Hearts and roses

David A. Meyer

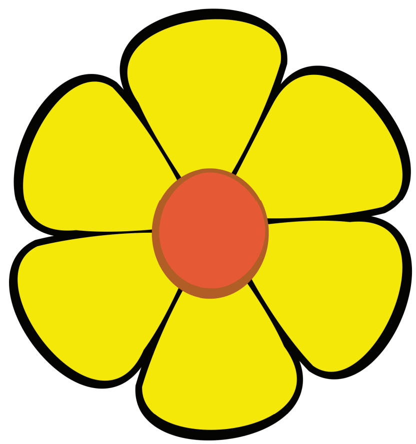
with Grant Allen and Eleanor Meyer

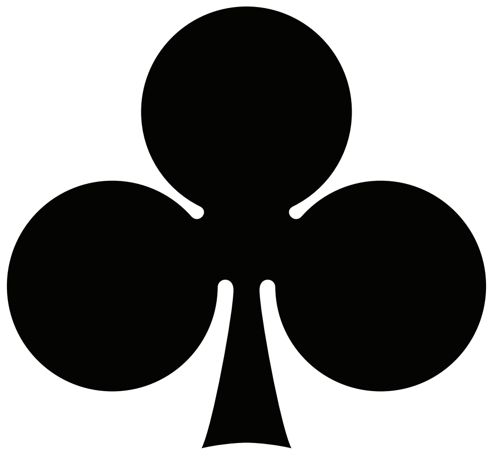
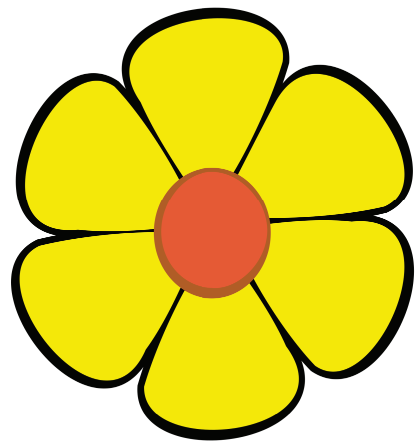
Mathematics Department, UC San Diego

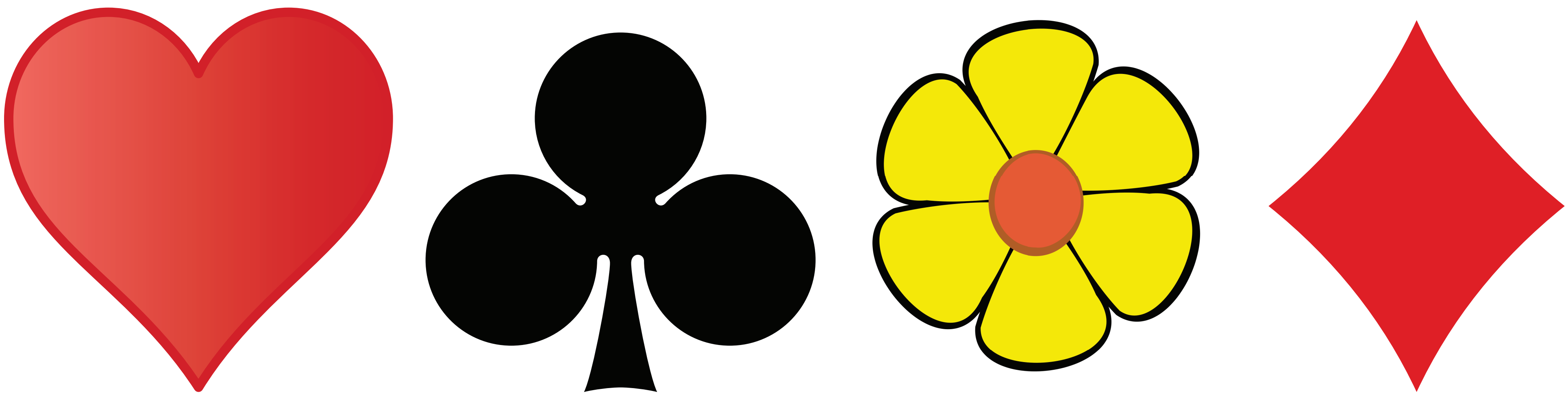
dmeyer@math.ucsd.edu

@dajmeyer

Student Colloquium
University of California, San Diego
La Jolla, CA, the day before Valentine's Day, 2018



„Ach! wunderselig ist die Braut,
Die's Krönlein tragen soll.
Ach, schenkte mir der Ritter traut
Ein Kränzlein nur von Rosen,
Wie wär' ich freudenvoll!“

Nicht lang, der Ritter trat herein,
Das Kränzlein wohl beschaut':
„O fasse, lieber Goldschmied mein,
Ein Ringlein mit Demanten
Für meine süße Braut!“

— Ludwig Uhland (1815)
„*Des Goldschmieds Töchterlein*“

‘Ah! wondrous happy lot is thine,
Who shall this chaplet wear;
Ah! what delight, what joy were mine
Gave he me but a chaplet
Of roses, I might wear.’

Not long before the knight came back,
Approved the wreath and cried,
‘I would, Sir goldsmith! ye would make
A wedding-ring with diamonds
For my enchanting bride!’

— Ludwig Uhland (1815)
„*Des Goldschmieds Töchterlein*“
translated by James Joseph Sylvester (1870)
as “The goldsmith’s daughter”

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a field where Geometry, Algebra, and the Theory of Numbers melt in a surprising manner into one another, like sun set tints or the colours of the dying dolphin, 'the last still loveliest'

— James Joseph Sylvester (1869)
Inaugural Presidential Address
to the Mathematical and Physical Section
of the British Association at Exeter

parting day
Dies like the dolphin, whom each pang imbues
With a new colour as it gasps away,
The last still loveliest, till—'tis gone—and all is gray.

— George Gordon, Lord Byron (1818)
Childe Harold's Pilgrimage, Canto the Fourth

I had often heard of the changing colors of a dying dolphin and now I was to witness them for the first time. No one can exaggerate the weird beauty of the sight as the fish in its last struggles changes through all its various hues. One can see the colors disappear, to be followed by others. Beginning with the head, they seem to sweep as a wave over the body. Blue gives place to white, then a light yellow, which in turn changes to a golden, and following this a copper-colored tint; and so on through all conceivable dues, until finally, the end having come, change is interrupted in its course, and two tints are left in possession of the body — one in the act of disappearing, the other about to spread itself over the surface.

— Ralph S. Tarr (1889)
"Animal life in the Gulf Stream"
The Popular Science Monthly

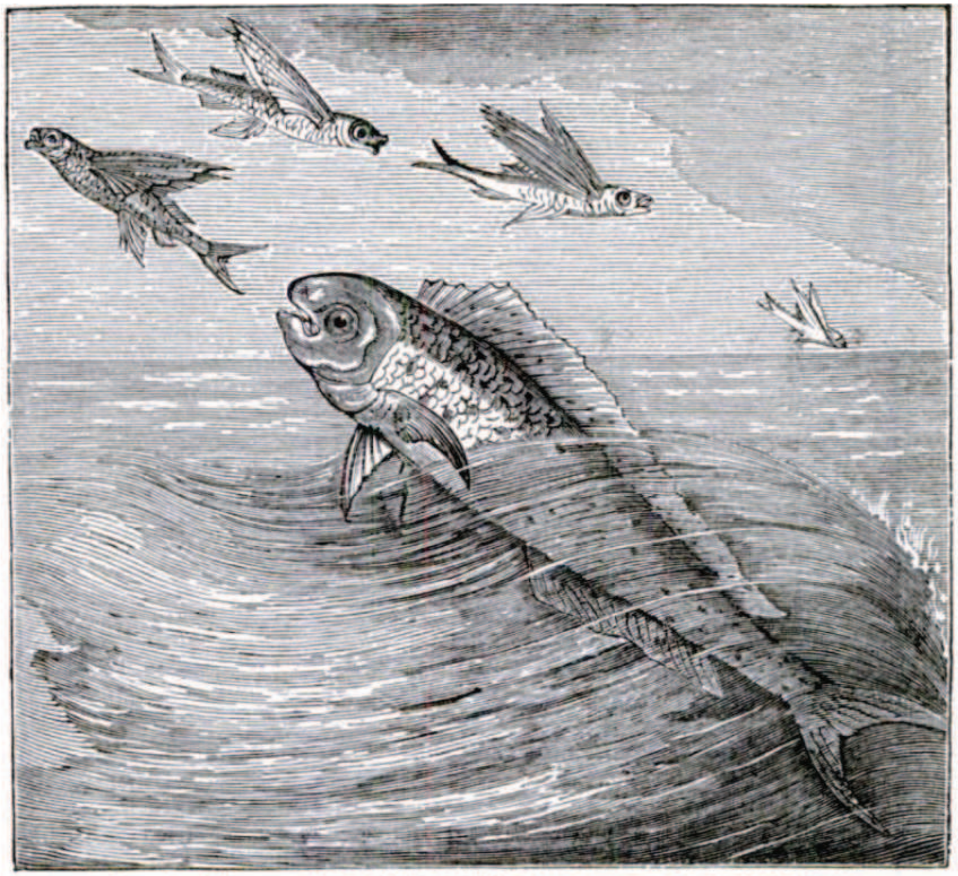

Fig. 4.—Flying-Fish (*Exocœtus*) pursued by the Dolphin.

— Ralph S. Tarr (1889)
"Animal life in the Gulf Stream"
The Popular Science Monthly

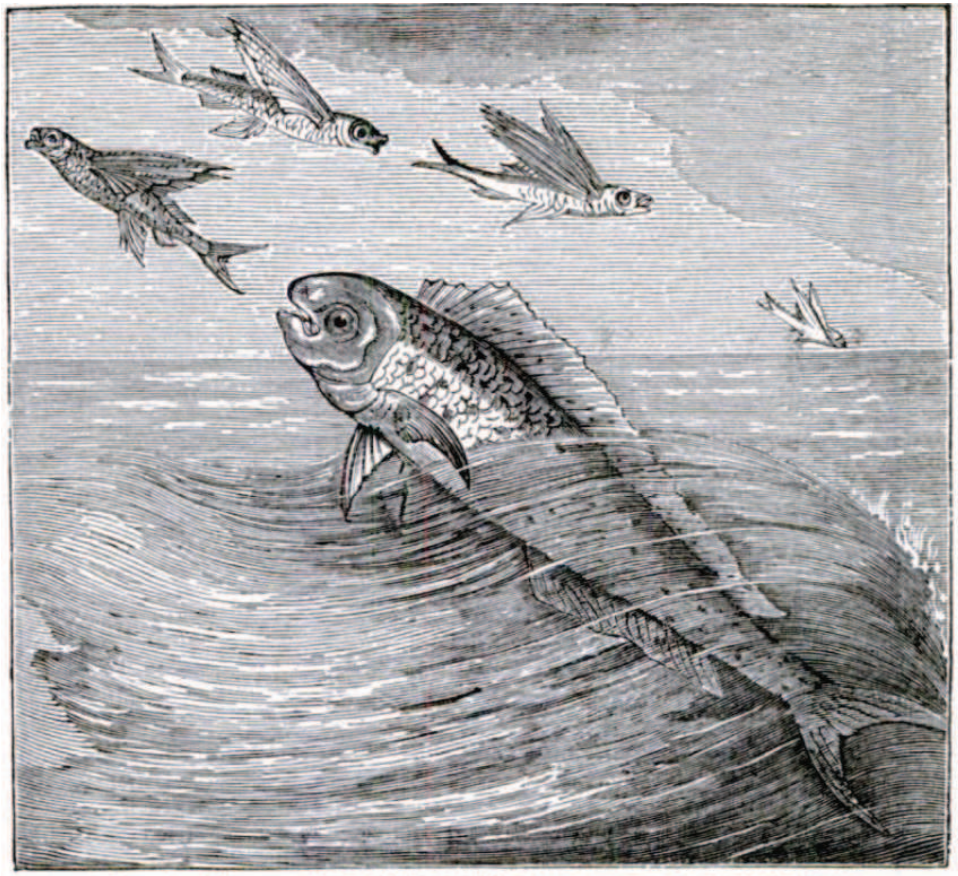

FIG. 4.—FLYING-FISH (*Exocœtus*) PURSUED BY THE DOLPHIN.

