Problem Set #2

(0.5.4) This exercise illustrates how complicated convergence can be when a series is not absolutely convergent. Let

$$a_n = \frac{(-1)^{n+1}}{n}, \quad \text{for } n = 1, 2, \ldots$$

(a) Show that the series $\sum a_n$ is convergent

We use the alternating series test. Since the a's are alternately positive and negative, and $\lim_{n\to\infty} a_n = 0$, the series converges by the alternating series test

(b) Show that $\sum_{n=1}^{\infty} a_n = \ln 2$.

Recall the Taylor series for $\ln(1+x)$. The general formula for the Taylor series of a function $f(x)$ centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)\cdot x^n}{n!}$$

Now let $f(x) = \ln(1+x)$, so that

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f^{(3)}(x) = \frac{+2}{(1+x)^3}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

Then $f^{(n)}(0) = (-1)^{n+1}(n-1)!$

Now plug this into the general Taylor formula to get (note n=0 term is zero)

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!\; x^n}{(n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\; x^n}{n}$$

This series converges for all x (check an intro calculus book), so for x=1, we get

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} a_n$$

continued →

(0.5.4 continued)

(c) Explain how to rearrange the terms in the series so that it converges to 5.

I will not give an explicit formula, but I will explain the process. The positive a_n's are

$$\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \dots \quad , \text{ that is } \frac{1}{\text{odd}}$$

and the negative a_n's are

$$-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \dots \quad , \text{ that is, } \frac{-1}{\text{even}}$$

We know that the sum of all the positive terms is infinity, and likewise the sum of all the negative terms is minus infinity. Now add positive terms (in order!) until the sum is greater than 5. Then add negative terms (in order) until the sum is less than 5. Now repeat this procedure. We use up all of the positive terms and all of the negative terms, and the limit will be 5.

(d) Explain how to rearrange the terms in the series so that it diverges.

We arrange the terms in "blocks", each of which will contain a single negative term, and enough positive terms (in order) so that the positives sum to at least 1:

first block: $1 - \frac{1}{2}$

second block: $\underbrace{\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \dots +}_{\text{enough positives to sum to 1}} \quad -\frac{1}{4}$

Then adding these blocks together in this order, each block sums to at least $\frac{1}{2}$, so the full list sums to infinity.

(0.5.5) Show that if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then any rearrangement of the series is still convergent and converges to the same limit.

Hint. For any $\varepsilon > 0$, there exist N such that $\sum_{n=N+1}^{\infty} |a_n| < \varepsilon$. For any rearrangement $\sum_{n=1}^{\infty} b_n$ of the series, there exists an M such that all of $a_1, \ldots, a_N$ appear among $b_1, \ldots, b_M$. Show that $\left|\sum_{n=1}^{n} a_n - \sum_{n=1}^{M} b_n\right| < \varepsilon$

First we will show their hint. Since $\sum_{n=1}^{\infty}$ is absolutely convergent, it is true that for any $\varepsilon > 0$, $\exists$ N such that $\sum_{n=N+1}^{\infty} |a_n| < \varepsilon$ This really just says that since $\sum_{n=1}^{\infty} |a_n|$ is finite, if you go out far enough, (so N sufficiently large) then the remainder of the sum is less than ε.

Now let $\{b_n\}$ be a reordering of the $\{a_n\}$. That is, for each a_n, a_n is sent to some b_{i_n} (the index changed, the value did not). So $a_1 \longmapsto b_{i_1}$, $a_2 \longmapsto b_{i_2}$, ... etc. Let $M = \max_{n \leq N} \{i_n\}$, that is, M will be the maximum index appearing when we send $a_1, \ldots, a_N$ to the reordered b's. Then all of the $a_1, \ldots, a_N$ appear among $b_1, \ldots, b_M$.

Now consider $\left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{M} b_n\right|$.

