Problem Set #1

1.5.2 For each of the following subsets, state whether it is open or closed (or both or neither) and say why.

(a) (x,y)-plane in $\mathbb{R}^3$

The (x,y)-plane is the set of all points in $\mathbb{R}^3$ with z coordinate equal to zero. Bear in mind that in $\mathbb{R}^3$, an open ball is 3-dimensional. Now pick a point in the (x,y)-plane, say the origin $(0,0,0)$. Then any open ball, say of radius $\varepsilon>0$, $B_\varepsilon((0,0,0))$ must contain points both above and below the (x,y)-plane. In particular, $(0,0,\varepsilon/2)\in B_\varepsilon((0,0,0))$, but $(0,0,\varepsilon/2)\notin (x,y)$-plane. Thus $B_\varepsilon((0,0,0))$ is not contained in the (x,y)-plane for any $\varepsilon>0$. Thus the (x,y)-plane is not open.

To decide whether the (x,y)-plane is closed, we will instead ask whether the complement is open. Let (x_0,y_0,z_0) be a point in the complement, so that $z_0\neq 0$. Then any point in $B_{|z_0|/2}(x_0,y_0,z_0)$ has z-component nonzero (specifically has z-component with absolute value between $|z_0|/2$ and $3|z_0|/2$). Thus the complement is open, so the (x,y)-plane is closed in $\mathbb{R}^3$

I will not go through this level of detail for the remaining parts

(b) $\mathbb{R}\subset\mathbb{C}$

Recall that we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the identification

$$x+iy \leftrightarrow (x,y)$$

so this really becomes a question of $\mathbb{R}\subset\mathbb{R}^2$, where $\mathbb{R}$ means the x-axis

Just as in part (a), any open ball around a point in the x-axis must contain points above and below, so $\mathbb{R}\subset\mathbb{C}$ is not open. Also similar to part (a), the complement is open since, for (x_0,y_0) in the complement, $B_{|y_0|/2}((x_0,y_0))$ is entirely contained in the complement, so $\mathbb{R}\subset\mathbb{C}$ is closed.

(c) The line $x=5$ in the (x,y)-plane.

This is almost the same as part (b), still a line in $\mathbb{R}^2$. Note $B_\varepsilon((5,0))\not\subset\{\text{the line } x=5\}$ for any ε, so the line is not open. If we look at the complement, consider the point (x_0,y_0) with $x_0\neq 5$. Then let $\varepsilon=|x_0-5|/2$, so that $B_\varepsilon((x_0,y_0))$ contains no points on the line $x=5$. Hence the complement is open, so the line $x=5$ is closed in the (x,y)-plane.

(1.5.2 continued)

(d) $(0,1) \subset \mathbb{C}$

Here we again identify $\mathbb{C}$ with $\mathbb{R}^2$, and $(0,1)$ corresponds to an interval. They don't specify, but I'll assume that this is the interval $(0,1)$ on the x-axis

This subset is not open, for if we pick the point $(\frac{1}{2},0)$, then $B_\varepsilon((\frac{1}{2},0)) \not\subset (0,1)$ for any ε since $B_\varepsilon((\frac{1}{2},0))$ contains complex numbers above and below the real axis.

However, $(0,1)$ is not closed either, since the point $(0,0)$ is in the complement, but $B_\varepsilon((0,0))$ contains points of the interval $(0,1)$ for <u>any</u> $\varepsilon > 0$ Hence the complement is not open, so $(0,1)$ is not closed

I suppose it is possible that we could read $(0,1)$ as the <u>point</u> corresponding to $i \in \mathbb{C}$. In this case, $(0,1)$ is closed but not open.

(e) $\mathbb{R}^n \subset \mathbb{R}^n$

Any open ball is in $\mathbb{R}^n$, so $\mathbb{R}^n$ is open. The complement is the empty set (which believe it or not is considered open). Thus $\mathbb{R}^n$ is both open and closed in $\mathbb{R}^n$ itself.