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"Animal life in the Gulf Stream"
The Popular Science Monthly

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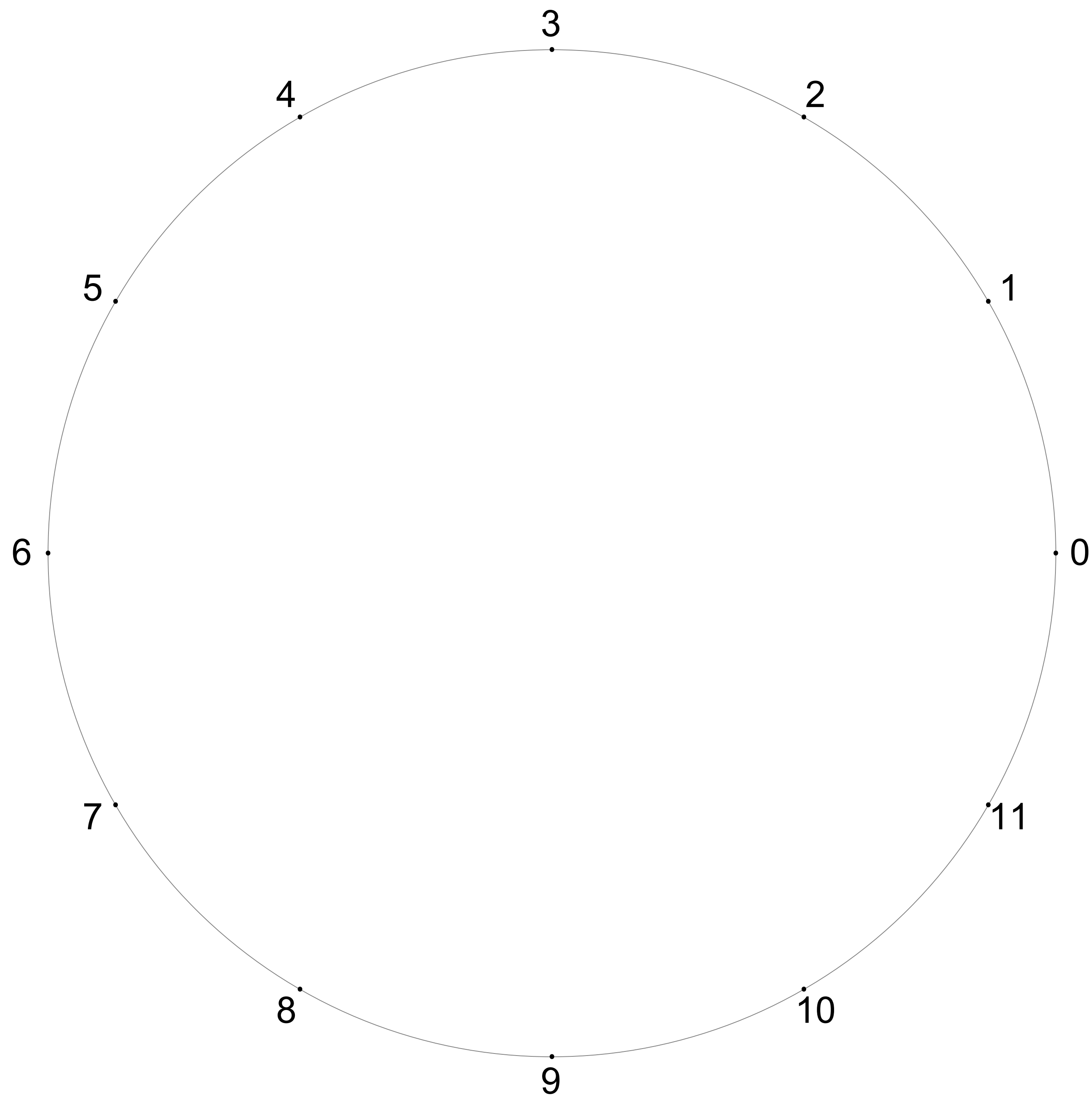
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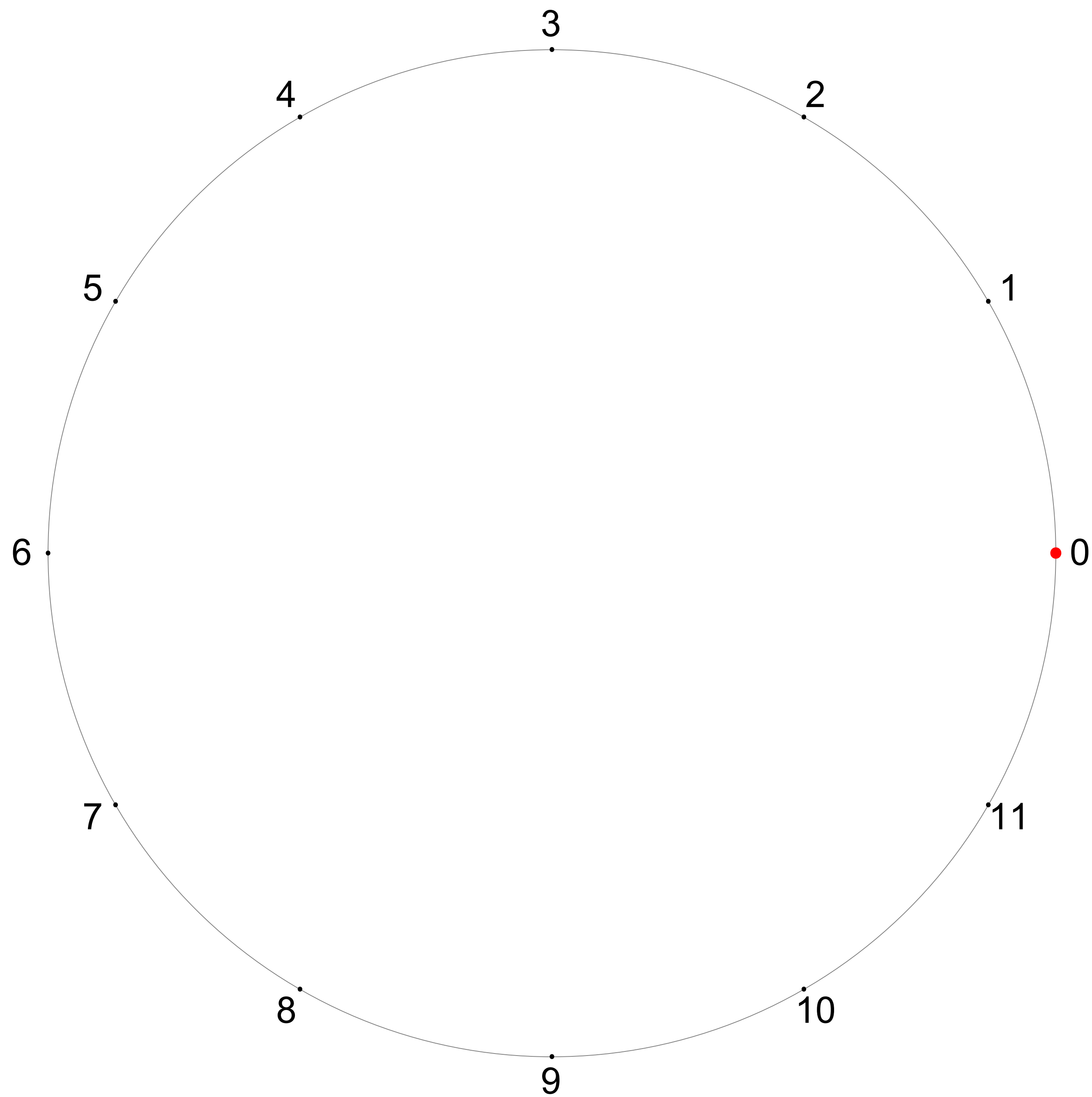
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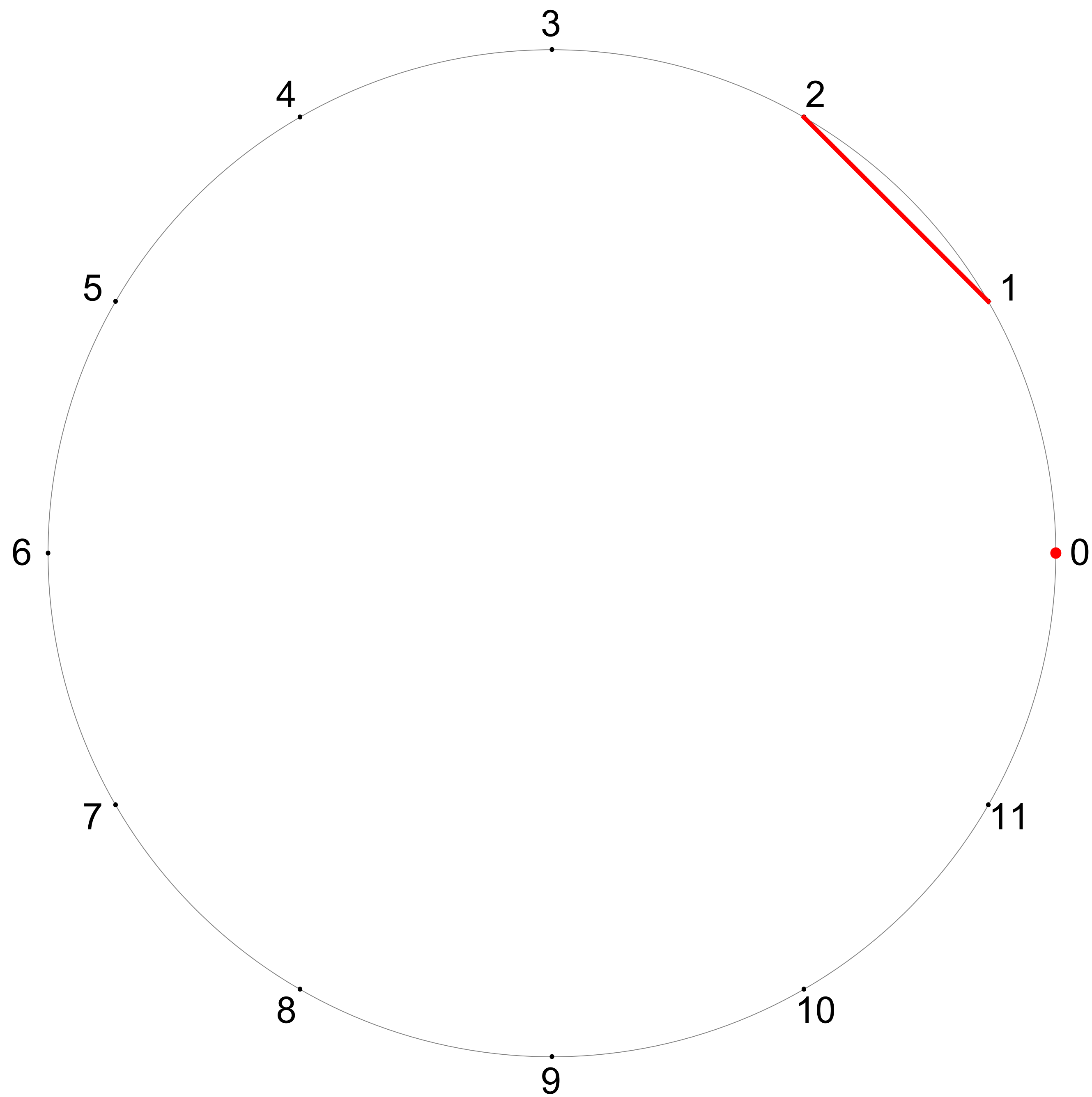

