



# Fibonacci numbers

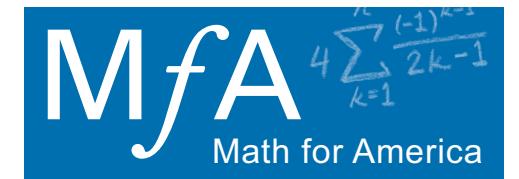
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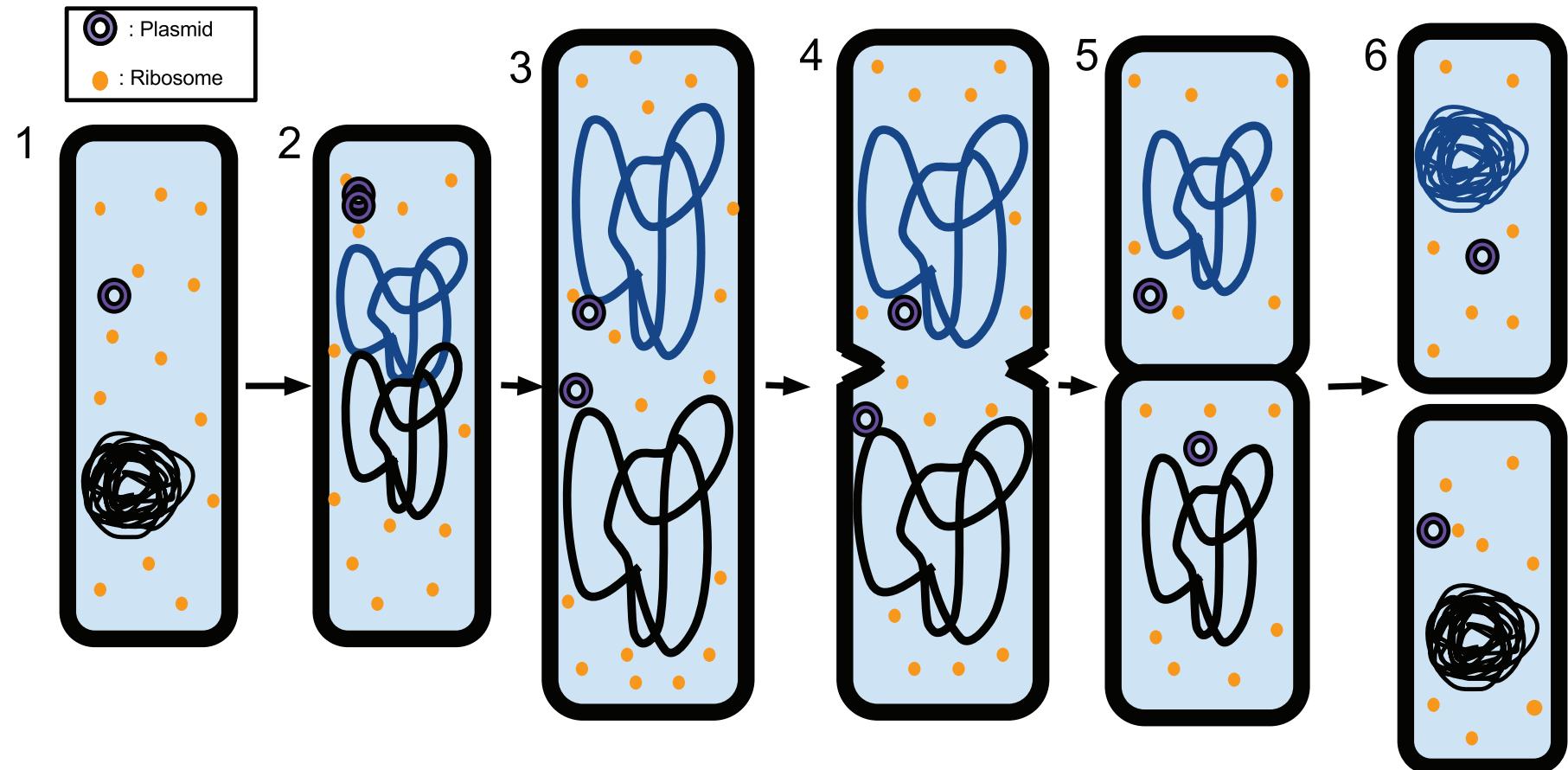
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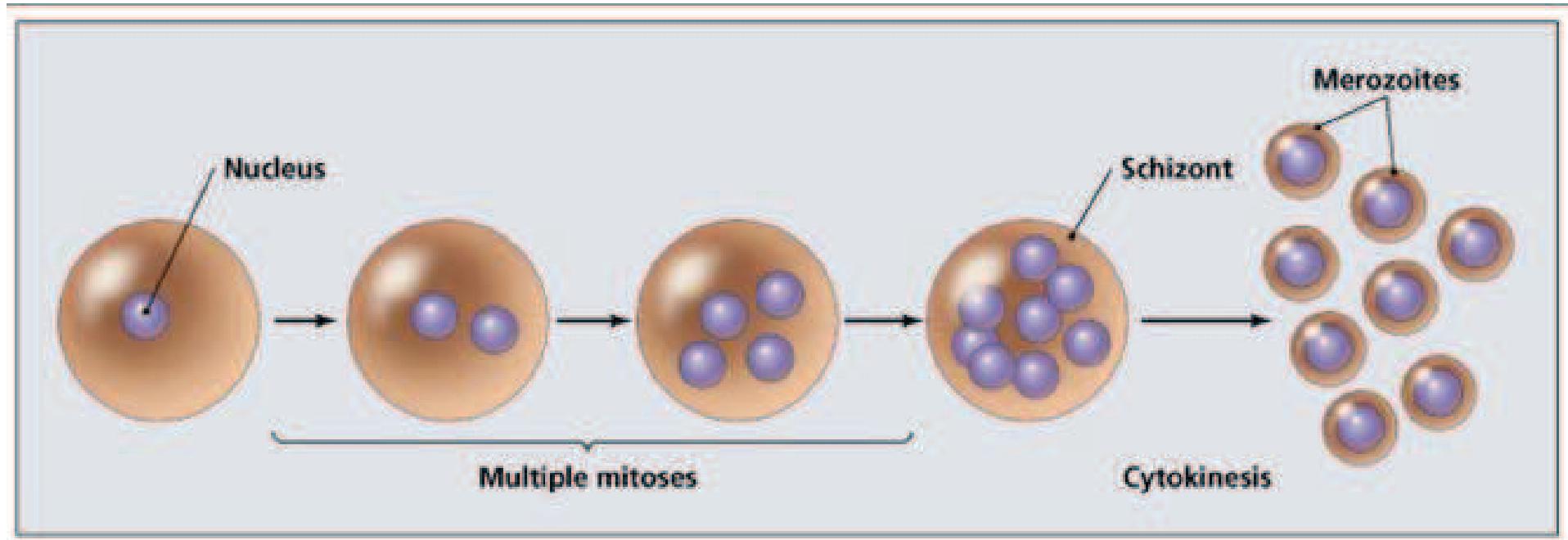


