Problem Set #5

(1.7.22) Let U be an open subset of $\mathbb{R}^n$ and let $f: U \to \mathbb{R}$ be differentiable at $\bar{a} \in U$. Show that if $\bar{v}$ is a unit vector making an angle θ with the gradient $\vec{\nabla} f(\bar{a})$, then

$$[Df(\bar{a})]\bar{v} = |\vec{\nabla} f(\bar{a})| \cos\theta$$

Why does this justify saying that $\vec{\nabla} f(\bar{a})$ points in the direction in which f increases fastest, and that $|\vec{\nabla} f(\bar{a})|$ is this fastest rate of increase?

Note that the gradient $\vec{\nabla} f(\bar{a})$ is just the Jacobian $[Df(\bar{a})]$ in the special case when the image of f is $\mathbb{R}^1$ (so that $[Df(\bar{a})]$ is a $1 \times n$ matrix, which by transpose I can think of as a vector. Then

$$\begin{aligned}[Df(\bar{a})]\bar{v} &= \vec{\nabla} f(\bar{a}) \cdot \bar{v} && \text{(the dot product)}\\ &= |\vec{\nabla} f(\bar{a})| \cdot |\bar{v}| \cos\theta && \text{(by Definition 1.4.6)}\\ &= |\vec{\nabla} f(\bar{a})| \cos\theta && \text{(since } \bar{v} \text{ is a unit vector)}\end{aligned}$$

Now because f is differentiable at $\bar{a} \in U$, the directional derivative in the direction of $\bar{v}$ is just equal to $[Df(\bar{a})]\bar{v}$. To maximize this, we fix a length on v (making it a unit vector), so that we want to maximize

$$|\vec{\nabla} f(\bar{a})| \cos\theta$$

This is greatest when $\cos\theta = 1$, or when $\theta = 0$ so that $\bar{v}$ points in the direction of $\vec{\nabla} f(\bar{a})$. Hence $\vec{\nabla} f(\bar{a})$ points in the direction in which f increases fastest (maximizing the directional derivative), and this maximum rate of increase is

$$|\vec{\nabla} f(\bar{a})| \cos 0 = |\vec{\nabla} f(\bar{a})|.$$

(1.8.3) Is $f\begin{pmatrix}x\\y\end{pmatrix} = \sin(e^{xy})$ differentiable at $\begin{pmatrix}0\\0\end{pmatrix}$?

Yes, f is differentiable at $\begin{pmatrix}0\\0\end{pmatrix}$. To prove this, simply note that xy, $\sin(z)$, e^w are all differentiable, and f is simply a composition, which is differentable by Theorem 1.8.3 (the Chain Rule).

(1.8.4)(a) What compositions can you form of the following functions?

(i) $f\begin{pmatrix}x\\y\\z\end{pmatrix}=x^2+y^2$ (ii) $g\begin{pmatrix}a\\b\end{pmatrix}=2a+b^2$ (iii) $\vec{f}(t)=\begin{pmatrix}1\\2t\\t^2\end{pmatrix}$ (iv) $\vec{g}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\cos x\\x+y\\\sin y\end{pmatrix}$

The following compositions make sense:

$f(\vec{f}(t))$ $\qquad$ $\vec{f}\left(g\begin{pmatrix}a\\b\end{pmatrix}\right)$

$f\left(\vec{g}\begin{pmatrix}x\\y\end{pmatrix}\right)$

$\vec{f}\left(f\begin{pmatrix}x\\y\\z\end{pmatrix}\right)$

(b) Compute these compositions.

$$f(\vec{f}(t)) = f\begin{pmatrix}1\\2t\\t^2\end{pmatrix} = 1+4t^2 \qquad \vec{f}\left(f\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \vec{f}\,(x^2+y^2) = \begin{pmatrix}1\\2(x^2+y^2)\\(x^2+y^2)^2\end{pmatrix}$$

$$f\left(\vec{g}\begin{pmatrix}x\\y\end{pmatrix}\right) = f\begin{pmatrix}\cos x\\x+y\\\sin y\end{pmatrix} = \cos^2 x+(x+y)^2 \qquad \vec{f}\left(g\begin{pmatrix}a\\b\end{pmatrix}\right) = \vec{f}(2a+b^2) = \begin{pmatrix}1\\4a+2b^2\\(2a+b^2)^2\end{pmatrix}$$

(c) Compute their derivatives, both by using the chain rule and directly from the compositions.