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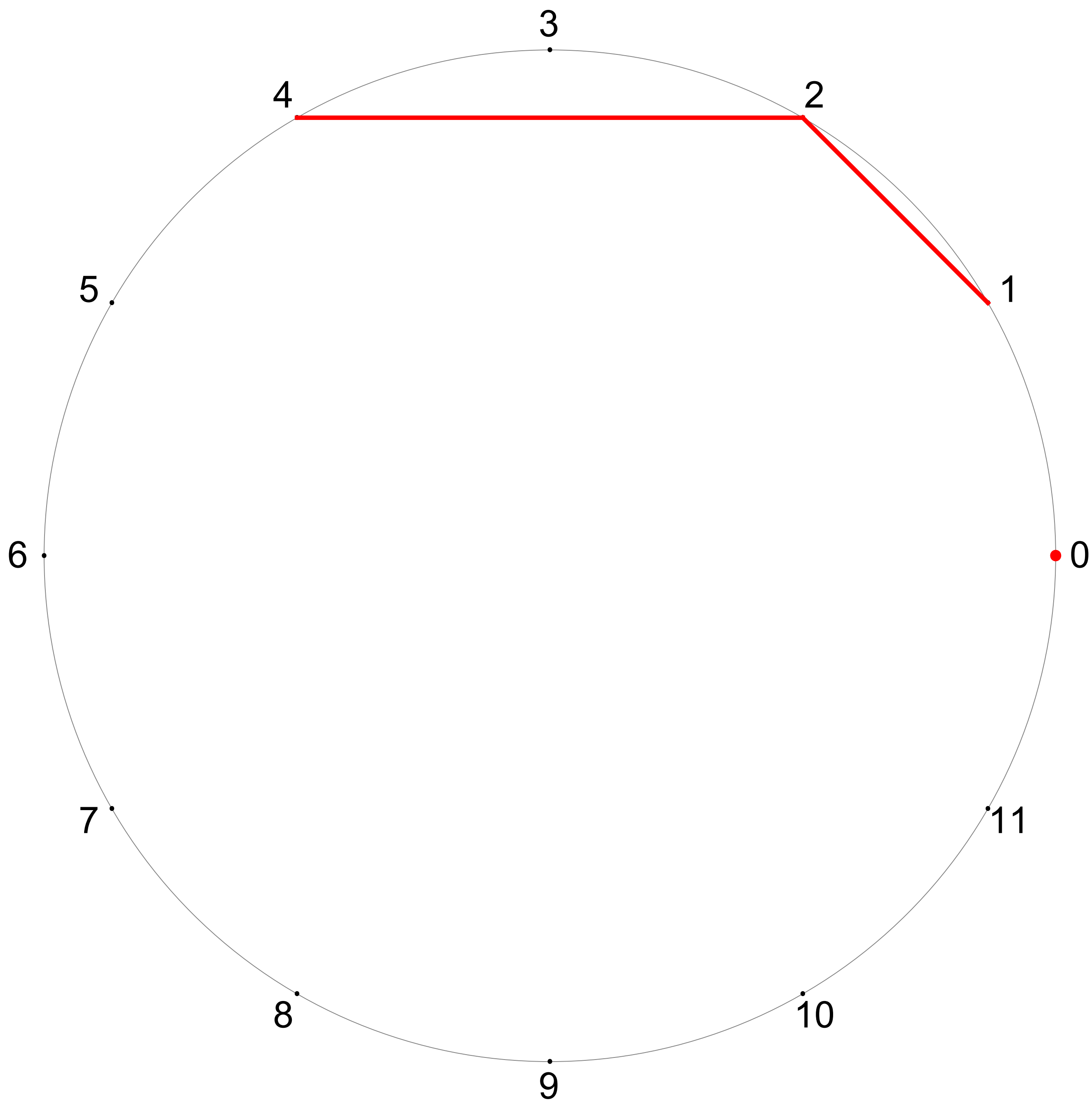
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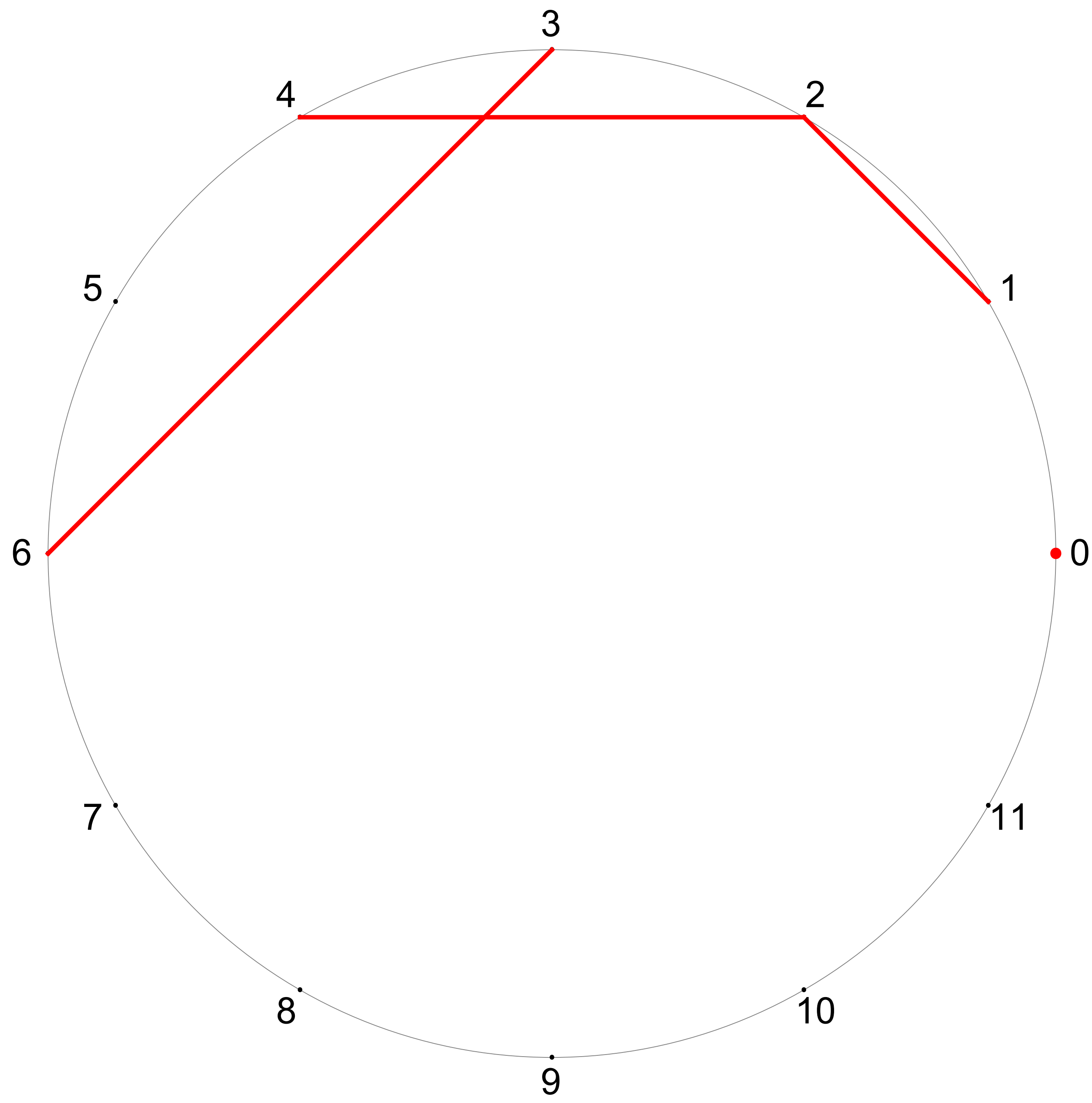