Every term in the first sum appears in the second sum with a negative sign, so all of the a_n's cancel with some of the b_n's. Thus the terms remaining in the b sum are some of those found among $a_{N+1}, \ldots$, out to a_∞. Hence

$$\left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{M} b_n\right| \leq \sum_{n=N+1}^{\infty} |a_n| < \varepsilon$$

Now we finish the proof. We know that $\sum_{n=1}^{\infty} a_n$ converges, say to L. We want to show that $\sum_{n=1}^{\infty} b_n$ converges to L as well. Let $S_k = \sum_{n=1}^{k} b_n$ be the k^{th} partial sum, then we need to show that $\lim_{k \to \infty} S_k = L$, that is, that for any $\varepsilon > 0$, $\exists$ $l > 0$ such that for $k > l$, $|S_k - L| < \varepsilon$

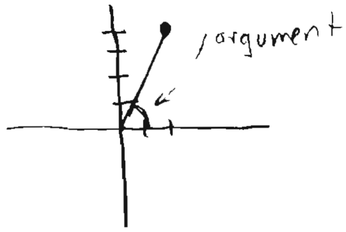
So let $\varepsilon > 0$. Then let N and M be as in the hint. The

$$|L - S_M| = \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{M} b_n\right| \leq \left|\sum_{n=N+1}^{\infty} a_n\right| + \left|\sum_{n=1}^{N} a_n - \sum_{n=1}^{M} b_n\right|$$
$$\leq \sum_{n=N+1}^{\infty} |a_n| + \varepsilon$$
$$\leq 2\varepsilon$$

This holds for $k \geq M$, so the proof is complete.

(0.7.3) Find the absolute value and argument of each complex number

(a) $2+4i$



The absolute value is $\sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$.
By the picture, the argument is $\theta = \arctan(4/2) = \arccos\left(\frac{1}{\sqrt{5}}\right)$
If you want this explicitly, use a calculator (the given form is fine)

(b) $(3+4i)^{-1}$

The inverse of a complex number has absolute value $= \frac{1}{\text{absolute value of}}$ reciprocal, and negative the argument. Thus $|3+4i| = \sqrt{3^2+4^2} = 5$, so $|3+4i|^{-1} = 1/5$

Argument of $3+4i$ is $\arctan(4/3) = \arccos(3/5)$, so the argument of $(3+4i)^{-1}$ is $-\arctan(4/3) = -\arccos(3/5)$

(c) $(1+i)^5$

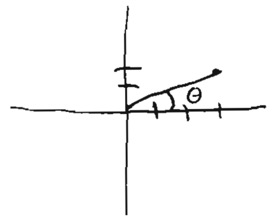
First note $|1+i| = \sqrt{2}$, so $|(1+i)^5| = (\sqrt{2})^5$.
Also the argument of $1+i$ is $\arctan(1) = \pi/4$ radians, so the argument of $(1+i)^5$ is $5\pi/4$ radians by De Moivre's Formula.

(d) $1+4i$

The absolute value is $|1+4i| = \sqrt{1^2+4^2} = \sqrt{17}$.
The argument is $\arctan(4) = \arccos(1/\sqrt{17})$.

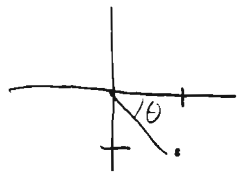
(0.7.4) Find the modulus and polar angle of the following complex numbers.

(a) $3+2i$



The modulus, or length is $\sqrt{3^2+2^2} = \sqrt{13}$. The polar angle will be given by $\arctan(\frac{2}{3})$ or $\arccos(\frac{3}{\sqrt{13}})$.

(b) $(1-i)^4$



Let $z = 1-i$. Then $|z| = \sqrt{1+1} = \sqrt{2}$, so $(1-i)^4$ has modulus $(\sqrt{2})^4 = 4$
The polar angle of z is $\arctan(-1) = -\pi/4$, so the polar angle of $(1-i)^4$ is π (or $-\pi$, something).

(c) $2+i$

The modulus is $\sqrt{4+1} = \sqrt{5}$ and the polar angle is $\arctan(\frac{1}{2})$.

(d) $\sqrt[7]{3+4i}$

There are seven distinct seventh roots of any nonzero complex number, as we see in Proposition 0.7.7. They all have the same length, so since $|3+4i| = \sqrt{9+16} = 5$, the modulus of $\sqrt[7]{3+4i}$ is $\sqrt[7]{5}$.