First let's do each directly:

$$\left[D(f\circ\vec{f})(t)\right] = [8t]$$

$$\left[D(f\circ\vec{g})\begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{bmatrix}-2\cos x\sin x+2(x+y) & 2(x+y)\end{bmatrix}$$

$$\left[D(\vec{f}\circ f)\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}0 & 0 & 0\\ 4x & 4y & 0\\ 4x(x^2+y^2) & 4y(x^2+y^2) & 0\end{bmatrix}$$

$$\left[D(\vec{f}\circ g)\begin{pmatrix}a\\b\end{pmatrix}\right] = \begin{bmatrix}0 & 0\\ 4 & 4b\\ 4(2a+b^2) & 4b(2a+b^2)\end{bmatrix}$$

Now we still need to compute via the chain rule. (see next page)

(continued) ⟶

(1.8.4 (c) continued)

$$[\mathbf{D}(f\circ\vec{f})(t)] = [\mathbf{D}f(\vec{f}(t))]\circ[\mathbf{D}\vec{f}(t)]$$
$$= \begin{bmatrix} 2x & 2y & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}$$

plug in new variables

$x = 1$
$y = 2t$
$z = 0$

$$= [4y]$$
$$= [8t]$$

$$[\mathbf{D}(f\circ\vec{g})\begin{pmatrix} x \\ y \end{pmatrix}] = [\mathbf{D}f(\vec{g}\begin{pmatrix} x \\ y \end{pmatrix})]\circ[\mathbf{D}\vec{g}\begin{pmatrix} x \\ y \end{pmatrix}]$$
$$= \begin{bmatrix} 2x' & 2y' & 0 \end{bmatrix}\begin{bmatrix} -\sin x & 0 \\ 1 & 1 \\ 0 & \cos y \end{bmatrix}$$

replace

$x' = \cos x$
$y' = x+y$
$z' = 0$

$$= \begin{bmatrix} -2\cos x\sin x + 2(x+y) & 2(x+y) \end{bmatrix}$$

$$[\mathbf{D}(\vec{f}\circ f)\begin{pmatrix} x \\ y \\ z \end{pmatrix}] = [\mathbf{D}\vec{f}(f\begin{pmatrix} x \\ y \\ z \end{pmatrix})]\circ[\mathbf{D}f\begin{pmatrix} x \\ y \\ z \end{pmatrix}]$$
$$= \begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}\begin{bmatrix} 2x & 2y & 0 \end{bmatrix}$$

← replace

$t = x^2+y^2$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 4x & 4y & 0 \\ 4x(x^2+y^2) & 4y(x^2+y^2) & 0 \end{bmatrix}$$

$$[\mathbf{D}(\vec{f}\circ g)\begin{pmatrix} a \\ b \end{pmatrix}] = [\mathbf{D}\vec{f}(g\begin{pmatrix} a \\ b \end{pmatrix})][\mathbf{D}g\begin{pmatrix} a \\ b \end{pmatrix}]$$
$$= \begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}\begin{bmatrix} 2 & 2b \end{bmatrix}$$

$t = 2a+b^2$

$$= \begin{bmatrix} 0 & 0 \\ 4 & 4b \\ 4(2a+b^2) & 4b(2a+b^2) \end{bmatrix}$$

These agree with our earlier calculations.

(1.8.6)(a) Prove the rule for differentiating dot products (part 7 of Theorem 1.8.1) directly from the definition of derivative.

We want to show that $[D(\bar{f}\cdot\bar{g})(\bar{a})]\bar{v} = [D\bar{f}(\bar{a})]\bar{v}\cdot\bar{g}(\bar{a}) + \bar{f}(\bar{a})\cdot[D\bar{g}(\bar{a})]\bar{v}$. Said another way, we want to show that

$$\lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big((\bar{f}\cdot\bar{g})(\bar{a}+\bar{v}) - (\bar{f}\cdot\bar{g})(\bar{a}) - [D\bar{f}(\bar{a})]\bar{v}\cdot\bar{g}(\bar{a}) - \bar{f}(\bar{a})\cdot[D\bar{g}(\bar{a})]\bar{v} \Big) = \bar{0}$$

