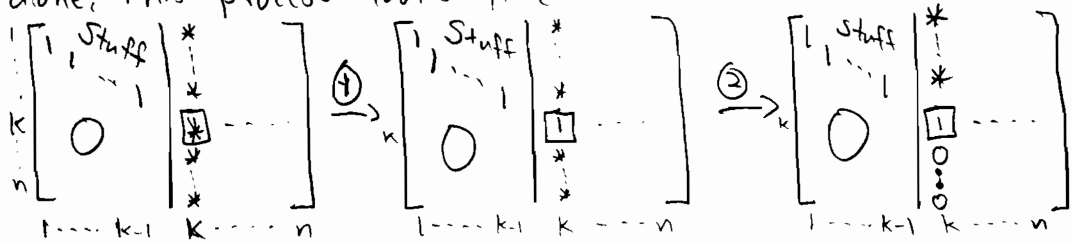
When they ask us to show that $Q(n) < R(n)$ for $n \geq 3$, I think they mean to plug in the formulas. So we want to show
$$\tfrac{2}{3}n^3 + \tfrac{3}{2}n^2 - \tfrac{7}{6}n < n^3 + \tfrac{n^2}{2} - \tfrac{n}{2}$$
This is equivalent to showing
$$\tfrac{1}{3}n^3 - n^2 + \tfrac{2}{3}n > 0$$
which is equivalent to $n^3 - 3n^2 + 2n > 0$ (by multiplying by 3).

Now $n^3 - 3n^2 + 2n = n(n^2 - 3n + 2) = n(n-1)(n-2)$, so if we consider the continuous analog of this function, $x^3 - 3x^2 + 2x$, this function only has three zeros, namely at $x=0$, $x=1$, and $x=2$. Hence $x^3 - 3x^2 + 2x$ cannot cross the x-axis again for $x \geq 3$, so it is either always positive or always negative. Plugging in $x=3$, we get 6, so $x^3 - 3x^2 + 2x > 0$ for $x \geq 3$. By our logic, $n^3 - 3n^2 + 2n > 0$ for $n \geq 3$, so $Q(n) < R(n)$ for $n \geq 3$.

(2.2.11 continued)

(e) Following the same steps as in part (b), show that the number of operations needed to go from the $(k-1)^{th}$ step to the k^{th} step of partial row reduction is $(n-k+1)(2n-2k+1)$.

Here we are only putting the k^{th} column into the desired form of partial row reduction, clearing the entries below the pivot but leaving those above the pivot alone. This process looks like



The logic is almost the same as in part (b). In step ① we multiply row k by an appropriate constant to get a 1 in the boxed pivot position. This takes $n-k+1$ operations as before.

In step ② we add multiples of row k to each of the rows below it to get zeros in the k^{th} column below the pivot. This takes $2(n-k+1)$ operations per row, as before, but now we only need to perform this step for each row below the pivot. There are $n-k$ rows below the pivot, so step ② takes $2(n-k)(n-k+1)$ operations.

Hence the total number of operations to partially row reduce the k^{th} column is $(n-k+1)+2(n-k)(n-k+1) = (n-k+1)(2n-2k+1)$.

(f) Show that $\sum_{k=1}^{n}(n-k+1)(2n-2k+1) = \frac{2}{3}n^3+\frac{1}{2}n^2-\frac{1}{6}n$.

In part (e) we showed that the number of operations to partially row reduce the k^{th} column is $(n-k+1)(2n-2k+1)$, so the sum on the left is the total number of operations to partially row reduce an $n\times n$ matrix (without also back substituting).

This proof is almost exactly like part (c). Split the sum into manageable terms and add them up. The only other fact you need is that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Check this by induction just as we showed that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ in part (c).

(2.2.11 continued)

(g) Show that back substitution requires $n^2 - n$ operations.