(f) The unit sphere in $\mathbb{R}^3$.

Here the unit sphere refers only to the shell of points in $\mathbb{R}^3$ which lie <u>exactly</u> distance 1 from the origin For <u>basically</u> the same reasons as in part (c), this subset is closed but not open.

1.5.3 Prove the following statements for open subsets of $\mathbb{R}^n$.
(a) Any union of open sets is open

Let I be any index set. By this we mean that we will use the elements of I to label our collection of open sets (think of $I = \mathbb{N}$ if you like). Then let U_i be an open set for each $i \in I$. Our goal is to show that

$$\text{union: } \bigcup_{i\in I} U_i \text{ is open. I will denote } \bigcup_{i\in I} U_i \text{ by } V$$

Let $x \in V$, so $x \in U_{i_0}$ for some $i \in I$. Then there is an open ball of some radius around x in U_{i_0} since U_{i_0} is open. Call this open ball W. Then $W \subseteq V$ since $W \subseteq U_{i_0}$ and $U_{i_0} \subseteq V$. Hence there is an open neighborhood of x in V. Since $x \in V$ was arbitrary, this shows that V is open, so the proof is complete.

(b) A finite intersection of open sets is open

Let $U_1, \dots, U_m$ be a finite collection of open sets in $\mathbb{R}^n$. We want to show

$$W = \bigcap_{i=1}^{m} U_i \text{ is open}$$

Let $x \in W$, so that $x \in U_i$ for each i. Since each U_i is open there is an open ball of some radius ε_i about x in U_i. Denote these balls by $B_{\varepsilon_i}(x)$. Now let $\varepsilon = \min_{1\le i\le m} \varepsilon_i$, the smallest of these radii. Then $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x)$ for each i, so $B_\varepsilon(x) \subseteq U_i$ for each i, so $B_\varepsilon(x)$ is an open ball around x in the intersection of the U_i. Hence (since x was any point) W is open.

(c) An infinite intersection of open sets is not necessarily open.

Let's work in just one-dimensional $\mathbb{R}$, so that our open balls are just intervals. Define U_m to be the interval $(-\frac{1}{m}, \frac{1}{m})$ for each natural number. Now let's consider the intersection

$$\bigcap_{m=1}^{\infty} (-1/m, 1/m)$$

These open intervals get smaller as m gets large. The $U_1 = (-1, 1)$, $U_2 = (-1/2, 1/2)$, etc. The only point in all of them (and thus in the intersection) is the origin. So

$$\bigcap_{m=1}^{\infty} (-1/m, 1/m) = \{0\}$$

But points are not open (I think we proved this elsewhere, if not it's easy to check), so the proof is complete.

1.5.4(a) Show that the interior of A is the biggest open set contained in A

Recall that the interior of A is the set of all $x \in \mathbb{R}^n$ such that there exists an $r>0$ such that $B_r(x) \subset A$

Let $\mathring{A}$ be the interior of A, and let $x \in \mathring{A}$, so that there is some $r>0$ such that $B_r(x) \subset A$ This is true for each point $x \in \mathring{A}$, so now take the union of all of these open balls as x varies. This union is $\mathring{A}$. By problem 1.5.3(a), any union of open sets is open Thus $\mathring{A}$ is an open subset of A

Now we'll show that that $\mathring{A}$ is the largest open subset of A. Suppose that there is a larger open set contained in A, which we'll call D, so that

$$\mathring{A} \subset D \subset A$$

Then there must be some point $y \in D \setminus \mathring{A}$. Then y is contained in an open set in A, so by the definition, there is some $r>0$ such that $B_r(y) \subset D \subset A$. This contradicts the assumption that $y \notin \mathring{A}$, since $\mathring{A}$ is the collection of all points with open balls around them in A Hence no such D can exist, so the proof is complete.

(b) Show that the closure of A is the smallest closed set containing A.