For the angle, let's first find one angle that works. Let $\theta = \arctan(4/3)$ be the polar angle of $3+4i$. Then $\theta/7$ is one candidate for the polar angle of $\sqrt[7]{3+4i}$. By Prop. 0.7.7, the full list of angles for the seventh roots will be

$$\frac{\theta}{7},\ \frac{\theta}{7}+\frac{2\pi}{7},\ \frac{\theta+4\pi}{7},\ \frac{\theta+6\pi}{7},\ \frac{\theta+8\pi}{7},\ \frac{\theta+10\pi}{7},\ \frac{\theta+12\pi}{7}$$

After this point, if we continued the pattern we would be repeating, so this is the full list.

(0.7.10) Describe the set of all complex numbers $z = x+iy$ such that

(a) $\bar{z} = -z$

Note that $\bar{z} = x - iy$ and $-z = -x - iy$, so if

$$x - iy = -x - iy$$

then $x = 0$. This corresponds then to the imaginary axis, that is, all complex numbers of the form $z = iy$.

(b) $|z-a| = |z-b|$ for any $a, b \in \mathbb{C}$

Let $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$. Assume that $a \neq b$. Then

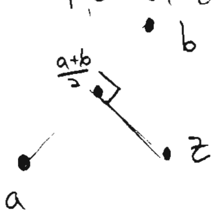
$$\begin{aligned} |z-a| &= |x+iy - a_1 - a_2 i| \\ &= |(x-a_1) + (y-a_2)i| \\ &= \sqrt{(x-a_1)^2 + (y-a_2)^2} \end{aligned}$$

similarly $\quad |z-b| = \sqrt{(x-b_1)^2 + (y-b_2)^2}$

So if $|z-a| = |z-b|$, then $\sqrt{(x-a_1)^2 + (y-a_2)^2} = \sqrt{(x-b_1)^2 + (y-b_2)^2}$

$$\begin{aligned} &\Rightarrow x-a_1)^2 + (y-a_2)^2 = (x-b_1)^2 + (y-b_2)^2 \\ &\Rightarrow x^2 - 2ax + a_1^2 + y^2 - 2a_2 y + a_2^2 = x^2 - 2b_1 x + b_1^2 + y^2 - 2b_2 y + b_2^2 \\ &\Rightarrow -2a_1 x + a_1^2 - 2a_2 y + a_2^2 = -2b_1 x + b_1^2 - 2b_2 y + b_2^2 \\ &\Rightarrow 2x(b_1 - a_1) + 2y(b_2 - a_2) = b_1^2 - a_1^2 + b_2^2 - a_2^2 \\ &\Rightarrow 2x(b_1 - a_1) + 2y(b_2 - a_2) = (b_1 - a_1)(b_1 + a_1) + (b_2 - a_2)(b_2 + a_2) \\ &\Rightarrow (b_1 - a_1)\big(2x - (b_1 + a_1)\big) + (b_2 - a_2)\big(2y - (b_2 + a_2)\big) = 0 \end{aligned}$$

This says that the vector $b-a$ is perpendicular to the vector $z - \frac{a+b}{2}$, since our last equation is the dot product of these two vectors



Thus z must lie on the line which perpendicularly bisects the segment from a to b at its midpoint. If $a = b$, then z can be anything.

(c) $\bar{z} = z^{-1}$

Write $z = r(\cos\theta + i\sin\theta)$, so $\bar{z} = r(\cos\theta - i\sin\theta)$ and $z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$

Now $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, so

$$r(\cos\theta - i\sin\theta) = \frac{1}{r}(\cos\theta - i\sin\theta) \Rightarrow r = 1$$

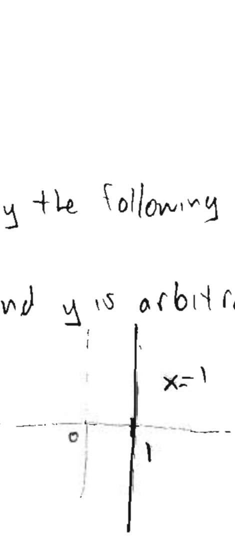
So this is just the unit circle in $\mathbb{C}$

(0.7.11)

(a) Sketch the loci in $\mathbb{C}$ given by the following equations

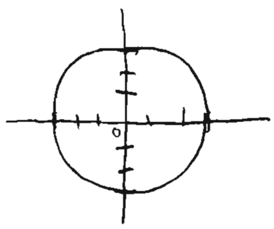
(i) $\operatorname{Re} z = 1$

Let $z = x+iy$. Then $x=1$ and y is arbitrary, so this is the line



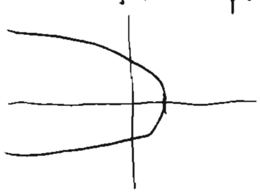
(ii) $|z| = 3$

This just says that if $z = x+yi$, then $\sqrt{x^2+y^2} = 3$, so $x^2+y^2=9$. This is the circle of radius 3.