# Binary fission



bacterial reproduction

# Multiple fission



parasite reproduction

# Leonardo of Pisa — Fibonacci (c.1170–c.1250)



# Liber Abaci (1202)

# Liber Abaci (1202)

*Quot paria coniculorum in uno  
anno ex uno pario germinentur.*

Quidam posuit unum par cuniculorum in quodam loco, qui erat undique pariete circundatus, ut sciret, quot ex eo paria germinarentur in uno anno, cum natura eorum sit per singulum mensem aliud par germinare, et in secundo mense ab eorum nativitate germinant.

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# Liber Abaci (1202)

*How many pairs of rabbits are created by one pair in one year?*

A man placed one pair of rabbits together in an enclosure, so he might know how many are created from a pair in one year, when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

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# Phyllotaxis



pinecone

# Phyllotaxis



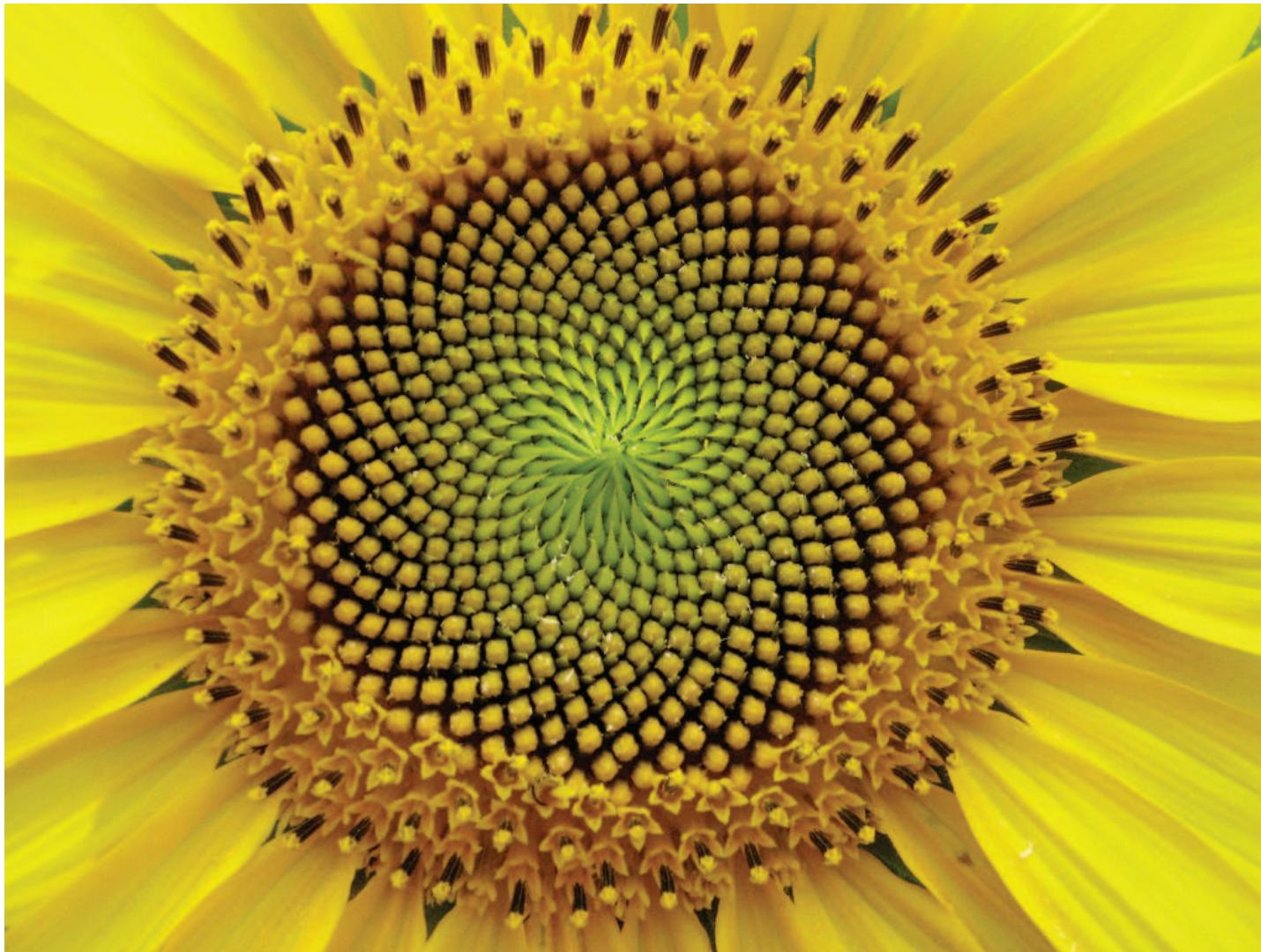
8 spirals counterclockwise

# Phyllotaxis



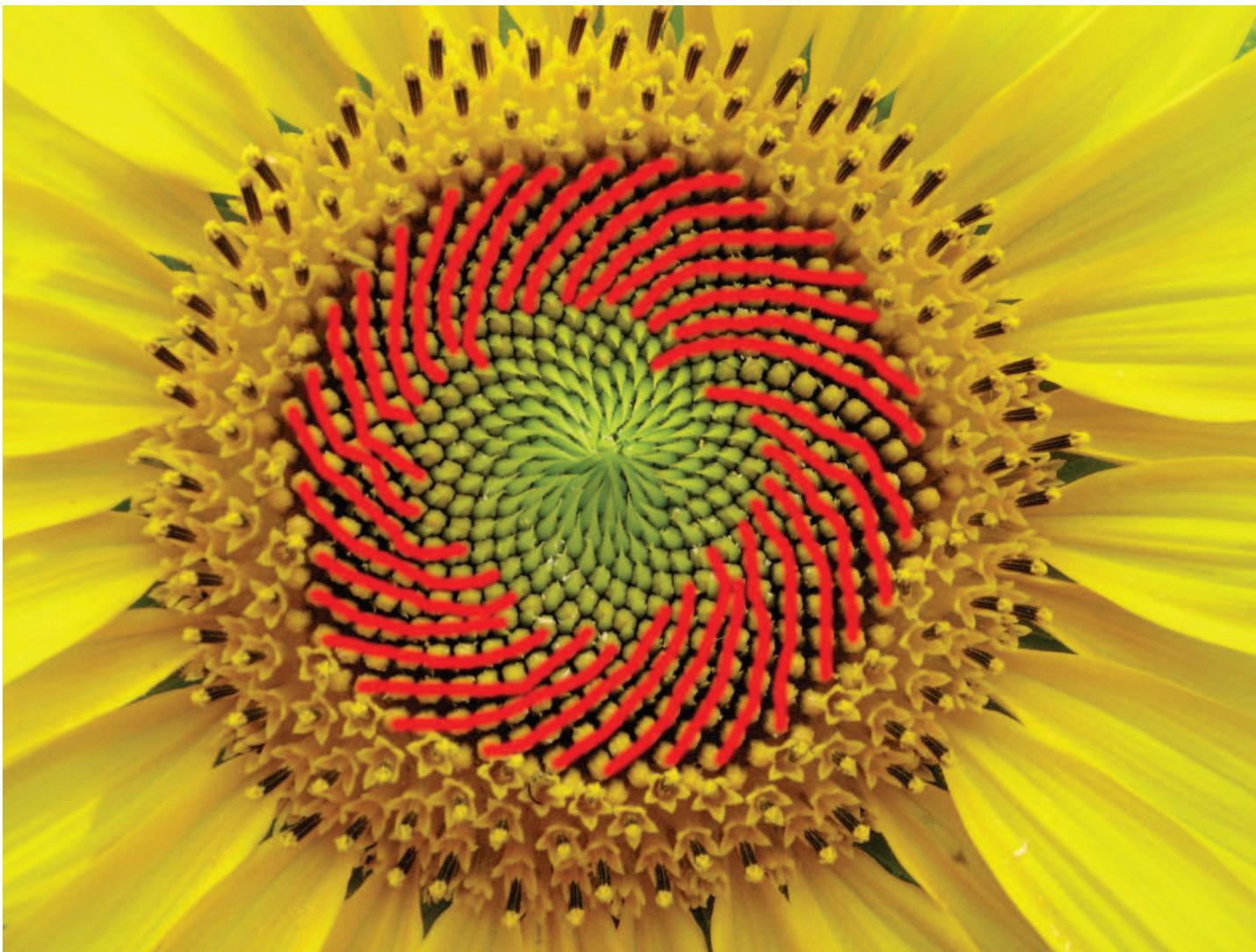
13 spirals clockwise

# Phyllotaxis



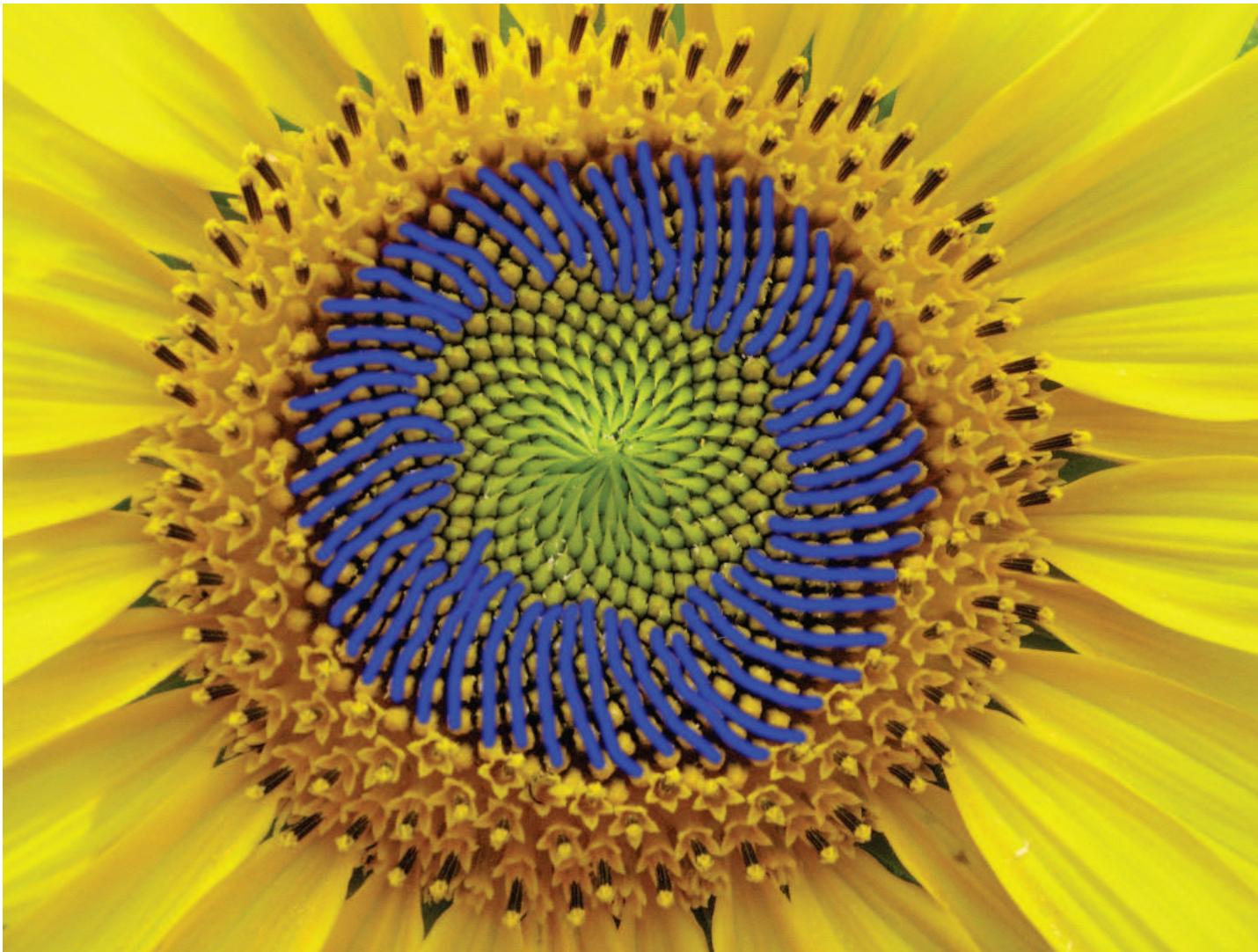
sunflower

# Phyllotaxis



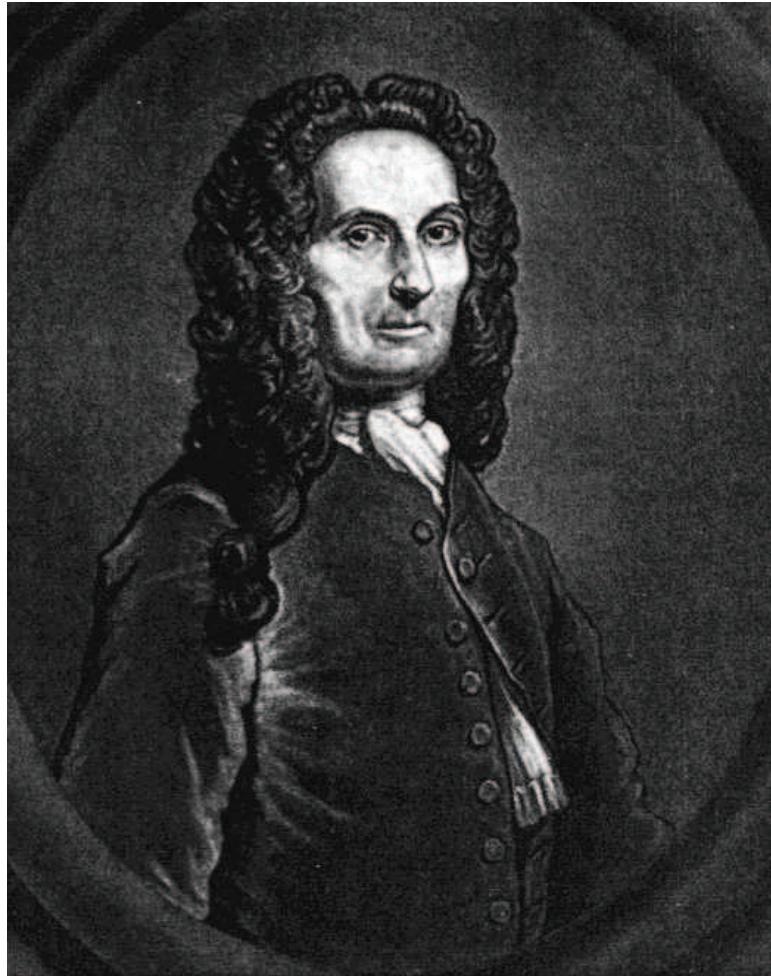
34 spirals clockwise

# Phyllotaxis



55 spirals counterclockwise

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$



Abraham de Moivre  
1667–1754



Jacques Philippe Marie Binet  
1786–1856

## Questions

1. What is the parity (evenness/oddness) of  $F_n$ ?
2. Use the recurrence relation to define  $F_n$  for negative integers  $n$ .
3. Find a formula for finite Fibonacci series:  $F_1 + F_2 + \cdots + F_n$ .
4. Find a formula for:  $F_1 + F_3 + \cdots + F_{2n-1}$ .
5. Find a formula for:  $F_1^2 + F_2^2 + \cdots + F_n^2$ .
6. Use the fact that  $\phi^2 = \phi + 1$  to find a formula for  $\phi^n$ .
7. Find a formula for  $F_n$  involving  $\phi$ , but not  $1/\phi$ .
8. What is  $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}$ ?

