Name: **31A MIDTERM 1**

1. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & -5 \end{bmatrix}$ and let $S = \{\vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0}\}$.

   a. [10 points] Is S a subspace of $\mathbb{R}^3$?

   Yes, S is a subspace of $\mathbb{R}^3$.

   b. [15 points] Explain your answer to part (a).

   Suppose $\vec{x}, \vec{y} \in S$. Then $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ (because multiplication by a matrix is a linear transformation). But $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$ since $\vec{x}, \vec{y} \in S$, so $A(\vec{x}+\vec{y}) = \vec{0}+\vec{0} = \vec{0}$, so S is closed under addition. Similarly, for $\alpha \in \mathbb{R}$, $A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha\vec{0} = \vec{0}$, so S is also closed under scalar multiplication. Thus S is a subspace of $\mathbb{R}^3$.

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1	
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2. Let A and S be as defined in problem 1. Let $\vec{a}^{\mathsf{T}}$ and $\vec{b}^{\mathsf{T}}$ be the first and second rows of A, respectively.

   a. [5 points] Compute $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} \det \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix} \\ -\det \begin{bmatrix} 1 & -2 \\ 1 & -5 \end{bmatrix} \\ \det \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix}$$

   b. [10 points] Explain why $\vec{a} \times \vec{b} \in S$.

   $(A\vec{x})_1 = \vec{a} \cdot \vec{x}$ and $(A\vec{x})_2 = \vec{b} \cdot \vec{x}$. But $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0 = \vec{b} \cdot (\vec{a} \times \vec{b})$, so $A(\vec{a} \times \vec{b}) = \vec{0}$, which means $\vec{a} \times \vec{b} \in S$.

   c. [10 points] Use your answers for parts (a) and (b) to write a different description of the set S than is given in problem 1.

   S consists exactly of all vectors orthogonal to both $\vec{a}$ and $\vec{b}$, namely

$$\{\alpha(\vec{a} \times \vec{b}) \mid \alpha \in \mathbb{R}\} = \left\{ \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\}.$$

3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let B be an $n \times n$ matrix. Define the *tensor product* $A \otimes B$ to be the $2n \times 2n$ matrix $A \otimes B = \left[\begin{array}{c|c} aB & bB \\ \hline cB & dB \end{array}\right]$.

   a. [5 points] For $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, compute $H \otimes H$.

$$H \otimes H = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} H & H \\ \hline H & -H \end{array}\right] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

   b. [10 points] Let $\vec{h}_i$ be the i^{th} column of $H \otimes H$. Show that $\vec{h}_i$ and $\vec{h}_j$ are orthogonal if $i \neq j$.

   For each pair of two different columns $\vec{h}_i$ and $\vec{h}_j$ of $H \otimes H$, there are two pairs of entries in which the corresponding entries are the same, and two pairs in which they are negatives of one another. Adding the products thus gives 0 for each pair of columns, so $\vec{h}_i \cdot \vec{h}_j = 0$, which means each pair of different columns is orthogonal.

   c. [10 points] Find $(H \otimes H)^{-1}$.

   Since $(H \otimes H)^{\mathsf{T}} = H \otimes H$, from part (b), all the off-diagonal elements in $H \otimes H(H \otimes H)$ are 0. It is easy to check that the diagonal elements are 1; thus $(H \otimes H)^{-1} = H \otimes H$.

4. [25 points] Let $M_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation reflecting across the line making angle α with $\vec{e}_1$, and let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation rotating counterclockwise by angle θ. Recall that

$$[M_\alpha] = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \quad \text{and} \quad [R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Show that the composition $M_\beta \circ M_\alpha$ is $R_{2(\beta-\alpha)}$.

$$\begin{aligned}
[M_\beta \circ M_\alpha] = [M_\beta][M_\alpha] &= \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\beta \cos 2\alpha + \sin 2\beta \sin 2\alpha & \cos 2\beta \sin 2\alpha - \sin 2\beta \cos 2\alpha \\ \sin 2\beta \cos 2\alpha - \cos 2\beta \sin 2\alpha & \sin 2\beta \sin 2\alpha + \cos 2\beta \cos 2\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\beta-\alpha) & -\sin 2(\beta-\alpha) \\ \sin 2(\beta-\alpha) & \cos 2(\beta-\alpha) \end{bmatrix} = [R_{2(\beta-\alpha)}]
\end{aligned}$$

Since the matrices representing these linear transformations are equal, the linear transformations are the same.

Extra Credit. Let $C : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $C(\vec{u}, \vec{v}) = \vec{u} \times \vec{v}$.

a. [5 points] Is C a linear transformation? Explain your answer.

C is not a linear transformation. We can think of it as a function $C : \mathbb{R}^6 \to \mathbb{R}^3$, for $\vec{x} = (\vec{u}, \vec{v}) \in \mathbb{R}^6$, but it fails both conditions for linearity. *E.g.*, $C(\alpha\vec{x}) = C(\alpha\vec{u}, \alpha\vec{v}) = (\alpha\vec{u}) \times (\alpha\vec{v}) = \alpha^2(\vec{u} \times \vec{v}) \neq \alpha(\vec{u} \times \vec{v})$.

b. [10 points] Is C onto? For $\vec{w} \in \mathbb{R}^3$, give a geometrical description of $C^{-1}(\{\vec{w}\})$.

C is onto: Let $\vec{u} \perp \vec{v}$ lie in the plane orthogonal to $\vec{w}$ and let $(\vec{u}, \vec{v}, \vec{w})$ have the same orientation as $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Choose their lengths so that $|\vec{u}||\vec{v}| = |\vec{w}|$. Then $\vec{u} \times \vec{v} = \vec{w}$. $C^{-1}(\{\vec{w}\})$ is all pairs of vectors in the plane orthogonal to $\vec{w}$ with this orientation and such that $|\vec{u}||\vec{v}| \sin\theta = |\vec{w}|$, where θ is the angle from $\vec{u}$ to $\vec{v}$.