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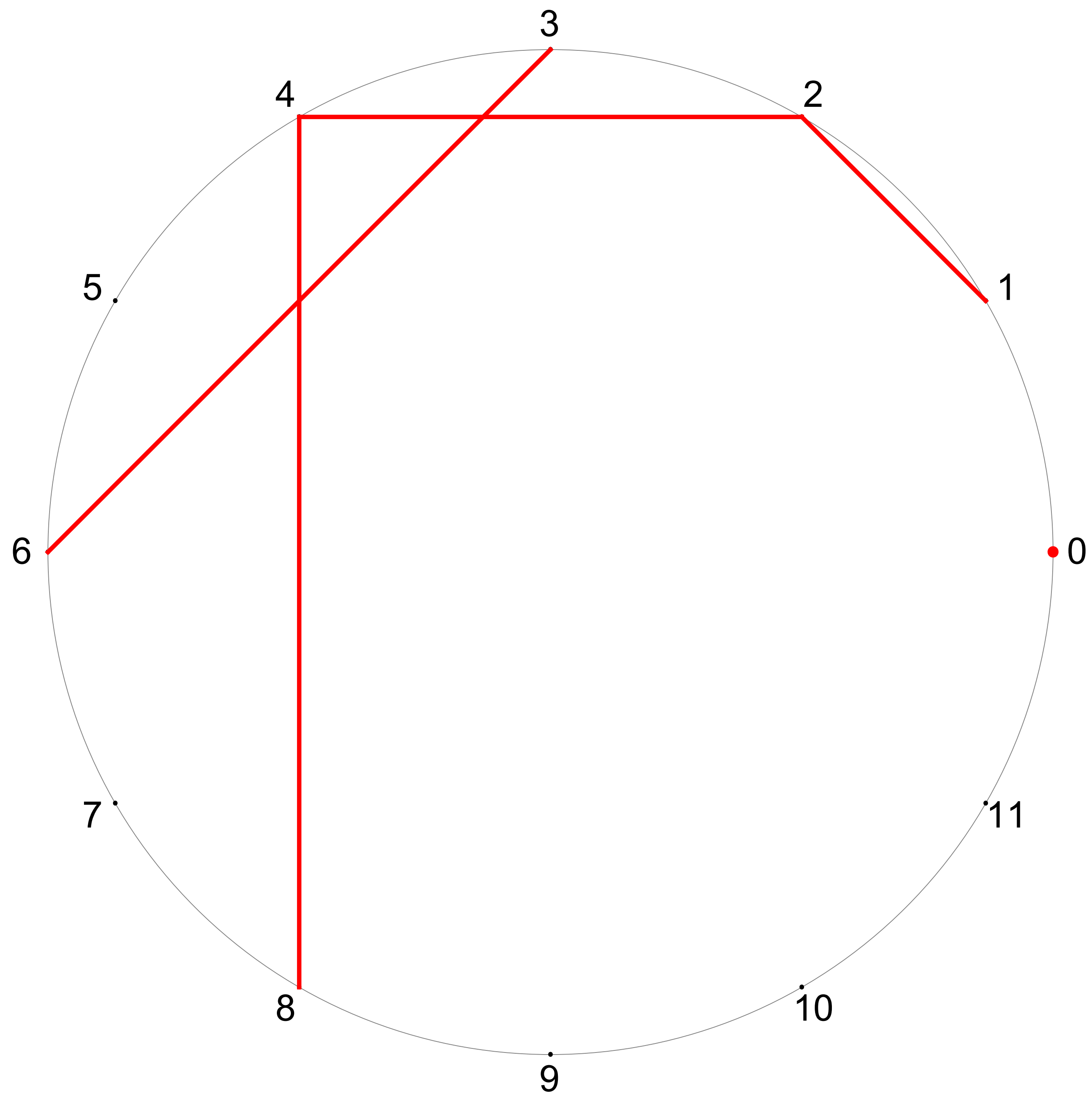
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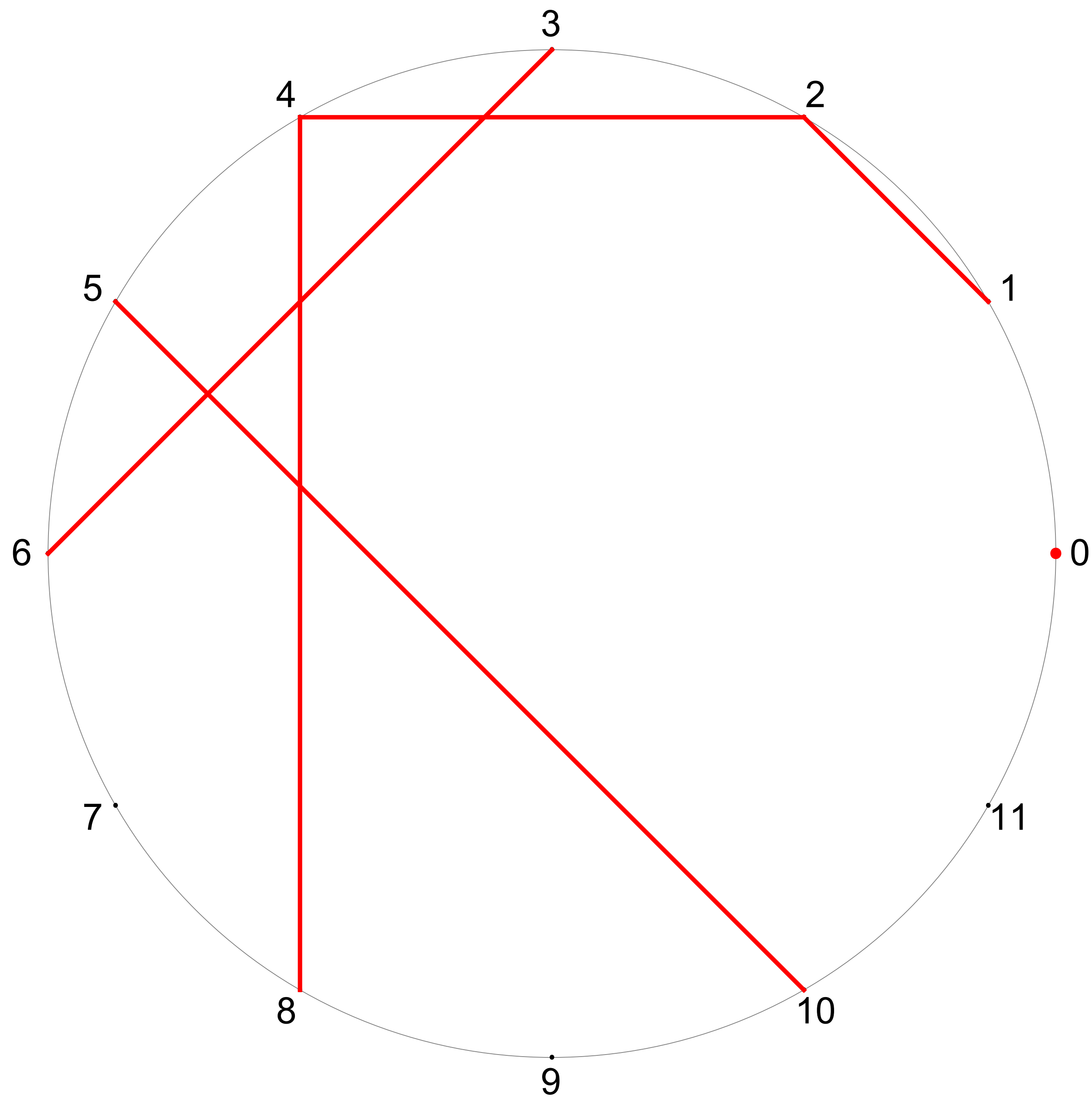
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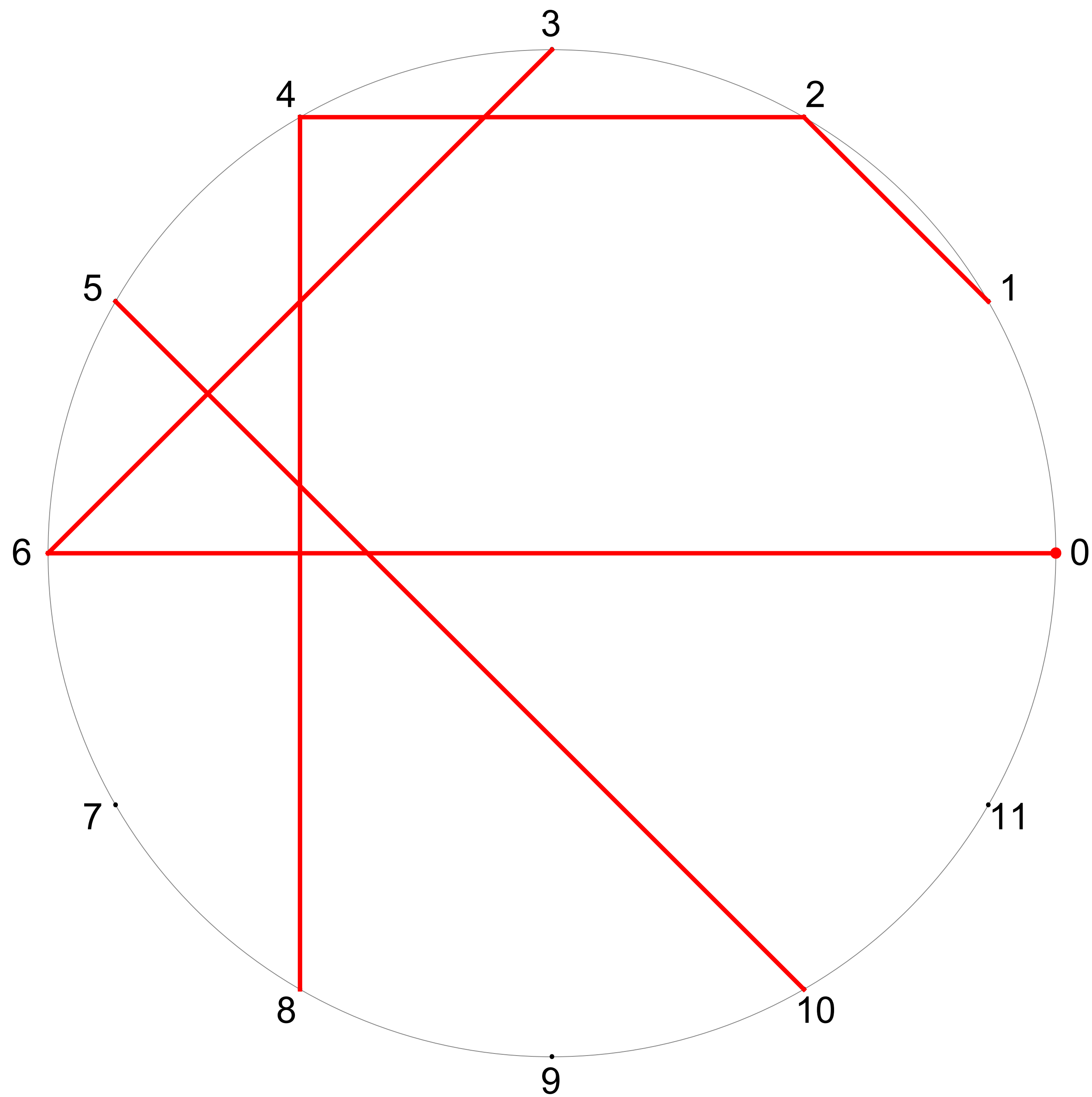
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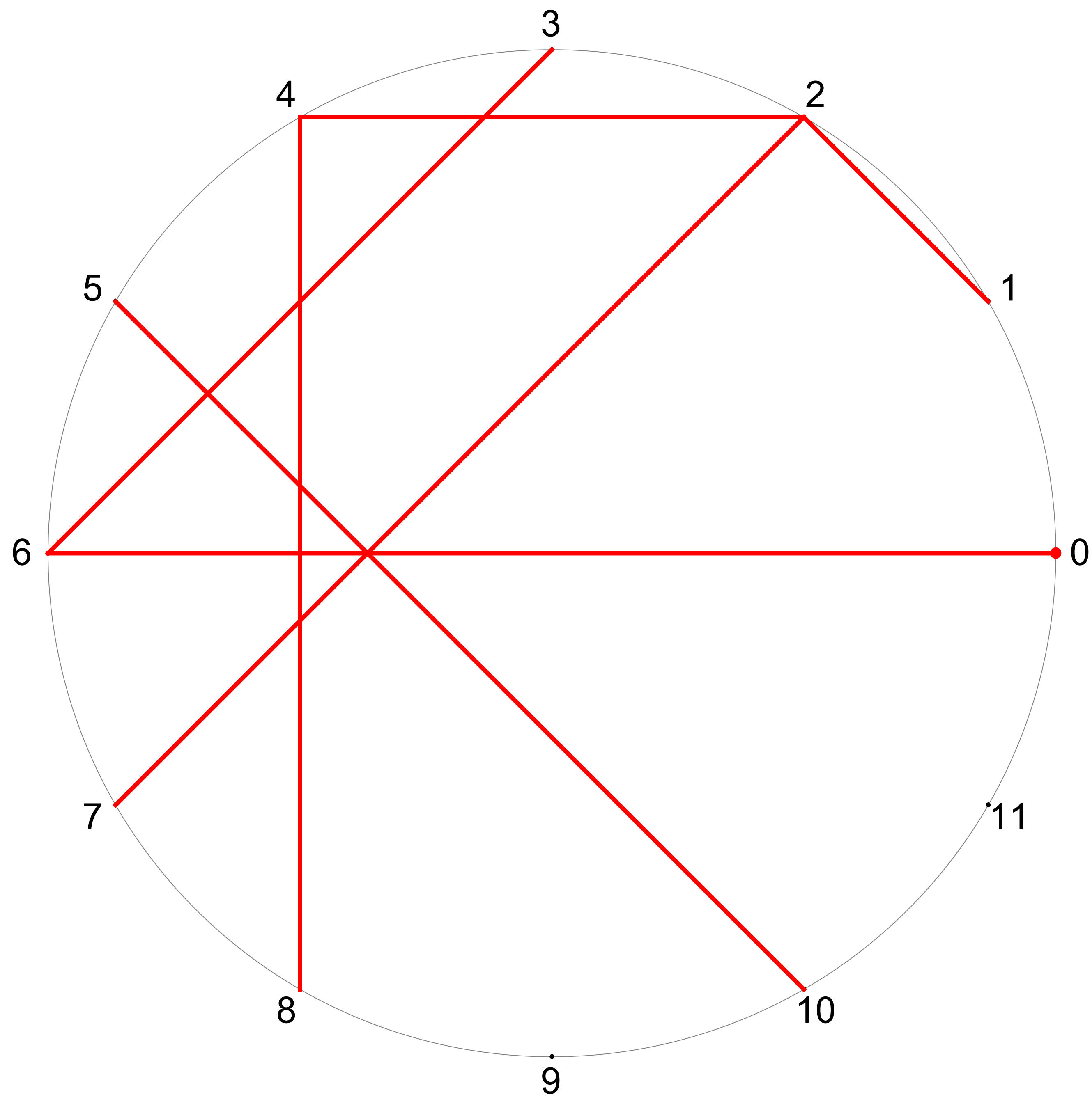