(b) What is the image of each locus under the mapping $z \mapsto z^2$?

(i) If $z = 1+yi$, then $z^2 = 1^2 + 2yi - y^2 = (1-y^2)+2yi$. Then the x-coordinate is $x = 1-y^2$, a parabola opening to the left as y varies



(ii) If $x^2+y^2=9$, then $(x^2+y^2)^2 = 81$, so the circle of radius 3 maps to the circle of radius 9 (double wrapped)

(c) What is the inverse image of each locus under $z \mapsto z^2$?

(i) If $z = 1+yi$ is mapped to, then $z = \sqrt{1+y^2}\left(\frac{1}{\sqrt{1+y^2}} + \frac{y}{\sqrt{1+y^2}}i\right)$
$= \sqrt{1+y^2}\left(\cos(\arctan y) + i\sin(\arctan y)\right)$

Then the things mapping to z are $w = \sqrt[4]{1+y^2}\left(\cos(\tfrac{1}{2}\arctan y) + i\sin(\tfrac{1}{2}\arctan y)\right)$
and $\sqrt[4]{1+y^2}\left(\cos(\tfrac{1}{2}\arctan y + \theta) + i\sin(\tfrac{1}{2}\arctan y + \theta)\right)$

This is the hyperbola $x^2-y^2=1$.

(ii) The inverse image of the circle of radius 3 is the circle of radius $\sqrt{3}$.

(0.7.13)

(a) Find all the cubic roots of 1

The modulus of 1 is 1, so by Proposition 0.77, the modulus of any root of 1 (cubic or otherwise) is also 1. Then also by Prop 0.77, since the angle of 1 is $\theta = 0$, the cubic roots of 1 are

$$\cos 0 + i\sin\theta = 1$$
$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$
$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

(b) Find the 4^{th} roots of 1

By the same procedure, the roots are

$$\cos 0 + i\sin 0 = 1$$
$$\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$
$$\cos\pi + i\sin\pi = -1$$
$$\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

(c) Find the 6^{th} roots of 1.

Same procedure.

$$\cos 0 + i\sin 0 = 1$$
$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$
$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$
$$\cos\pi + i\sin\pi = -1$$
$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$
$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

(1.5.8) (a) Find the inverse of the matrix B at left by finding the matrix A such that $B = I - A$ and computing the value of the series $S = I + A + A^2 + A^3 + \cdots$

$$B = \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so} \quad A = I - B = \begin{bmatrix} 0 & -\varepsilon & -\varepsilon \\ 0 & 0 & -\varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } A^2 = \begin{bmatrix} 0 & 0 & \varepsilon^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A^3 = \text{zero matrix}$$

Then $S = I + A + A^2 = \begin{bmatrix} 1 & -\varepsilon & -\varepsilon+\varepsilon^2 \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{bmatrix}$

A quick check shows that, indeed, $S = B^{-1}$!

(b) Compute the inverse of the matrix C in the margin, where $|\varepsilon| < 1$.

$$C = \begin{bmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{bmatrix}. \quad \text{Let } D = I - C = \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix}$$

Then Consider $S = I + D + D^2 + D^3 + \cdots$

$$D = \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix}, \quad D^2 = \begin{bmatrix} -\varepsilon^2 & 0 \\ 0 & -\varepsilon^2 \end{bmatrix}, \quad D^3 = \begin{bmatrix} 0 & -\varepsilon^3 \\ \varepsilon^3 & 0 \end{bmatrix}, \quad \cdots$$

If you like, you can show by induction that for even powers

$$D^n = (-\varepsilon^2)^{\frac{n}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and for odd powers,

$$D^n = (-\varepsilon^2)^{\frac{n-1}{2}} \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix}$$