At this point we have an $n \times n$ matrix in the form

$$\begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{bmatrix}$$

We back substitute from right to left. Clearing above the last pivot gives us

$$\begin{bmatrix} 1 & * & \cdots & * & 0 \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & * & \vdots \\ 0 & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

This required us to clear out $n-1$ rows in the last column. For each row, we add and appropriate multiple of the last row to get the zero, which takes two operations per row. (one for multiplication and one for addition). Hence, it takes $2(n-1)$ operations to clear the last column. From there it will take $2(n-2)$ operations to clear the next column (one fewer $*$'s), and in general, $2(n-k)$ operations to clear the k^{th} column from the right.

Hence the full back substitution is the sum of these numbers for each column, so back substitution takes

$$\sum_{k=1}^{n} 2(n-k) = \sum_{k=1}^{n} 2n - \sum_{k=1}^{n} 2k$$

$$= 2n^2 - 2\sum_{k=1}^{n} k \qquad \text{We proved } \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ in part (c).}$$

$$= 2n^2 - 2\left(\frac{n(n+1)}{2}\right)$$

$$= 2n^2 - n^2 - n$$

$$= n^2 - n$$

(h) Compute $Q(n)$

$Q(n)$ is the total number of operations needed to do partial row reduction and then back substitution. Here it is the sum of the answers we got in parts (f) and (g)

Thus $\quad Q(n) = \left(\frac{2}{3} n^3 + \frac{1}{2} n^3 - \frac{1}{6} n\right) + (n^2 - n) = \frac{2}{3} n^3 + \frac{3}{2} n - \frac{7}{6} n$, as desired.

(2.3.2) Find the inverse, or show that it does not exist, for each of the following matrices

(a) I will use the process outlined in this section (Theorem 2.3.3) to check for inverses of square matrices. Non square matrices cannot have inverses, as we will show in problem 2.3.3. This also follows from Proposition 2.3.2.

$$\left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 9 & 9 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & 54 & -9 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{54} \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{5}{54} \\ 0 & 1 & -\frac{1}{6} & \frac{1}{54} \end{array}\right]$$

So by Theorem 2.3.3, the inverse of $\begin{bmatrix} 1 & -5 \\ 9 & 9 \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{6} & \frac{5}{54} \\ -\frac{1}{6} & \frac{1}{54} \end{bmatrix}$.

(b) $$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

The matrix $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ does not row reduce to the identity, so there is no inverse.

(c) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{4} & -\frac{9}{4} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

The right half of this augmented matrix is the inverse.

(d) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$ This matrix is not square, so it cannot row reduce to the identity, so by Proposition 2.3.2, it has no inverse.

(e) $$\left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 8 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 8 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{35}{3} & -\frac{8}{3} & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & \frac{8}{7} & 0 & -\frac{3}{7} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -6 & \frac{8}{7} & -1 & -\frac{3}{7} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{4}{21} & \frac{1}{6} & \frac{1}{14} \end{array}\right]$$

(continues →)

(2.3.2(e) continued)

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & \frac{17}{63} & \frac{1}{18} & \frac{1}{42} \\ 0 & 1 & 0 & \frac{4}{21} & \frac{5}{6} & -\frac{1}{14} \\ 0 & 0 & 1 & -\frac{4}{21} & \frac{1}{6} & \frac{1}{14} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{7} & -\frac{1}{2} & \frac{1}{14} \\ 0 & 1 & 0 & \frac{4}{21} & \frac{5}{6} & -\frac{1}{14} \\ 0 & 0 & 1 & -\frac{4}{21} & \frac{1}{6} & \frac{1}{14} \end{array}\right]$$

The right half is the inverse.

(f)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

The right half is the inverse.

(g)

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 1 & 3 & 6 & 10 & 0 & 0 & 1 & 0 \\ 1 & 4 & 10 & 20 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & 9 & -1 & 0 & 1 & 0 \\ 0 & 3 & 9 & 19 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 10 & 2 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

Looks like alternating binomial expansions. Kinda cool.