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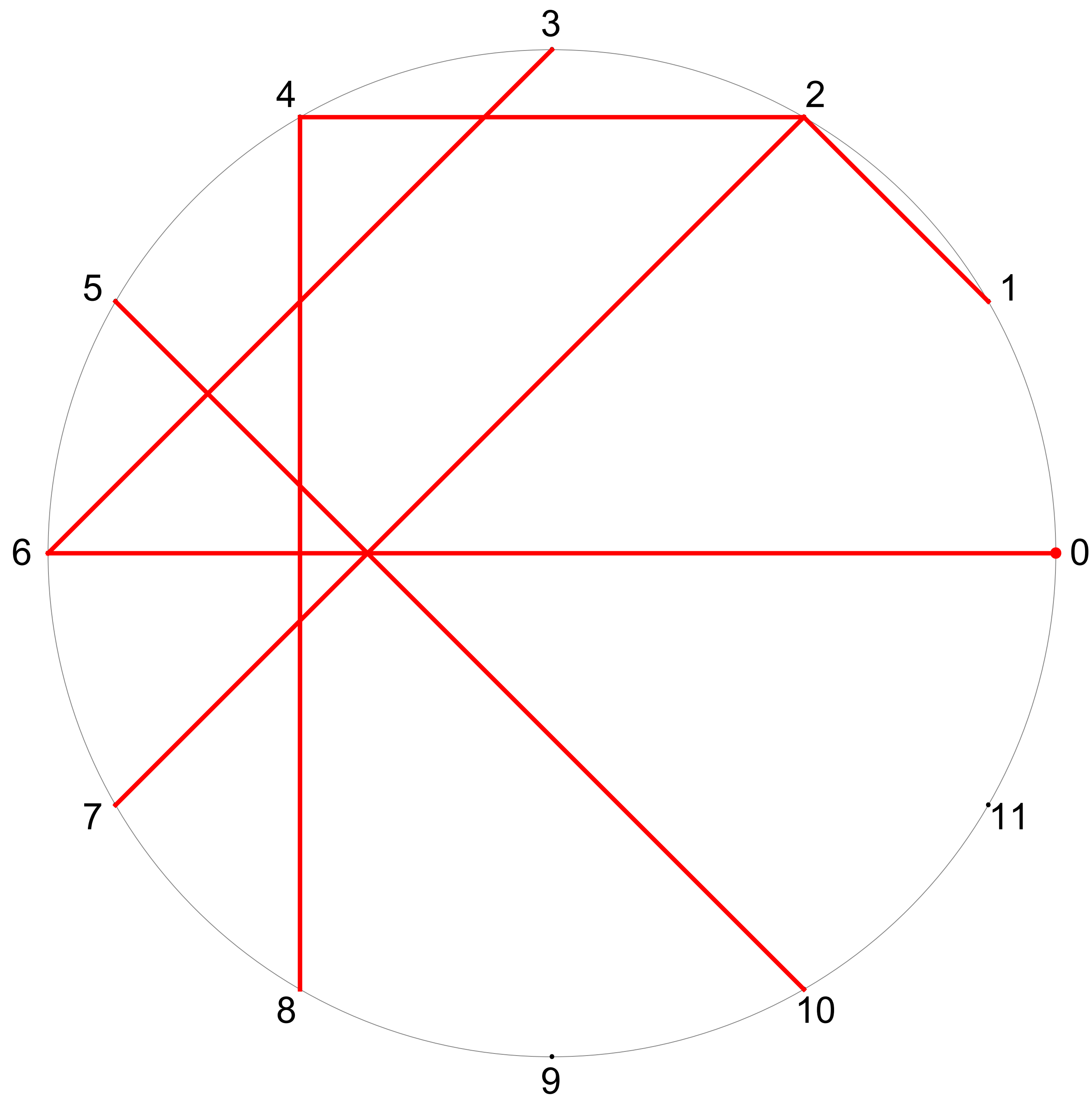
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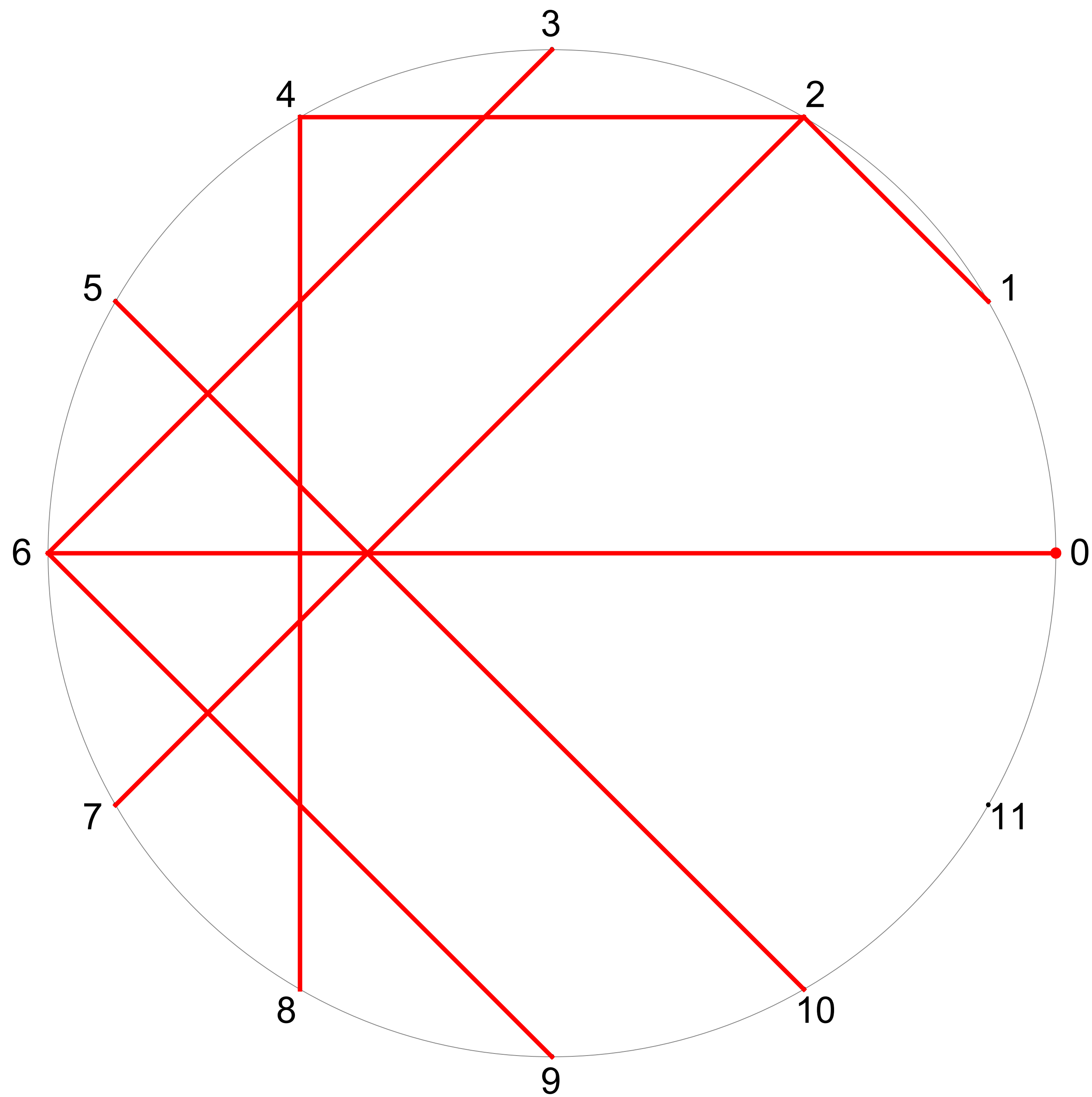
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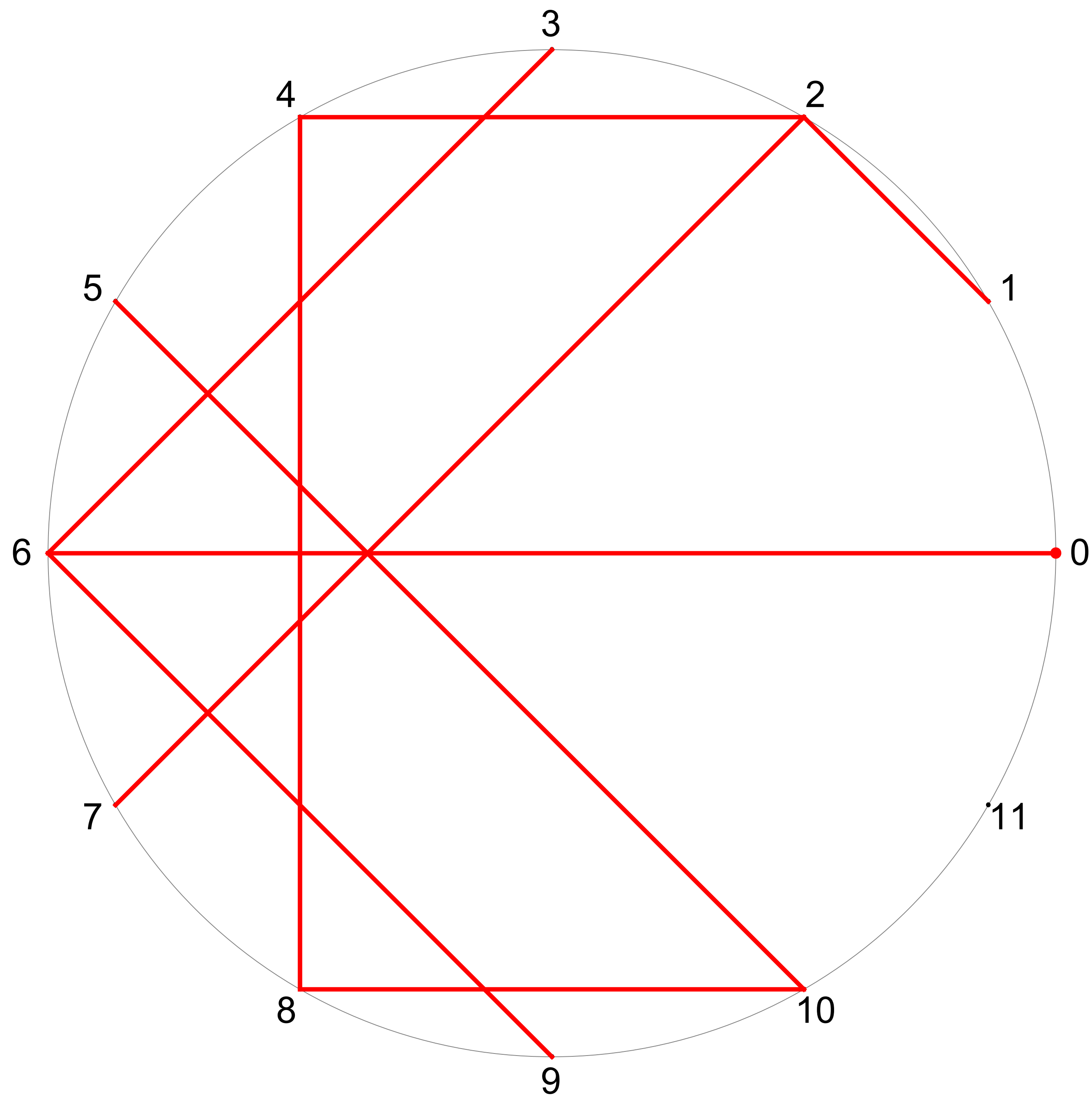