Note that, by definition,

$$(\bar{f}\cdot\bar{g})(\bar{a}+\bar{v}) = \bar{f}(\bar{a}+\bar{v})\cdot\bar{g}(\bar{a}+\bar{v}) \text{ and } (\bar{f}\cdot\bar{g})(\bar{a}) = \bar{f}(\bar{a})\cdot\bar{g}(\bar{a}).$$

Thus our initial limit becomes

$$\lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{f}(\bar{a}+\bar{v})\cdot\bar{g}(\bar{a}+\bar{v}) - \bar{f}(\bar{a})\cdot\bar{g}(\bar{a}) - [D\bar{f}(\bar{a})]\bar{v}\cdot\bar{g}(\bar{a}) - \bar{f}(\bar{a})\cdot[D\bar{g}(\bar{a})]\bar{v} \Big)$$

$$= \lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{f}(\bar{a}+\bar{v})\cdot\bar{g}(\bar{a}+\bar{v}) - \bar{f}(\bar{a}+\bar{v})\cdot g(\bar{a}) + \bar{f}(\bar{a}+\bar{v})\cdot g(\bar{a}) - \bar{f}(\bar{a})\cdot g(\bar{a})$$
$$- [D\bar{f}(\bar{a})]\bar{v}\cdot\bar{g}(\bar{a}) - \bar{f}(\bar{a})\cdot[D\bar{g}(\bar{a})]\bar{v} \Big)$$

add and subtract $\bar{f}(\bar{a}+\bar{v})\cdot g(\bar{a})$

$$= \lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{f}(\bar{a}+\bar{v})\cdot\bar{g}(\bar{a}+\bar{v}) - \bar{f}(\bar{a}+\bar{v})\cdot g(\bar{a}) - \bar{f}(\bar{a})\cdot[D\bar{g}(\bar{a})]\bar{v}$$
$$+ \bar{f}(\bar{a}+\bar{v})\cdot\bar{g}(\bar{a}) - \bar{f}(\bar{a})\cdot g(\bar{a}) - [D\bar{f}(\bar{a})]\bar{v}\cdot\bar{g}(\bar{a}) \Big)$$

reorder terms

Now split this into the sum of two limits, namely the top row and the bottom row of the last line in our equality. Then

$$\lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{f}(\bar{a}+\bar{v})\cdot g(\bar{a}+\bar{v}) - f(\bar{a}+\bar{v})\cdot g(\bar{a}) - f(\bar{a})\cdot[D\bar{g}(\bar{a})]\bar{v} \Big)$$

$$= \lim_{\bar{v}\to\bar{0}} \bar{f}(\bar{a}+\bar{v})\cdot \lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\big(\bar{g}(\bar{a}+\bar{v}) - \bar{g}(\bar{a})\big) - \bar{f}(\bar{a})\cdot \lim_{\bar{v}\to\bar{0}} [D\bar{g}(\bar{a})]\bar{v} \Big)$$

$$= \bar{f}(\bar{a})\cdot \lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{g}(\bar{a}+\bar{v}) - \bar{g}(\bar{a}) - [D\bar{g}(\bar{a})]\bar{v} \Big)$$

$= \bar{f}(\bar{a})\cdot\bar{0}$ since the last limit is equal to $\bar{0}$ by the assumption that g is differentiable at $\bar{a}$.

$= \bar{0}$

Similarly, $\displaystyle\lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{f}(\bar{a}+\bar{v})\cdot\bar{g}(\bar{a}) - \bar{f}(\bar{a})\cdot\bar{g}(\bar{a}) - [D\bar{f}(\bar{a})]\bar{v}\cdot\bar{g}(\bar{a}) \Big)$

$$= \bar{g}(\bar{a})\cdot \lim_{\bar{v}\to\bar{0}} \frac{1}{|\bar{v}|}\Big(\bar{f}(\bar{a}+\bar{v}) - \bar{f}(\bar{a}) - [D\bar{f}(\bar{a})]\bar{v} \Big)$$

$= \bar{g}(\bar{a})\cdot\bar{0}$

$= \bar{0}.$

(continued) →

(1.8.6(a) continued)

Thus the limit is equal to zero, so we have verified the differentiating of dot products formula. Note that this is <u>exactly</u> the proof of the standard product rule in calculus, but with dot products.