# Phyllotaxis



calla lily: 1 petal

# Phyllotaxis



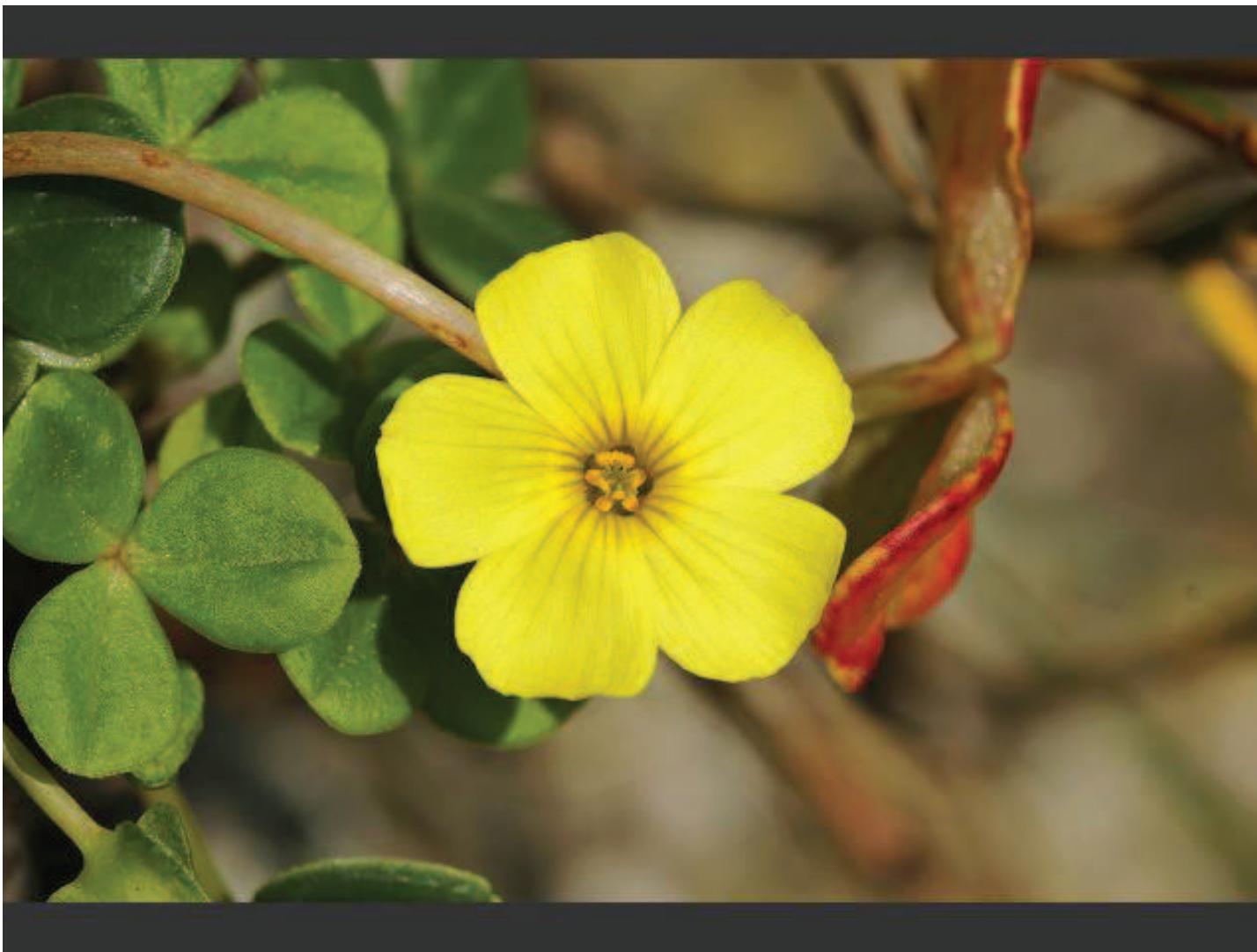
euphorbia: 2 petals

# Phyllotaxis



trillium: 3 petals

# Phyllotaxis



buttercup: 5 petals

# Phyllotaxis



columbine: 5 petals

# Phyllotaxis



delphinium: 5 petals

# Phyllotaxis



bloodroot: 8 petals

# Phyllotaxis



delphinium: 8 petals