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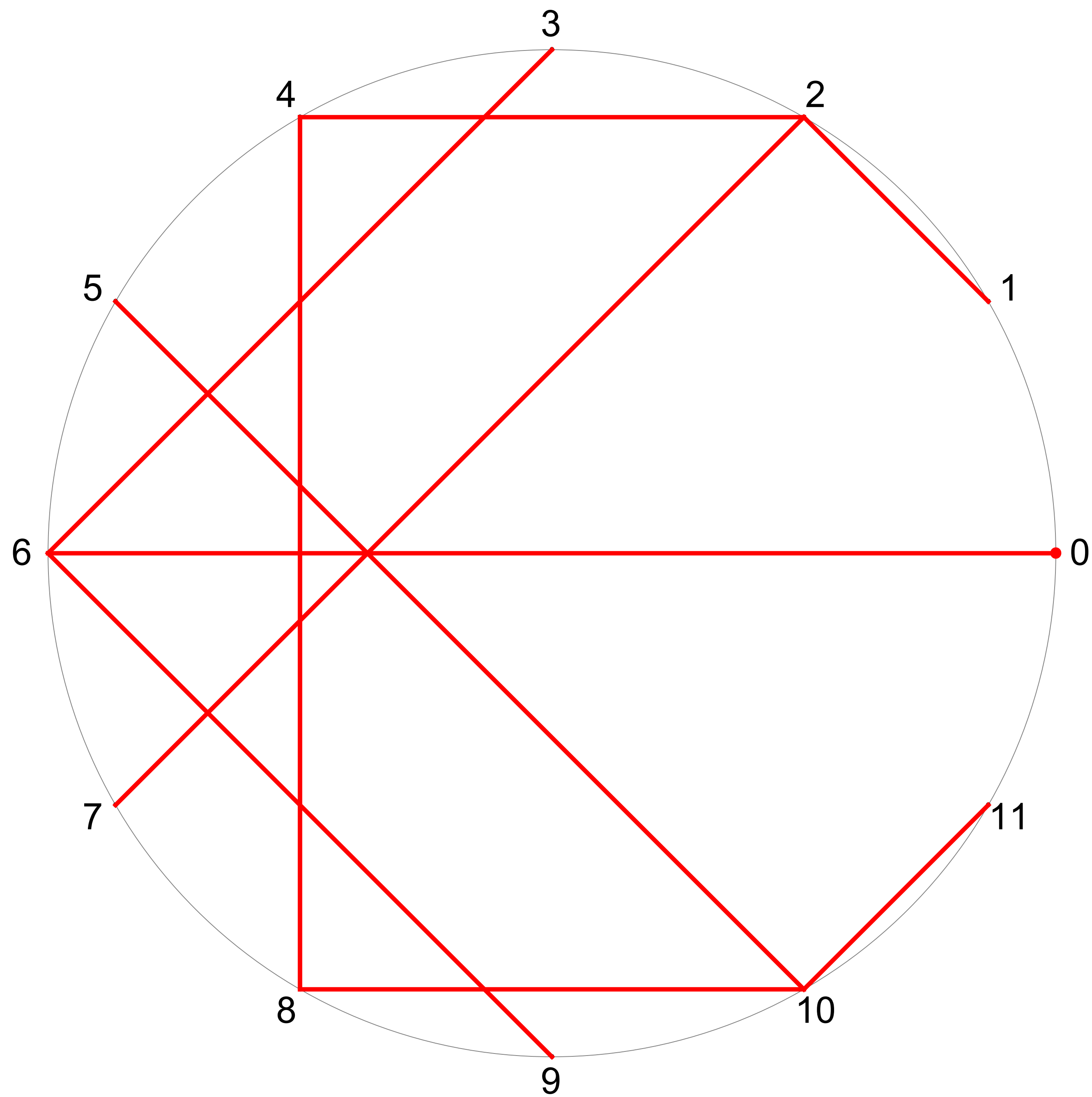