The closure of A, denoted $\bar{A}$ is the set of all $x \in \mathbb{R}^n$ such that $\forall r>0$, $B_r(x) \cap A \neq \emptyset$. Said another way, $\bar{A}$ is the set of all x such that x is not contained in the interior of the complement of A (for if x was in $(\mathring{A^c})$, then there would be some $r>0$ such that $B_r(x) \subset A^c$, so that $B_r(x) \cap A = \emptyset$.

Now let's finish the proof using complements. We just explained that

$$\bar{A} = \{\text{all points not in } (\mathring{A^c})\}$$
$$= \left((\mathring{A^c})\right)^c$$

that is, the complement of the interior of the complement of A. Recall that $(\mathring{A^c})$ is open by part (a), so $\bar{A}$ is closed. Furthermore, if we had a smaller closed set containing A, call it D so $A \subset D \subset \bar{A}$, then by taking complements,

$$A^c \supset D^c \supset (\bar{A})^c = (\mathring{A^c})$$

Then D^c is open and it contains $(\mathring{A^c})$, so $D^c = (\mathring{A^c})$. Thus $D = \bar{A}$, so $\bar{A}$ is indeed the smallest closed set containing A.

(1.5.4 continued)

(c) Show that the closure of a set A is A plus its boundary: $\bar{A} = A \cup \partial A$

Recall that ∂A is defined to be those point $x \in \mathbb{R}^n$ such that every neighborhood of x intersects both A and and the complement. That is, $B_r(x) \cap A \neq \phi$ and $B_r(x) \cap A^c \neq \phi$ for all $r > 0$

First let's show that $A \cup \partial A$ is closed by showing that $(A \cup \partial A)^c$. Let $x \in (A \cup \partial A)^c$, so that $x \notin A$ and $x \notin \partial A$. Thus (by the definition of boundary) there is some $r > 0$ so that $B_r(x) \cap A = \phi$ or $B_r(x) \cap A^c = \phi$. But $B_r(x) \cap A^c$ cannot be the empty set, since $x \in A^c$, so $B_r(x) \cap A = \phi$.

If we can also show $B_r(x) \cap \partial A = \phi$, then we will have an open ball around x in $(A \cup \partial A)^c$, so $A \cup \partial A$ will be closed. But of course $B_r(x) \cap \partial A = \phi$, because if $B_r(x)$ contained a point $y \in \partial A$, then $B_r(x)$ would be an open neighborhood of y, so by the definition of boundary, it would intersect A. This would be a contradiction, so $A \cup \partial A$ is closed (and contains A), so $\bar{A} \subset A \cup \partial A$ (since $\bar{A}$ is the smallest closed set containing A). We still need to show $\bar{A} \supset A \cup \partial A$, so that $\bar{A} = A \cup \partial A$.

We already explained that if $x \in (A \cup \partial A)^c$, then there is some $r > 0$ such that $B_r(x) \subset A^c$. So $x \in (\mathring{A^c})$, the interior of A^c. Thus if $x \in A \cup \partial A$, then $x \in ((\mathring{A^c}))^c$, which from part (b) we know to be $\bar{A}$. Hence $\bar{A} = A \cup \partial A$.

(d) Show that the boundary is the closure minus the interior: $\partial A = \bar{A} - \mathring{A}$

We already know from (c) that $\partial A \subset \bar{A}$, so all we need to show is that if $x \in \partial A$, then $x \notin \mathring{A}$.

Let $x \in \partial A$, so that $B_r(x) \cap A^c \neq \phi$ for all $r > 0$. Then there is no $r > 0$ such that $B_r(x) \subset A$ (for then it wouldn't intersect A^c). Thus by part (a), $x \notin \mathring{A}$, so $\partial A = \bar{A} - \mathring{A}$.