(b) By a similar argument, show that if $\bar{f}, \bar{g} : U \to \mathbb{R}^3$ are both differentiable at $\bar{a}$, then so is the cross product $\bar{f} \times \bar{g} : U \to \mathbb{R}^3$. Find the formula for this derivative.

From part (a), we noticed that the rule for differentiating a dot product was just the standard product rule, so we might guess here that

$$[D(\bar{f} \times \bar{g})(\bar{a})]\bar{v} = [D\bar{f}(\bar{a})]\bar{v} \times \bar{g}(\bar{a}) + \bar{f}(\bar{a}) \times [D\bar{g}(\bar{a})]\bar{v}$$

The proof that this holds is exactly the same as the proof in part (a) with each dot product $\bullet$ replaced by a cross product $\times$.

(1.8.9) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any differentiable function. Show that the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = y\,\varphi(x^2 - y^2)$$

satisfies the equation

$$\frac{1}{x} D_1 f\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{y} D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{y^2} f\begin{pmatrix} x \\ y \end{pmatrix}.$$

First note that by the chain rule,

$$D_1 f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\partial}{\partial x} f\begin{pmatrix} x \\ y \end{pmatrix} = y\,\varphi'(x^2 - y^2) \cdot 2x$$

$$D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\partial}{\partial y} f\begin{pmatrix} x \\ y \end{pmatrix} = y\,\varphi'(x^2 - y^2) \cdot (-2y) + \varphi(x^2 - y^2)$$

Then

$$\begin{aligned} \frac{1}{x} D_1 f\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{y} D_2 f\begin{pmatrix} x \\ y \end{pmatrix} &= 2y\,\varphi'(x^2 - y^2) - 2y\,\varphi'(x^2 - y^2) + \frac{1}{y}\varphi(x^2 - y^2) \\ &= \frac{1}{y}\varphi(x^2 - y^2) \\ &= \frac{1}{y^2} f\begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

(1.8.11) If $f\binom{x}{y} = \varphi\left(\frac{x+y}{x-y}\right)$ for some differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$, show that

$$x D_1 f + y D_2 f = 0.$$

Note that

$$D_1 f = \frac{\partial f\binom{x}{y}}{\partial x} = \varphi'\left(\frac{x+y}{x-y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right)$$

$$= \varphi'\left(\frac{x+y}{x-y}\right) \cdot \frac{(x-y)\cdot 1 - (x+y)}{(x-y)^2}$$

$$= \varphi'\left(\frac{x+y}{x-y}\right) \cdot \left(\frac{-2y}{(x-y)^2}\right)$$

Similarly,

$$D_2 f = \frac{\partial f\binom{x}{y}}{\partial y} = \varphi'\left(\frac{x+y}{x-y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right)$$

$$= \varphi'\left(\frac{x+y}{x-y}\right) \cdot \frac{(x-y)\cdot 1 - (x+y)(-1)}{(x-y)^2}$$

$$= \varphi'\left(\frac{x+y}{x-y}\right) \cdot \left(\frac{2x}{(x-y)^2}\right)$$

Thus

$$x D_1 f + y D_2 f = \varphi'\left(\frac{x+y}{x-y}\right)\left(\frac{-2xy}{(x-y)^2}\right) + \varphi'\left(\frac{x+y}{x-y}\right)\left(\frac{2xy}{(x-y)^2}\right)$$

$$= 0.$$

(1.9.1) Show that the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} \begin{cases} \frac{x^4+y^4}{x^2+y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

is differentiable at every point in $\mathbb{R}^2$.

First, let $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that the Jacobian is given by

$$\left[Df\begin{pmatrix} x \\ y \end{pmatrix}\right] = \left[\frac{4x^3(x^2+y^2) - 2x(x^4+y^4)}{(x^2+y^2)^2} \quad \frac{4y^3(x^2+y^2) - 2y(x^4+y^4)}{(x^2+y^2)^2}\right] \qquad (*)$$

Away from the origin, these partial derivatives exist and are continuous by Corollary 1.5.30. Thus f is C^1 away from the origin, hence differentiable except possibly at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now we still need to show that f is differentiable at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We will do this by showing that the partial derivatives exist at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and that along with (*) above, we can show these partials to be continuous.