# Phyllotaxis



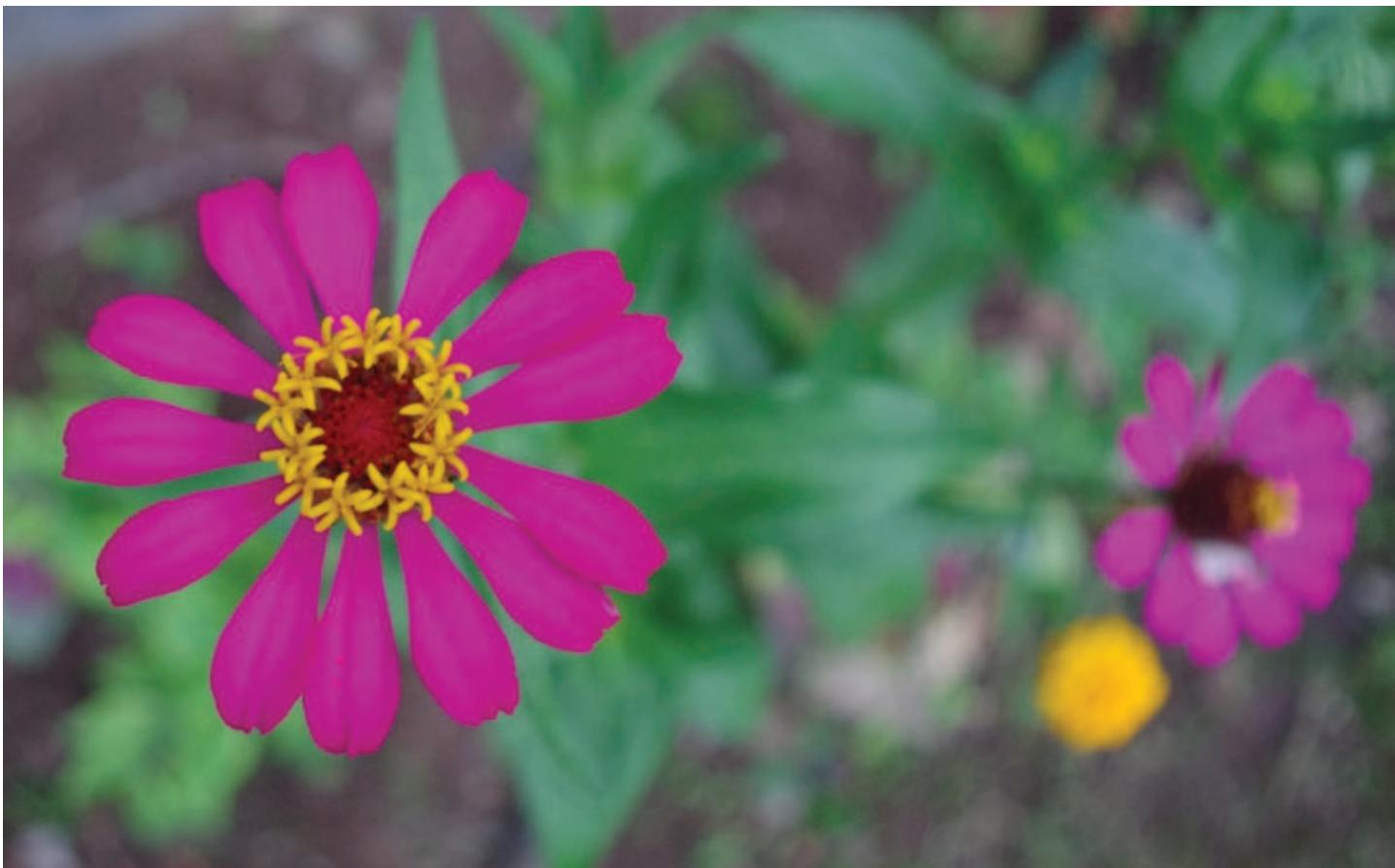
signet marigold: 8 petals

# Phyllotaxis



black-eyed Susan: 13 petals

# Phyllotaxis



13 petals

# Phyllotaxis



chrysanthemum carinatum: 21 petals

# Phyllotaxis



daisy: 21 petals

# Phyllotaxis

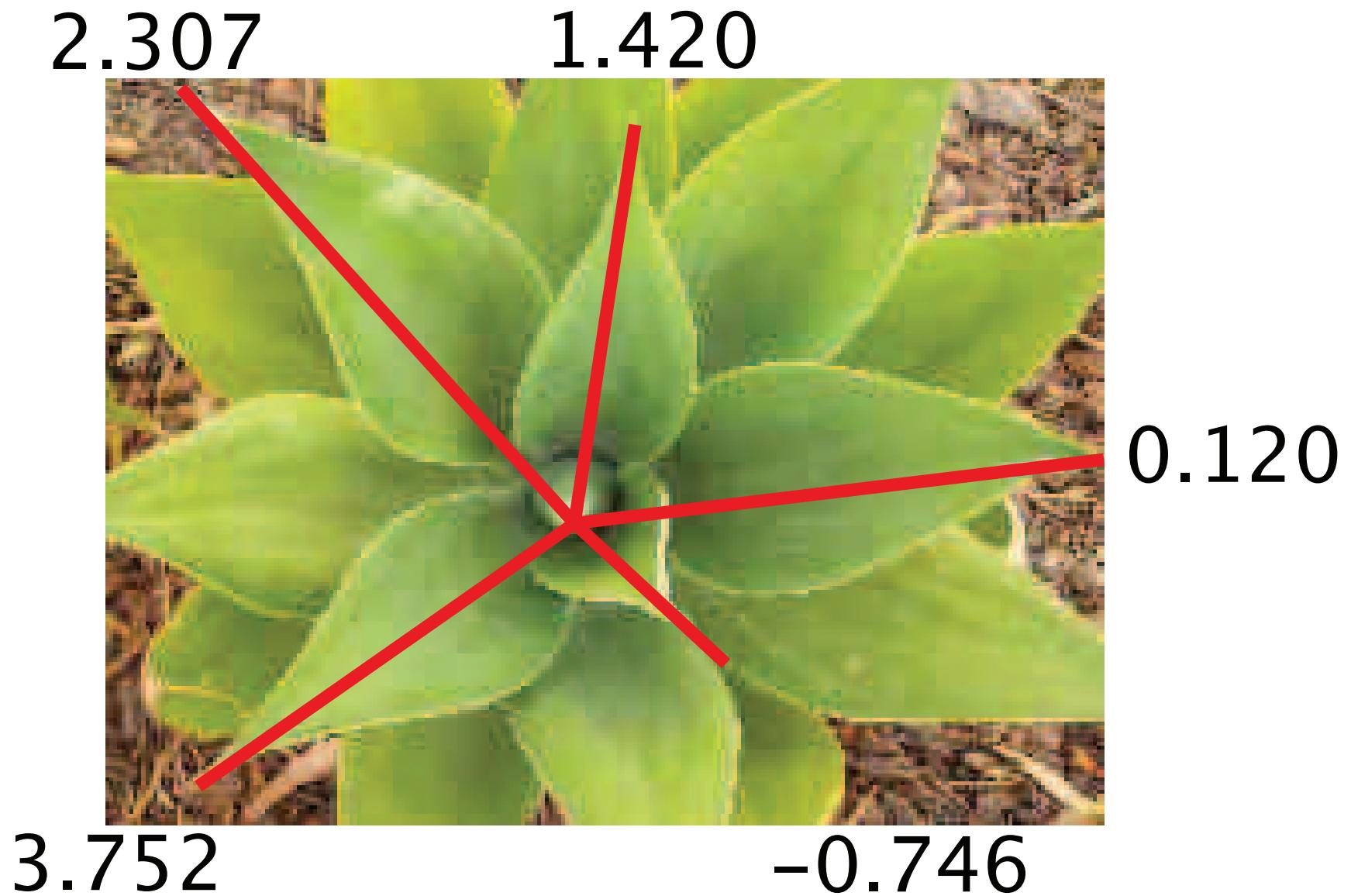


sunflower: 21 petals

# Phyllotaxis



# Phyllotaxis



## Questions

1. What are the first several convergents of  $\psi = 1 + \sqrt{2}$ ?
2. Find a quadratic equation satisfied by  $\psi$  (and  $-1/\psi$ ).
3. What is the corresponding recurrence relation?
4. What are the numbers analogous to the Fibonacci numbers?

And some geometry questions ...