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n &\longmapsto 2n \\
12 - n &\longmapsto 2(12 - n) \equiv 12 - 2n
\end{aligned}
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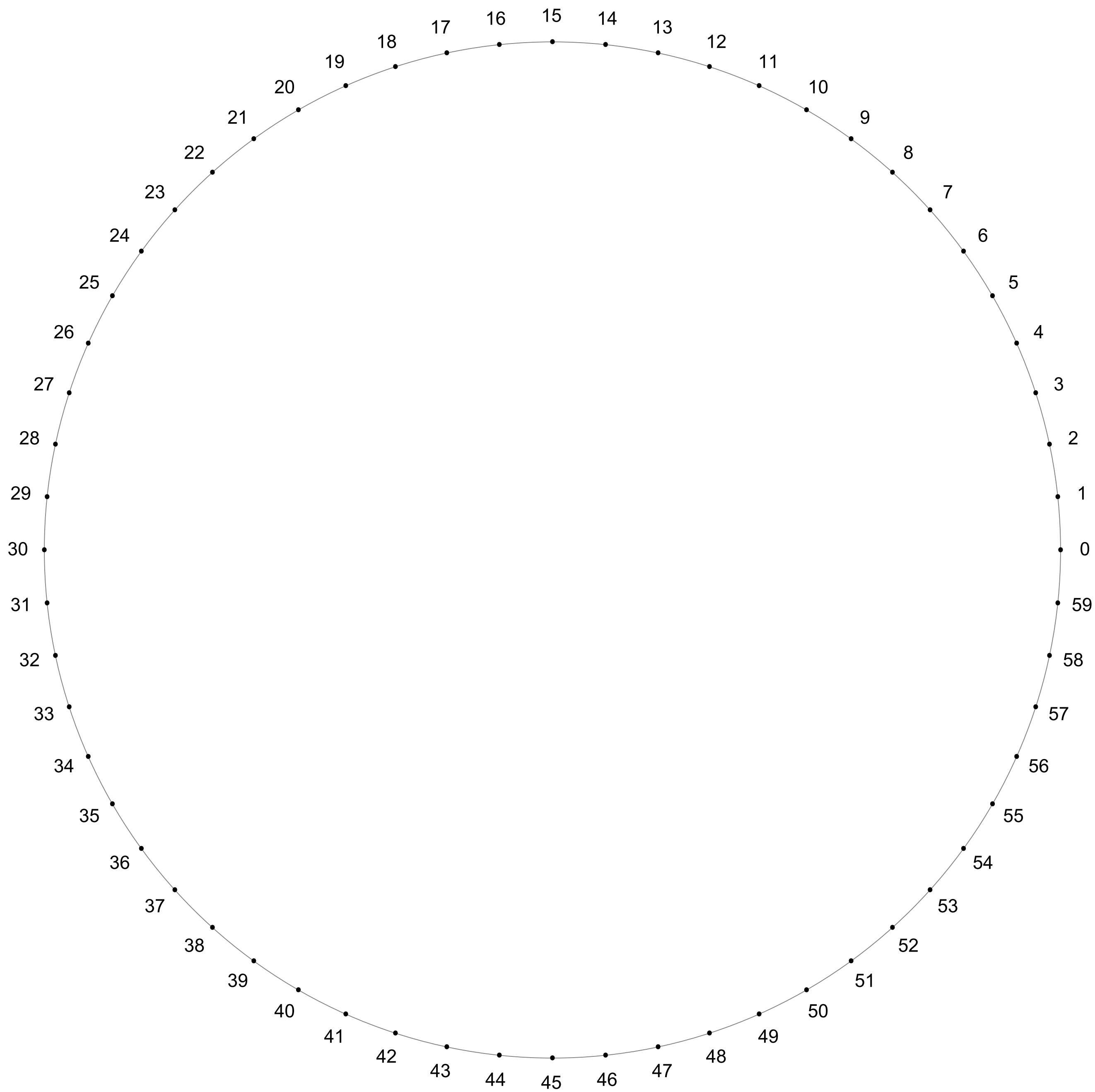
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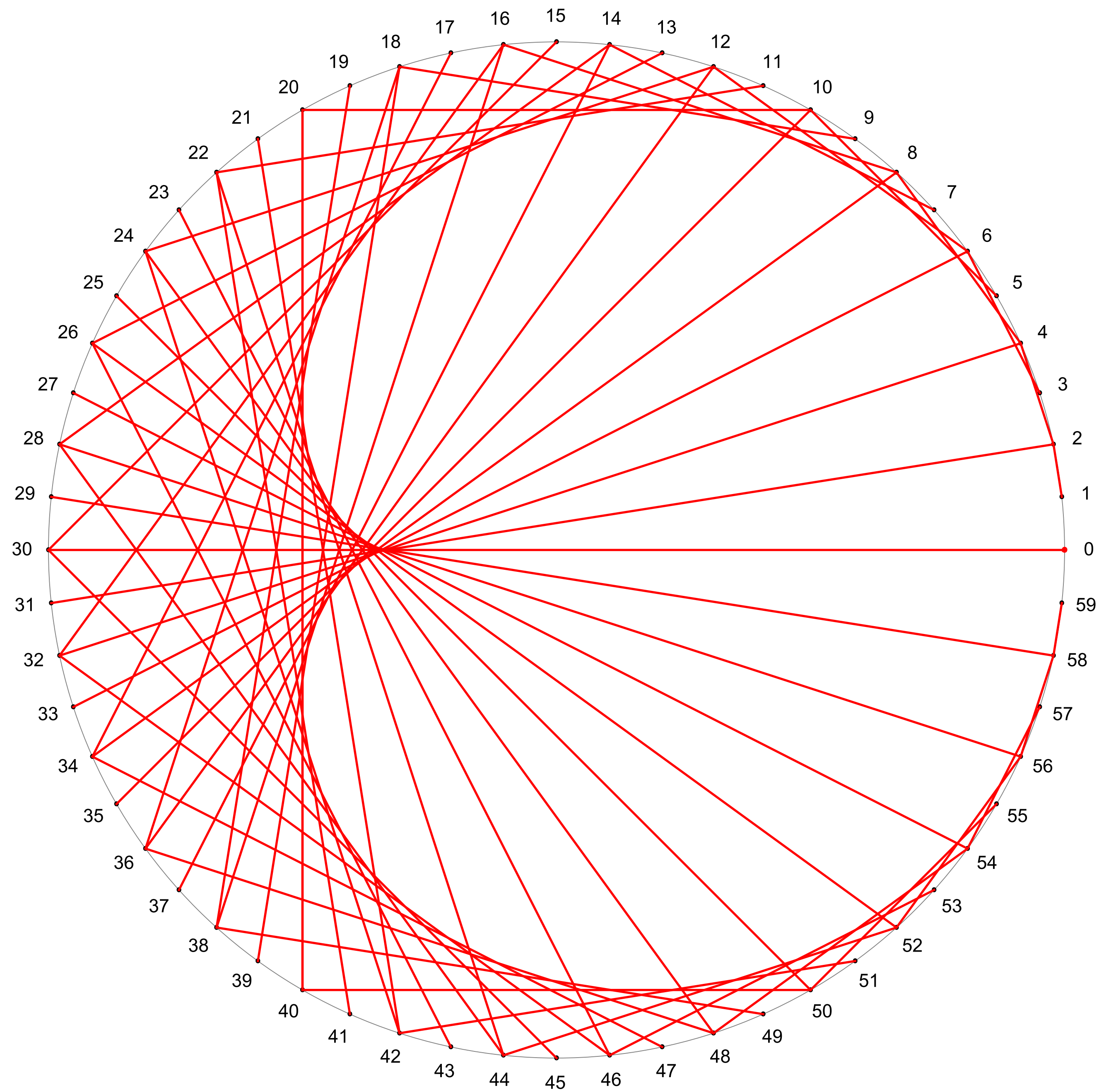
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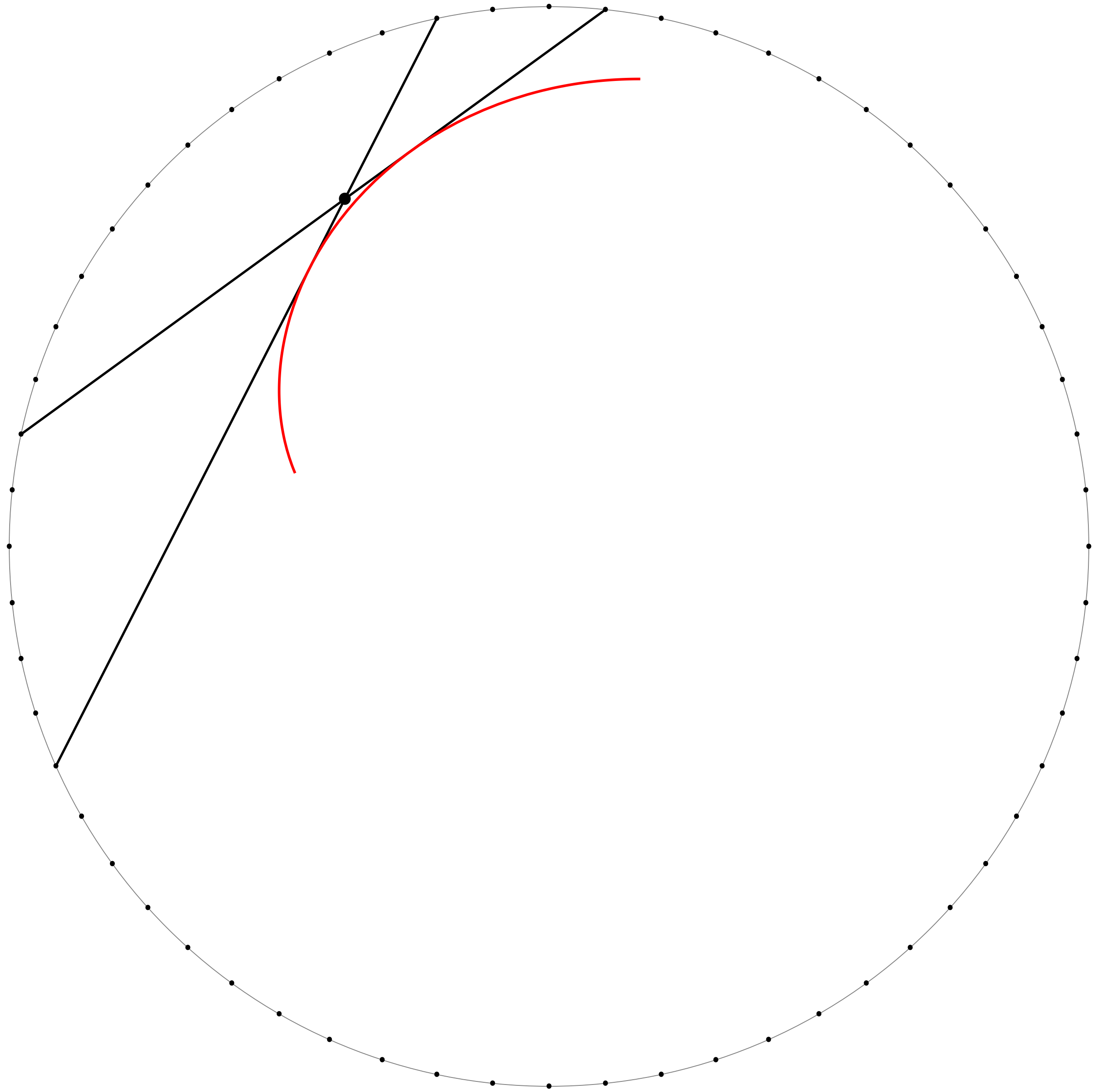
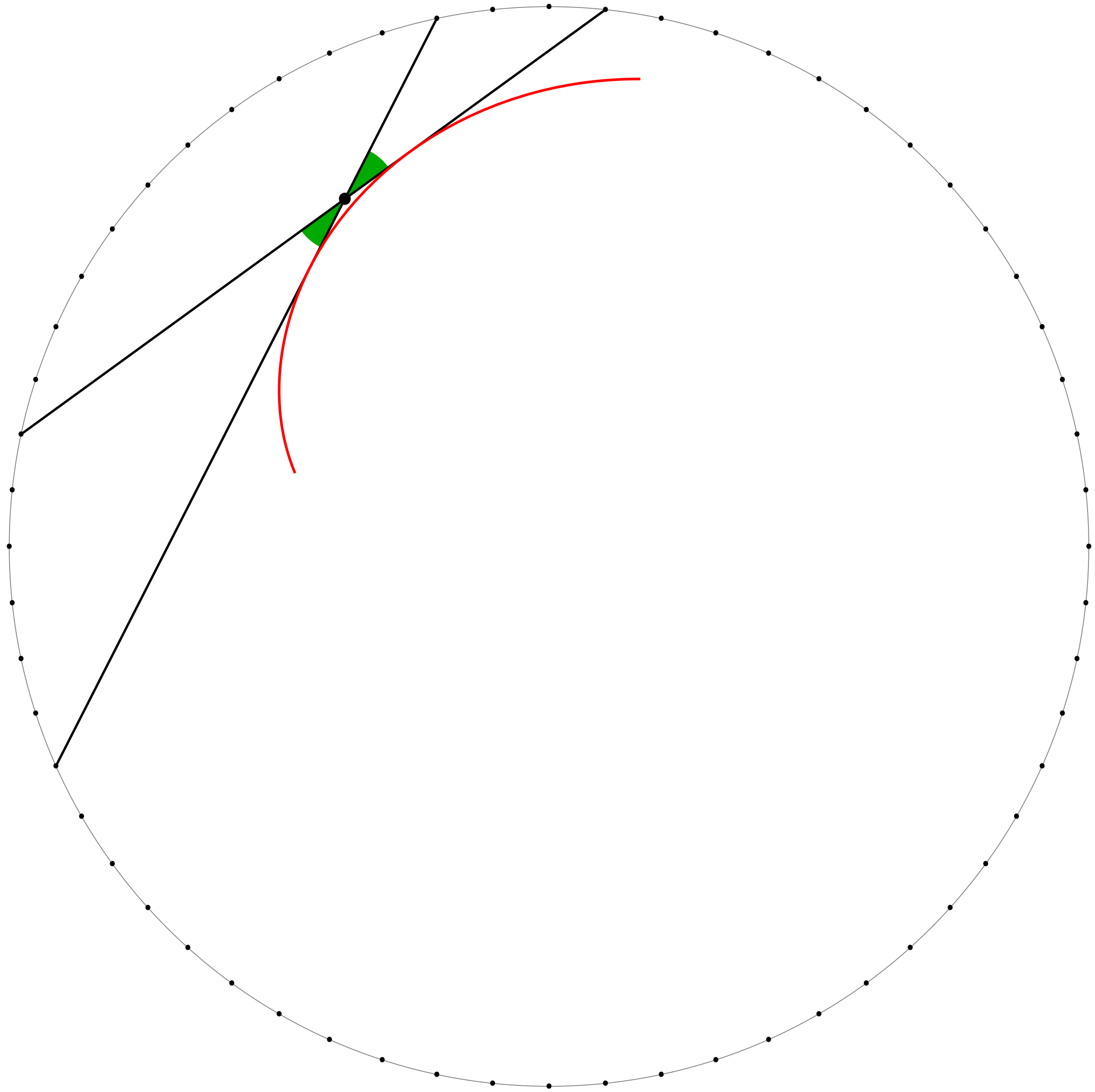
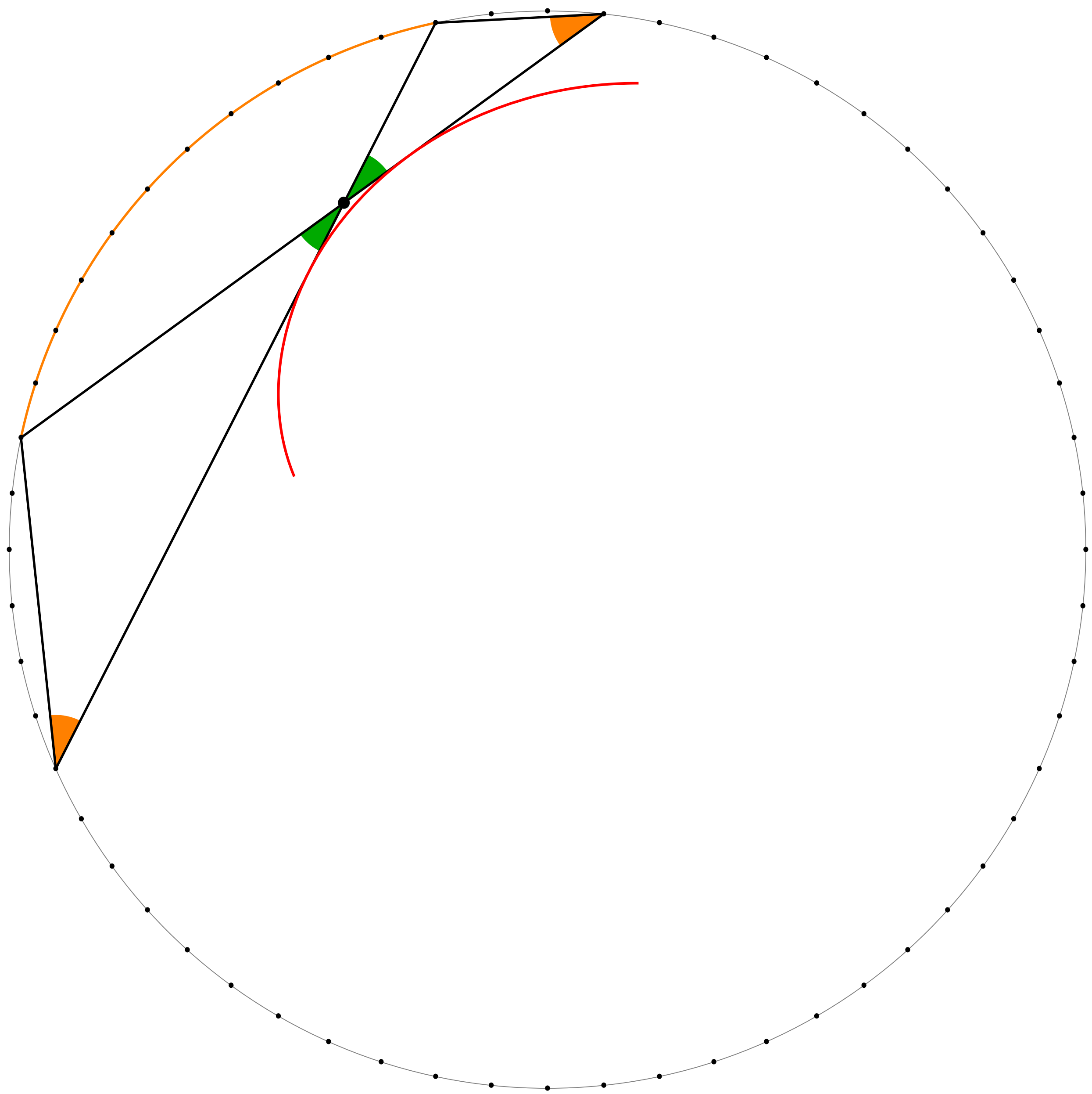
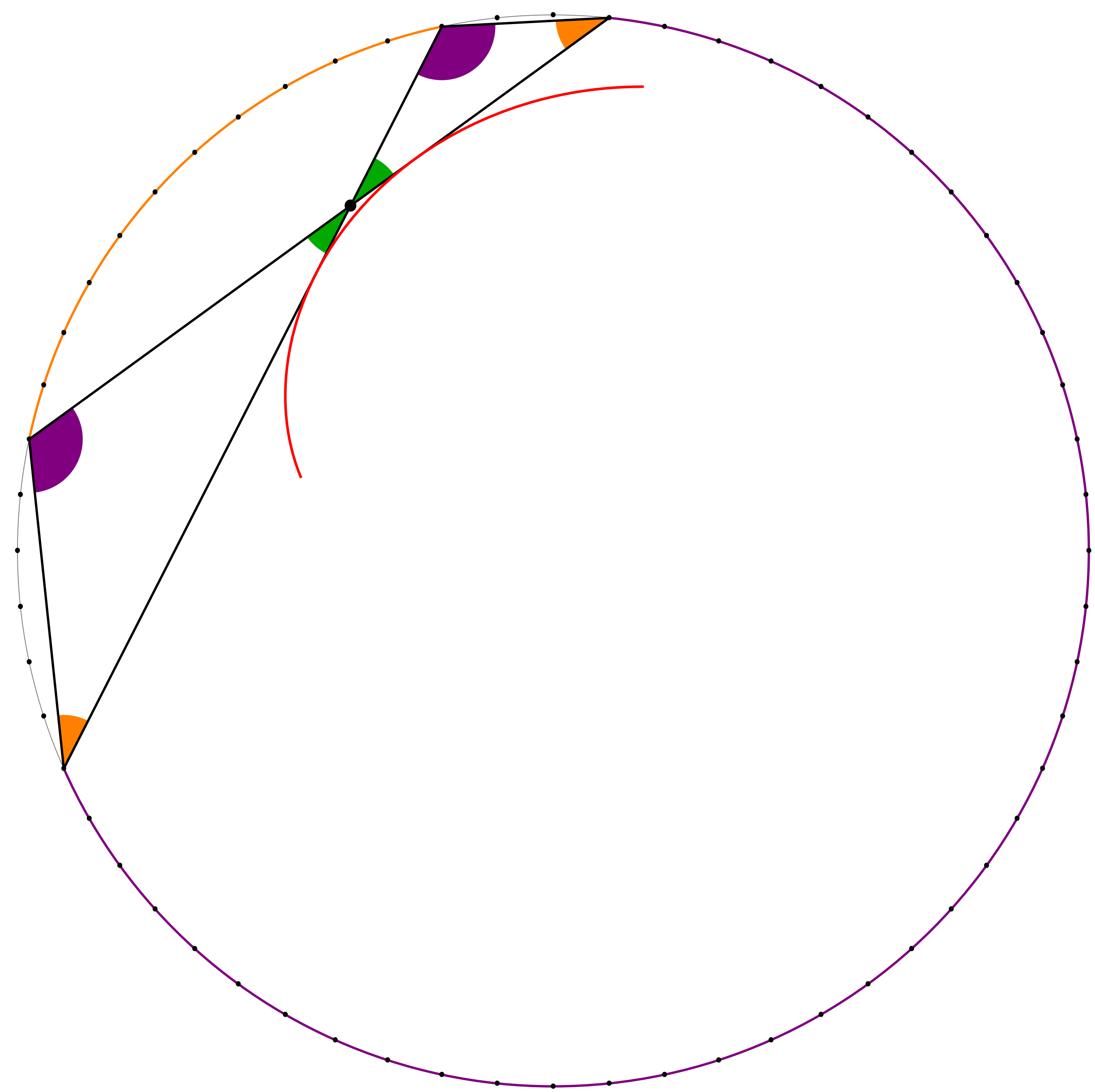
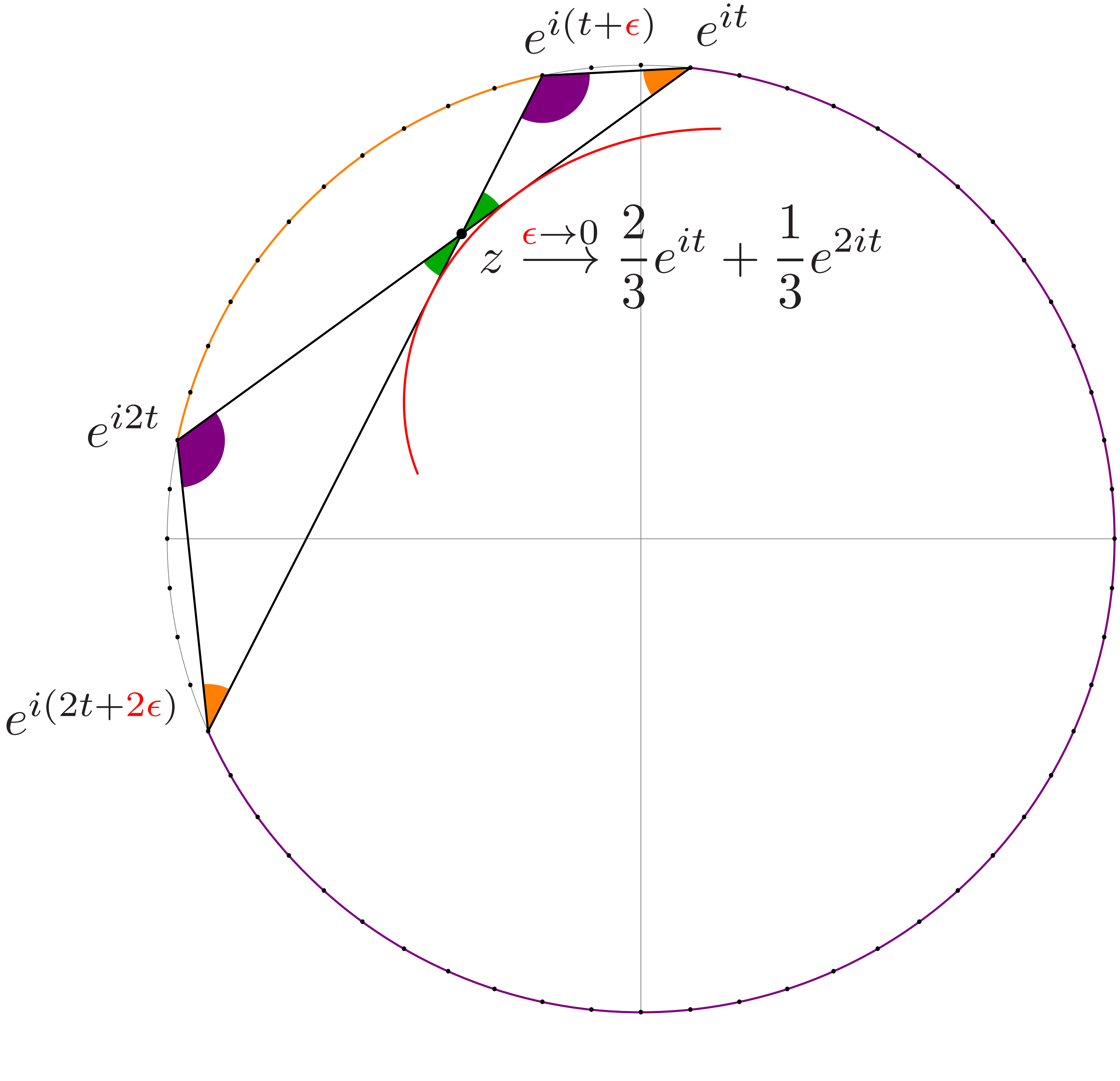


