# Practice Midterm 2.

(1) Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation whose kernel is equal to the span of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -4 \\ -13 \\ -7 \end{bmatrix}$$

What is the dimension of the image of $T$?

(2) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$ and let $A$ be the matrix whose columns are those vectors.
Consider the following statements:

A: The vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are linearly dependent.

B: There exist scalars $a, b \in \mathbb{R}$ such that $\vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$.

C. There exists a vector $\vec{x} \in \mathbb{R}^3$ such that $\vec{x} \neq \vec{0}$ and $A\vec{x} = \vec{0}$.

Which of the statements imply each other? For each potential implication, either justify your answer or provide a counterexample.

(3) Consider the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

(a) Find a basis for the kernel of $A$.

(b) Find a basis for the image of $A$.

(4) Explain why each of the following statements is true or false and justify your answer.

(a) There cannot exist five nonzero vectors $\vec{v}_1, \ldots, \vec{v}_5 \in \mathbb{R}^4$ such that the vectors are orthogonal.

(b) There exists a matrix $A$ such that $A\vec{x} = \vec{b}$ has either infinitely many solutions or a unique solution (depending on $\vec{b}$), but never zero solutions.

(c) Let $A$ and $B$ be $2 \times 2$ matrices such that $AB = I$. Then $B = A^{-1}$. (Note that we don't assume $BA = I$).

(d) Let $T : \mathbb{R}^7 \to \mathbb{R}^3$ be a linear transformation. Then there are exactly 8 possibilities for the dimension of the kernel.

(5) Consider the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

where the $n^{th}$ term in the sequence is the sum of the previous two. Said another way, if $a_n$ is the $n^{th}$ term in the sequence, then

$$a_n = a_{n-1} + a_{n-2}.$$

Let $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Show that $F^n$ can be writen explicitly in terms of the entries in the above sequence for all positive integers $n$.

This problem is fairly open ended. You'll need to find your own formula and then prove it.