First of all,

$$\lim_{h \to 0} \frac{1}{h}\Big(f(h,0) - f(0,0)\Big) = \lim_{h \to 0} \frac{1}{h}\left((0+h)^2 - 0\right) \qquad \text{(using Definition 1.7.13)}$$

$$= \lim_{h \to 0} h$$

$$= 0$$

Similarly, $\lim_{h \to 0} \frac{1}{h}\left(f(0,h) - f(0,0)\right) = \lim_{h \to 0} \frac{1}{h}\left((0+h)^2 - 0\right)$

$$= \lim_{h \to 0} h$$

$$= 0$$

Hence both partial derivatives exist at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and are equal to 0.

Now let's show that the two partial derivatives are continuous at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. For $\frac{\partial f}{\partial x}$,

$$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{4x^3(x^2+y^2) - 2x(x^4+y^4)}{(x^2+y^2)^2} = \lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{2x^5 + 4x^3y^2 - 2xy^4}{(x^2+y^2)^2}$$

$$= \lim_{r \to 0} \frac{r^5(2\cos^5\theta + 4\cos^3\theta\sin^2\theta - 2\cos\theta\sin^4\theta)}{r^4} \qquad \text{(polar coordinates)}$$

$$= 0$$

The partial $\frac{\partial f}{\partial y}$ is almost identical. Thus the partial derivatives are continuous at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so f is C^1 even at the origin, hence f is differentiable everywhere.

(1.9.2)(a) Show that for f defined below, all directional derivatives exist but that f is not differentiable at the origin:

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases} \dfrac{3x^2y-y^3}{x^2+y^2} & \text{if } \begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & \text{if } \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \end{cases}$$

If $\begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, then

$$\left[Df\begin{pmatrix}x\\y\end{pmatrix}\right] = \left[\frac{(x^2+y^2)6xy-(3x^2y-y^3)2x}{(x^2+y^2)^2} \quad \frac{(x^2+y^2)(3x^2-3y^2)-(3x^2y-y^3)2y}{(x^2+y^2)^2}\right]$$

These partial derivatives are continuous away from the origin, so f is C^1 except at $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

When $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, let's show that all directional derivatives exist by using the definition of directional derivative. Let $\vec{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ be any vector thought of as being based at the origin. Then the directional derivative of f at $\begin{pmatrix}0\\0\end{pmatrix}$ in the direction of $\vec{v}$ is

$$\begin{aligned}\lim_{h\to 0}\frac{f(\vec{0}+h\vec{v})-f(\vec{0})}{h} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{3h^3v_1^2v_2-h^3v_2^3}{h^2(v_1^2+v_2^2)}\right)\\ &= \lim_{h\to 0}\frac{3v_1^2v_2-v_2^3}{v_1^2+v_2^2}\\ &= \frac{3v_1^2v_2-v_2^3}{v_1^2+v_2^2} \qquad (*)\end{aligned}$$

In particular, the limit exists, so f has directional derivatives in every direction at the origin.

We still need to show that f is not differentiable at the origin. Suppose for a moment that f was differentiable at the origin (we'll use this to obtain a contradiction). Then

$$\left[Df\begin{pmatrix}0\\0\end{pmatrix}\right] = [0 \quad -1]$$

Here we just plugged in the partial derivatives at $\begin{pmatrix}0\\0\end{pmatrix}$ using the formula in (*). More specifically,

$$\begin{aligned}\frac{\partial f}{\partial x}\begin{pmatrix}0\\0\end{pmatrix} &= \lim_{h\to 0}\frac{1}{h}\left(f\begin{pmatrix}h\\0\end{pmatrix}-f\begin{pmatrix}0\\0\end{pmatrix}\right)\\ &= \lim_{h\to 0}\frac{1}{h}(0)\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}\begin{pmatrix}0\\0\end{pmatrix} &= \lim_{h\to 0}\frac{1}{h}\left(f\begin{pmatrix}0\\h\end{pmatrix}-f\begin{pmatrix}0\\0\end{pmatrix}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h^3}{h^2}\right)\\ &= -1\end{aligned}$$

If this holds, then the directional derivative at $\begin{pmatrix}0\\0\end{pmatrix}$ in the direction $\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$ is

$$\frac{3-1}{2} = 1 \neq [0 \quad -1]\begin{bmatrix}1\\1\end{bmatrix}$$

Thus f is not differentiable at the origin.