Thus if we sum just the even powers, we get

$$S^+ := \sum_{n \text{ even}} D^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-\varepsilon^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-\varepsilon^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \varepsilon^2 + \varepsilon^4 - \cdots & 0 \\ 0 & 1 - \varepsilon^2 + \varepsilon^4 - \cdots \end{bmatrix}$$

Now $1 - \varepsilon^2 + \varepsilon^4 - \cdots$ is the geometric series $\sum_{m=0}^{\infty} (-\varepsilon^2)^m = \frac{1}{1-(-\varepsilon^2)} = \frac{1}{1+\varepsilon^2}$

Thus $S^+ = \begin{bmatrix} \frac{1}{1+\varepsilon^2} & 0 \\ 0 & \frac{1}{1+\varepsilon^2} \end{bmatrix}$.

continued) → 9

(1.5.8 continued)

Similarly sum the odd powers, we get

$$S := \sum_{n \text{ odd}} D^n = \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix} + (-\varepsilon^2)\begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix} + (-\varepsilon^2)^2\begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix} + \cdots$$

$$= \left(\sum_{m=0}^{\infty} (-\varepsilon^2)^m \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix} \right)$$

$$= \frac{1}{1+\varepsilon^2} \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\varepsilon}{1+\varepsilon^2} \\ \frac{-\varepsilon}{1+\varepsilon^2} & 0 \end{bmatrix}$$

Putting these together, we find that $\begin{bmatrix} \frac{1}{1+\varepsilon^2} & \frac{\varepsilon}{1+\varepsilon^2} \\ \frac{-\varepsilon}{1+\varepsilon^2} & \frac{1}{1+\varepsilon^2} \end{bmatrix} = C^{-1}$

(1.5.20) (a) Let $Mat(n,m)$ denote the space of $n \times m$ matrices, which we identify with $\mathbb{R}^{nm}$. For what numbers $a \in \mathbb{R}$ does the sequence of matrices $A^k \in Mat(2,2)$ converge as $k \to \infty$, when $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$? What is the limit?

I claim that $A^k = \begin{bmatrix} 2^{k-1}a^k & 2^{k-1}a^k \\ 2^{k-1}a^k & 2^{k-1}a^k \end{bmatrix}$ $\forall k \geq 1$.

I'll prove this by induction. The base case $k=1$ is true. Now assume true for k, and we'll show true for $k+1$.

$$A^{k+1} = A^k \cdot A = \begin{bmatrix} 2^{k-1}a^k & 2^{k-1}a^k \\ 2^{k-1}a^k & 2^{k-1}a^k \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$= \begin{bmatrix} (2^{k-1}a^{k+1}) \cdot 2 & (2^{k-1}a^{k+1}) \cdot 2 \\ \text{same} & \text{same} \end{bmatrix}$$

$$= \begin{bmatrix} 2^k a^{k+1} & 2^k a^{k+1} \\ 2^k a^{k+1} & 2^k a^{k+1} \end{bmatrix}$$

Thus the claim is true $\forall k$.

Thus $\lim_{k \to \infty} A^k = \lim_{k \to \infty} 2^{k-1}a^k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

so the question really becomes: when does $\lim_{k \to \infty} 2^{k-1}a^k$ converge?

$$\lim_{k \to \infty} 2^{k-1}a^k = a \cdot \lim_{k \to \infty} (2k)^{k-1}$$

This converges to 0 for $|a| < \frac{1}{2}$. Thus for $|a| < \frac{1}{2}$, $\lim_{k \to \infty} A^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

But what about the boundary points (always check boundary points!)

When $a = \frac{1}{2}$, $A^k = A$ $\forall k$, so $\lim_{k \to \infty} A^k = A$. When $a = -\frac{1}{2}$, $A^k = (-1)^{k-1}A$, so no limit exists.

If $|a| > \frac{1}{2}$, then the sequence diverges.

(b) What about 3×3 or $n \times n$ matrices, where every entry is a?

The proof is almost identical, so I'll omit it. Feel free to ask me if you want. The result is that $\lim_{k \to \infty} A^k =$ the zero matrix if $|a| < \frac{1}{n}$ (for $n \times n$ matrix) and

$\lim_{k \to \infty} A^k = A$ if $a = \frac{1}{n}$

diverges otherwise.

(1.6.2) Let $A \subset \mathbb{R}^n$ be a subset that is not compact. Show that there exists a continuous unbounded function on A

Since A is not compact, then it is either not closed or not bounded (or both). We split the proof into cases.