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 2 & -3 & 3 & -1 \\ 0 & 1 & 2 & 0 & 2 & -8 & 9 & -3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -2 & 8 & -7 & 2 \\ 0 & 1 & 0 & 0 & -6 & 14 & -11 & 3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & -6 & 4 & -1 \\ 0 & 1 & 0 & 0 & -6 & 14 & -11 & 3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

The right half is the inverse. (Look at all the patterns!)

(2.3.3)(a) Derive from Theorem 2.2.1 the fact that only square matrices have inverses.

Suppose that A is a matrix with an inverse A^{-1}. Consider a matrix equation $A\vec{x}=\vec{b}$, where $\vec{b}$ is any vector so that the equation makes sense. Then

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Then $\vec{x}$ can be exactly one thing, namely $A^{-1}\vec{b}$, so $A\vec{x}=\vec{b}$ has a unique solution for any $\vec{b}$. (I suppose that I should mention that if A is $m\times n$, then $\vec{x}$ is $n\times 1$ so that the multiplication makes sense. Hence $\vec{b}$ is $m\times 1$, so $A\vec{x}=\vec{b}$ for any $\vec{b}\in\mathbb{R}^m$).

Now we use Theorem 2.2.1 to show that A is square. Under the assumptions above (A is $m\times n$ and invertible), consider the echelon form $[\widetilde{A}\,|\,\widetilde{b}]$. If $m<n$, then $\widetilde{A}$ would have at most m pivots (since $\widetilde{A}$ only has m rows), so not all columns of $\widetilde{A}$ would have pivots. Thus, whether $\widetilde{b}$ is a pivot column or not, Theorem 2.2.1 shows that $A\vec{x}=\vec{b}$ cannot have a unique solution. This would be a contradiction, so $m<n$ cannot happen.

Similarly, if $m>n$, then there are some nonpivot rows of $\widetilde{A}$. Then for certain choices of $\vec{b}$ (namely those that would have a nonzero entry in the last entry of $\widetilde{b}$), $\widetilde{b}$ would be a pivot column, so $A\vec{x}=\vec{b}$ would have no solutions. But we know that $A\vec{x}=\vec{b}$ has a unique solution for any $\vec{b}$ since A is invertible, so again, this would be a contradiction. Thus $m>n$ cannot happen either, so the only possibility remaining is that $m=n$ and A is square.

(b) Find matrices A, B where $AB=I$ but $BA\neq I$.

There are many examples that work. Here is one such example:

$A=\begin{bmatrix}1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}1\\0\end{bmatrix}$ $\quad$ Then $A\cdot B=\begin{bmatrix}1\end{bmatrix}=I$ but $BA\ \begin{bmatrix}1&0\\0&0\end{bmatrix}\neq I$

(2.3.5) Working by hand, solve the system of linear equations

$$3x - y + 3z = 1$$
$$2x + y - 2z = 1$$
$$x + y + z = 1$$

(a) By row reduction:

$$\left[\begin{array}{ccc|c} 3 & -1 & 3 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & -2 & 1 \\ 3 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & -4 & 0 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 16 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & \frac{1}{8} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{7}{8} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{8} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{8} \end{array}\right].$$

The unique solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{2} \\ \frac{1}{8} \end{bmatrix}$.

(b) By computing and using the matrix inverse:

$$\left[\begin{array}{ccc|ccc} 3 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 3 & -1 & 3 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -4 & 0 & 1 & -2 \\ 0 & -4 & 0 & 1 & 0 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 16 & 1 & -4 & 5 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 1 & \frac{1}{16} & -\frac{1}{4} & \frac{5}{16} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{16} & \frac{1}{4} & \frac{11}{16} \\ 0 & 1 & 0 & -\frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{16} & -\frac{1}{4} & \frac{5}{16} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{4} & -\frac{1}{16} \\ 0 & 1 & 0 & -\frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{16} & -\frac{1}{4} & \frac{5}{16} \end{array}\right]$$