1.5.14 State whether the following limits exist, and prove it

(a) $\lim_{\binom{x}{y} \to \binom{1}{2}} \frac{x^2}{x+y}$

By Corollary 1.5.30, any rational function (quotient of polynomials) is continuous on the subset of $\mathbb{R}^n$ where the denominator does not vanish. The denominator doesn't vanish here, so the function is continuous and the limit exists (and is equal to $\frac{1}{3}$).

(b) $\lim_{\binom{x}{y} \to \binom{0}{0}} \frac{\sqrt{|x|}\, y}{x^2+y^2}$

We will show that this limit does not exist by approaching the origin along two different paths and getting different results.

along $x=0$: limit becomes $\lim_{y \to 0} \frac{0}{y^2} = \lim_{y \to 0} 0 = 0$

along $y = \sqrt{|x|}$ (positive): limit becomes $\lim_{x \to 0} \frac{x}{x^2+x} = \lim_{x \to 0} \frac{1}{x+1} = 1$

So no limit exists.

(c) $\lim_{\binom{x}{y} \to \binom{0}{0}} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$

Same approach as in part (b).

along positive $x=y$: $\lim_{x \to 0} \frac{\sqrt{|x^2|}}{\sqrt{2x^2}} = \lim_{x \to 0} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

along $x=0$ (positive y): $\lim_{y \to 0} \frac{0}{\sqrt{y^2}} = \lim_{y \to 0} 0 = 0$

Thus the limit does not exist.

(d) $\lim_{\binom{x}{y} \to \binom{0}{0}} x^2+y^2-3$

Polynomials are continuous by Corollary 1.5.30 (a), so the limit exists (and is equal to -3).

1.5.16

(a) Let $D^* \subset \mathbb{R}^2$ be the region $0 < x^2 + y^2 + 1$, and let $f: D^* \to \mathbb{R}$ be a function. What does the following assertion mean?

$$\lim_{\binom{x}{y} \to \binom{0}{0}} f\binom{x}{y} = a$$

This means $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall (x,y) \in D^*$, $|\sqrt{x^2+y^2}| < \delta \Rightarrow |f\binom{x}{y} - a| < \varepsilon$.

(b) For the following two functions, defined on $\mathbb{R}^2 - \{\binom{0}{0}\}$, either show that the limit exists at $\binom{0}{0}$ and find it, or show that it does not exist.

$$\lim_{\binom{x}{y} \to \binom{0}{0}} \frac{\sin(x+y)}{\sqrt{x^2+y^2}}$$ Let's examine this limit along several paths

along $y=0$ (positive x-axis): $\lim_{x \to 0} \frac{\sin x}{x} \overset{\text{L'Hopital}}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1$

along $y = -x$ (positive x) : $\lim_{x \to 0} \frac{\sin(0)}{\sqrt{2x^2}} = \lim_{x \to 0} 0 = 0$

Thus the limit does not exist

$$\lim_{\binom{x}{y} \to \binom{0}{0}} (|x| + |y|) \ln(x^2 + y^4)$$

We'll show that the limit is 0. Note that it is enough to show that the limit of the absolute value is 0, since if the absolute value of a function tends to 0, then the function tends to 0.

$$\lim_{\binom{x}{y} \to \binom{0}{0}} (|x|+|y|)\ln(x^2+y^4) = \lim_{\binom{x}{y} \to \binom{0}{0}} |x| \ln(x^2+y^4) + \lim_{\binom{x}{y} \to \binom{0}{0}} |y| \ln(x^2+y^4)$$

$$\leq \lim_{\binom{x}{y} \to \binom{0}{0}} |x| \ln(x^2) + \lim_{\binom{x}{y} \to \binom{0}{0}} |y| \ln(y^4)$$

$$= \lim_{x \to 0^+} 2x \ln(x) + \lim_{y \to 0^+} 4y \ln(y)$$

Now using L'Hopital, $\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0$

Similarly $\lim_{y \to 0^+} 4y\ln(y) = 0$, so the full limit is 0.