(continued) →

(1.9.2 continued)

(b) Show that the function g defined below has directional derivatives at every point, but is not continuous.

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

First note that for $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$\left[Dg\begin{pmatrix} x \\ y \end{pmatrix}\right] = \left[\frac{(x^4+y^2)2xy - 4x^3(x^2y)}{(x^4+y^2)^2} \quad \frac{(x^4+y^2)x^2 - 2y(x^2y)}{(x^4+y^2)^2}\right]$$

Thus g is C^1 away from the origin, so in particular, directional derivatives exist.

At $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we use the definition of directional derivative to show that they exist. Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then

$$\lim_{h \to 0} \frac{1}{h}\left(g(h\vec{v}) - g\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{h^3 v_1^2 v_2}{h^2(h^2 v_1^4 + v_2^2)}\right)$$

$$= \lim_{h \to 0} \frac{v_1^2 v_2}{h_1^2 v_1^4 + v_2^2}$$

$$= \lim_{h \to 0} \frac{v_1^2 v_2}{v_2^2}$$

$$= \lim_{h \to 0} \frac{v_1^2}{v_2}$$

$$= \frac{v_1^2}{v_2}$$

Unless of course, $v_2 = 0$, in which case $\displaystyle\lim_{h \to 0} \frac{v_1^2 v_2}{h^2 v_1^4 + v_2^2} = \lim_{h \to 0} \frac{0}{h^2 v_1^4} = 0$.

Thus directional derivatives exist at every point.

Finally, we show that g is not continuous at the origin by showing that the limit attains different values on different approaches.

Along $x=0$: $\displaystyle\lim_{y \to 0} g\begin{pmatrix} 0 \\ y \end{pmatrix} = \lim_{y \to 0} \frac{0}{y^2} = 0$

Along $y = x^2$: $\displaystyle\lim_{\begin{pmatrix} x \\ x^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} g\begin{pmatrix} x \\ x^2 \end{pmatrix} = \lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

These limits are not equal, so the function is not continuous at the origin.

(1.9.2 continued)

(c) Show that the function h defined below has directional derivatives at all points but is not bounded in a neighborhood of $\bar{0}$.

$$h\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases} \dfrac{x^2y}{x^6+y^2} & \text{if } \begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & \text{if } \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \end{cases}$$

If $\begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, then $\left[Dh\begin{pmatrix}x\\y\end{pmatrix}\right] = \left[\dfrac{(x^6+y^2)2xy - x^2y\cdot 6x^5}{(x^6+y^2)^2} \quad \dfrac{(x^6+y^2)x^2 - x^2y\cdot 2y}{(x^6+y^2)^2}\right]$

Thus h is continuously differentiable away from the origin, so directional derivatives exist for all other points.

At the origin, we use the definition of directional derivative. Let $\bar{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$. Then

$$\lim_{\ell\to 0} \frac{1}{\ell}\left(h(\ell\bar{v}) - h\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$= \lim_{\ell\to 0} \frac{1}{\ell}\left(\frac{\ell^3 v_1^2 v_2}{\ell^6 v_1^6 + \ell^2 v_2^2}\right)$$

$$= \lim_{\ell\to 0} \frac{v_1^2 v_2}{\ell^4 v_1^6 + v_2^2}$$

$$= \frac{v_1^2 v_2}{v_2^2}$$

$$= \frac{v_1^2}{v_2} \quad \text{as long as } v_2 \neq 0.$$

If $v_2 = 0$, then $\displaystyle\lim_{\ell\to 0} \frac{v_1^2 v_2}{\ell^4 v_1^6 + v_2^2} = \lim_{\ell\to 0} \frac{0}{\ell^4 v_1^6} = 0$

Thus all directional derivatives exist at every point.

Finally, we need to show that h is unbounded in a neighborhood of $\bar{0}$. Consider the approach to the origin on which $y = x^3$ for positive values of x. On this path, h becomes

$$h\begin{pmatrix}x\\x^3\end{pmatrix} = \begin{cases} \dfrac{x^5}{2x^6} & \text{if } \begin{pmatrix}x\\x^3\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & \text{if } \begin{pmatrix}x\\x^3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \end{cases}$$

But $x^5/2x^6 = \frac{1}{2x}$ which tends to ∞ as $x \to 0$, so h is unbounded on this path to the origin, and hence unbounded in a neighborhood of the origin.