Case 1: A is not bounded. Consider $f(\vec{x}) = |\vec{x}|$. Since A is unbounded, A contains $\vec{x}$ which are arbitrarily far from the origin, so $|\vec{x}|$ is unbounded. Note that the absolute value is continuous, so we are done.

Case 2: A is not closed. If A is not closed, then A does not contain all of its boundary. Specifically, there is some boundary point $\vec{a}$ which is not in A. Then

$$f(\vec{x}) = \frac{1}{|\vec{x} - \vec{a}|}$$

is continuous on A since it is a rational function and the denominator is nonzero (on A at least). Also, $f(\vec{x})$ is unbounded since any neighborhood of $\vec{a}$ contains points in A, so $\vec{x} - \vec{a}$ can be arbitrarily small for $\vec{x} \in A$.

(1.6.6)

(a) Show that the function $f(x) = |x| e^{-|x|}$ has an absolute maximum at some $x > 0$.

First of all, we may restrict our attention to when x is positive, since negative x just mirror the positive. So redefine

$$f(x) = x e^{-x} \qquad x \geq 0$$

By L'Hopital's Rule,

$$\lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = \lim_{x \to \infty} e^{-x} = 0$$

Thus, for any $\varepsilon > 0$, there is some $\delta > 0$ such that $\delta < x \Rightarrow xe^{-x} < \varepsilon$.

Now pick ε to be less than any specific value of $x e^{-x}$, say $1 \cdot e^{-1} = 1/e$. Then $x > \delta \Rightarrow xe^{-x} < 1/\varepsilon$. So any max would have to occur for x in $[0, \delta]$. But $[0, \delta]$ is closed and bounded, so compact, so $f(x)$ does indeed take a maximum value within that range.

(b) Where is the maximum of this function? What is the maximum value?

Again, restrict to $x \geq 0$; we'll fill in the rest by symmetry. Any maximum (local or global) must occur as a location where the derivative is equal to zero.

$$f'(x) = -xe^{-x} + e^{-x}$$
$$= (1-x)e^{-x}$$

Now e^{-x} is never zero, so $x=1$ is the only (positive) x at which we have a maximum. Hence the max occurs at $x = 1, -1$ and the max value is $f(1) = 1/\varepsilon$.

(c) Show that the image of f is $[0, 1/\varepsilon]$.

We know that the function's max value is $1/\varepsilon$. Also, $f(x)$ is non-negative, so $f(x) \geq 0$, and $f(0) = 0$, so 0 is the global minimum. Then by the Intermediate Value Theorem, $f(x)$ takes all values between 0 and $1/\varepsilon$, so the image of f is $[0, 1/\varepsilon]$.

(1.6.7) Show that if f is differentiable on a neighborhood of $[a,b]$, and

$$f'(a) < m < f'(b)$$

then there exists $c \in (a,b)$ such that $f'(c) = m$

Suppose first that $f(a) \leq f(b)$, and consider $g(x) = f(x) - mx$. Now let's consider the minimum of $g(x)$ on $[a,b]$. Since $[a,b]$ is compact, $g(x)$ has a minimum. If the minimum were to occur at a, this would say that $f(x) - mx \geq f(a) - ma \quad \forall x \in [a,b]$, or equivalently,

$$f(x) - f(a) \geq mx - ma \qquad \forall x \in [a,b]$$

$$\Rightarrow \quad \frac{f(x) - f(a)}{x - a} \geq m \qquad \forall x \in [a,b]$$

$$\Rightarrow \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \geq m, \text{ a contradiction}$$

Similarly, the minimum can't occur at b, for then

$$f(x) - mx \geq f(b) - mb \qquad \forall x \in [a,b]$$

$$\Rightarrow \quad \frac{f(x) - f(b)}{x - b} \leq m \qquad (\text{since } x-b \text{ is negative})$$

Again, this leads to a contradiction.

Thus, the minimum of $g(x)$ occurs at some point $c \in (a,b)$, so $g'(c) = 0$, which implies

$$f'(c) - m = 0$$

$$\Rightarrow \quad f'(c) = m.$$

The case when $f(a) \geq f(b)$ is almost the same, just chang $g(x)$ to $mx - f(x)$ and repeat the argument.