The unique solution should be given by $A^{-1}\vec{b}$, so we compute

$$\underbrace{\begin{bmatrix} \frac{3}{16} & \frac{1}{4} & -\frac{1}{16} \\ -\frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{16} & -\frac{1}{4} & \frac{5}{16} \end{bmatrix}}_{A^{-1}} \overbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}^{\vec{b}} = \overbrace{\begin{bmatrix} \frac{3}{8} \\ \frac{1}{2} \\ \frac{1}{8} \end{bmatrix}}^{\vec{x}}$$

The two solutions agree, as they should.

(2.3.8) (a) Predict the effect of multiplying the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ by each of the elementary matrices below, with the elementary matrix on the left

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Multiplication by this elementary matrix should multiply the second row by 3.

(ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ Multiplication on the left by this matrix should transpose the second and third rows.

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ Multiplication on the left by this matrix should add twice the first row to the third row.

(b) Confirm your answer by carrying out the multiplication.

(i)
$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -1 \\ 6 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix} \end{array} \leftarrow \text{As expected}$$

(ii)
$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \end{array} \leftarrow \text{As expected}$$

(iii)
$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array} \leftarrow \text{As expected}$$

(c) Redo part (a) and part (b) by placing the elementary matrix on the right.

(i) Multiplying on the right by this matrix should multiply the second <u>column</u> by 3
(ii) Multiplication on the right by this matrix should transpose second and third <u>columns</u>.
(iii) This should add twice the third <u>column</u> to the first <u>column</u>.

(continued →)

(2.3.8(c) continued)

Checking predictions:

(i)

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 0 & 3 & 2 \end{bmatrix} \end{array} \leftarrow \text{As expected}$$

(ii)

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \end{array} \leftarrow \text{As expected}$$

(iii)

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} -1 & 0 & -1 \\ 4 & 1 & 1 \\ 4 & 1 & 2 \end{bmatrix} \end{array} \leftarrow \text{As expected}$$

(2.3.12) Prove Proposition 2.3.7

Proposition 2.3.7 states that
$$(1)\ (E_1(i,x))^{-1} = E_1(i,\tfrac{1}{x})$$
$$(2)\ (E_2(i,j,x))^{-1} = E_2(i,j,-x)$$
$$(3)\ (E_3(i,j))^{-1} = E_3(i,j)$$
where the notation is explained in Definition 2.3.6.

We could prove this just by multiplying (for example) $E_1(i,x)\cdot E_1(i,\frac{1}{x})$ and $E_1(i,\frac{1}{x})\cdot E_1(i,x)$, showing that we get the identity for each, and thus $(E_1(i,x))^{-1} = E_1(i,\frac{1}{x})$, then repeat this for each of (1), (2), (3). I don't particularly like this method, as it consists of multiplying lots of matrices filled with ...'s, which can be misleading. Instead, note that in exercise 2.1.5 (from a previous homework assignment), we proved that any row operation can be undone by another row operation. Explicitly, we showed that multiplying the i^{th} row by a nonzero constant x can be undone by multiplying the i^{th} row by $\frac{1}{x}$.

Similarly, adding x times the i^{th} row to the j^{th} row is undone by adding $-x$ times the i^{th} row to the j^{th} row, and switching the i^{th} and j^{th} rows is undone by switching the two rows again. By the text after definition 2.3.6, each of these processes corresponds to multiplying on the left by a particular elementary matrix.

Thus, multiplying the i^{th} row by x and then by $\frac{1}{x}$ gives us the original matrix, so this says $I = E_1(i,\frac{1}{x})\, E_1(i,x)$. Similarly, $I = E_1(i,x)\cdot E_1(i,\frac{1}{x})$. Hence $(E_1(i,x))^{-1} = E_1(i,\frac{1}{x})$.

The cases of (2) and (3) are already done as well. I described by the row operations are inverses, so the corresponding elementary matrices are inverses as well.