(2.10.2) Where is the mapping $\bar{f}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}xy\\x^2-y^2\end{pmatrix}$ guaranteed by the inverse function theorem to be locally invertible?

First we note that $\bar{f}$ is continuously differentiable, so by the inverse function theorem, $\bar{f}$ is locally invertible at every point $\bar{x}_0$ such that $\bar{f}(\bar{x}_0)$ is invertible. Thus we calculate

$$[D\bar{f}\begin{pmatrix}x\\y\end{pmatrix}] = \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix}$$

This derivative matrix is invertible as long as the determinant is nonzero, that is,

$$-2y^2 - 2x^2 \neq 0$$
$$-2(x^2+y^2) \neq 0$$

Thus we see that $\bar{f}$ is locally invertible except at the origin.

(2.10.6) Let $F : \mathbb{R}^2 - \begin{pmatrix}0\\0\end{pmatrix} \to \mathbb{R}^2$ be given by

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} \frac{(x^2-y^2)}{(x^2+y^2)} \\ \frac{xy}{(x^2+y^2)} \end{pmatrix}$$

Does F have a local inverse at every point of $\mathbb{R}^2$?

This is a somewhat odd question. Here $\mathbb{R}^2$ refers to the codomain of f, so we are asked if f is invertible at every point of the codomain. This doesn't make sense since f isn't even onto. In particular, $|xy| \leq x^2+y^2$, so $-1 \leq \frac{xy}{x^2+y^2} \leq 1$

Thus F does not map to any points in $\mathbb{R}^2$ with second coordinate greater than 1, so f cannot be invertible at such a point.

(2.10.8) Let $x^2+y^3+e^y=0$ define implicitly $y=f(x)$. Compute $f'(x)$ in terms of x and y.

First note that $x^2+y^3+e^y=0$ is differentiable by virtue of being a sum and composition of known differentiable functions. We write $F:\mathbb{R}^2\to\mathbb{R}$ by

$$F\begin{pmatrix}x\\y\end{pmatrix} = x^2+y^3+e^y$$

and let $\bar{c}$ be any point satisfying $F(\bar{c})=0$. Let's check that $[DF(\bar{c})]$ is onto.

$$[DF(\bar{c})] = \begin{bmatrix} 2x & 3y^2+e^y \end{bmatrix}$$

Note that $3y^2 \geq 0$ and $e^y > 0$, so this matrix is always onto.

We want to ~~write~~ think about $y=f(x)$, so think of x as the nonpivotal variable and y as the pivotal variable. Then there is some function f such that $y=f(x)$ in a neighborhood of $(x_0,y_0)=\bar{c}$ such that $F(\bar{c})=0$.

We don't know how large this neighborhood is without using the strong form of the implicit function theorem, but that's okay for our purposes. By the last line of Theorem 2.10.13,

$$f'(x) = -\frac{1}{3y^2+e^y}\cdot 2x$$

Thus we know the derivative without actually finding the implicit function $f(x)$.

(2.10.9) Does the system of equations
$$x+y+\sin(xy)=a$$
$$\sin(x^2+y)=2a$$
have a solution for sufficiently small a?

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y+\sin(xy) \\ \sin(x^2+y) \end{pmatrix}$$
and note that $f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then the question really becomes: Is f local invertible at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the origin in the codomain?

Note that $$\left[Df\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{bmatrix} 1+y\cos(xy) & 1+x\cos(xy) \\ 2x\cos(x^2+y) & \cos(x^2+y) \end{bmatrix}$$
so f is continuously differentiable. Furthermore, at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
$$\left[Df\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
which is invertible.

Thus, by Theorem 2.10.4, f is locally invertible with differentiable inverse in some neighborhood of $f\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If we wanted to know what that neighborhood is, then we would need to use Theorem 2.10.7, the full version of the inverse function theorem. As it stands, we already know that the system of equations
$$x+y+\sin(xy)=a$$
$$\sin(x^2+y)=2a$$
for sufficiently small a.