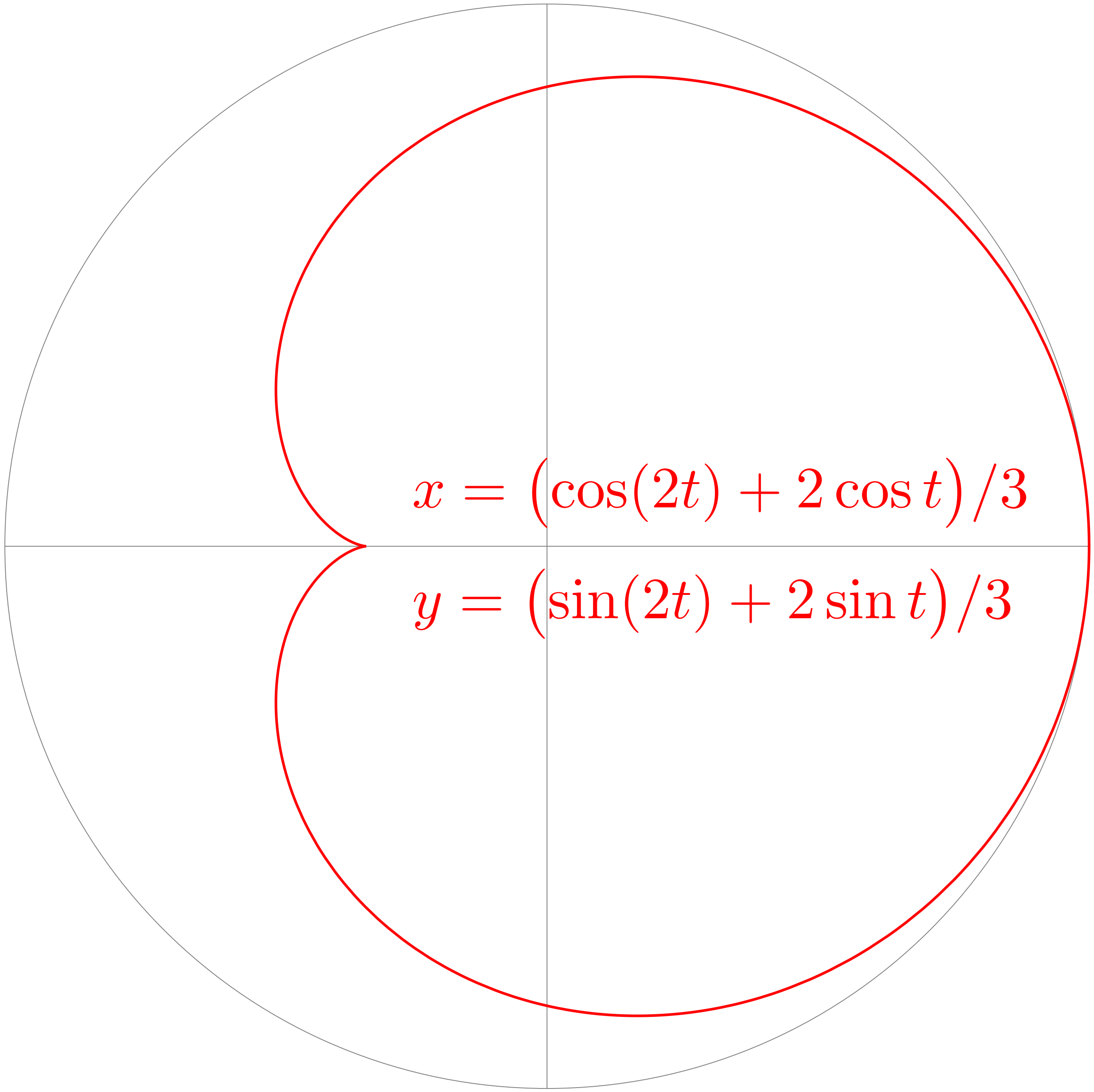




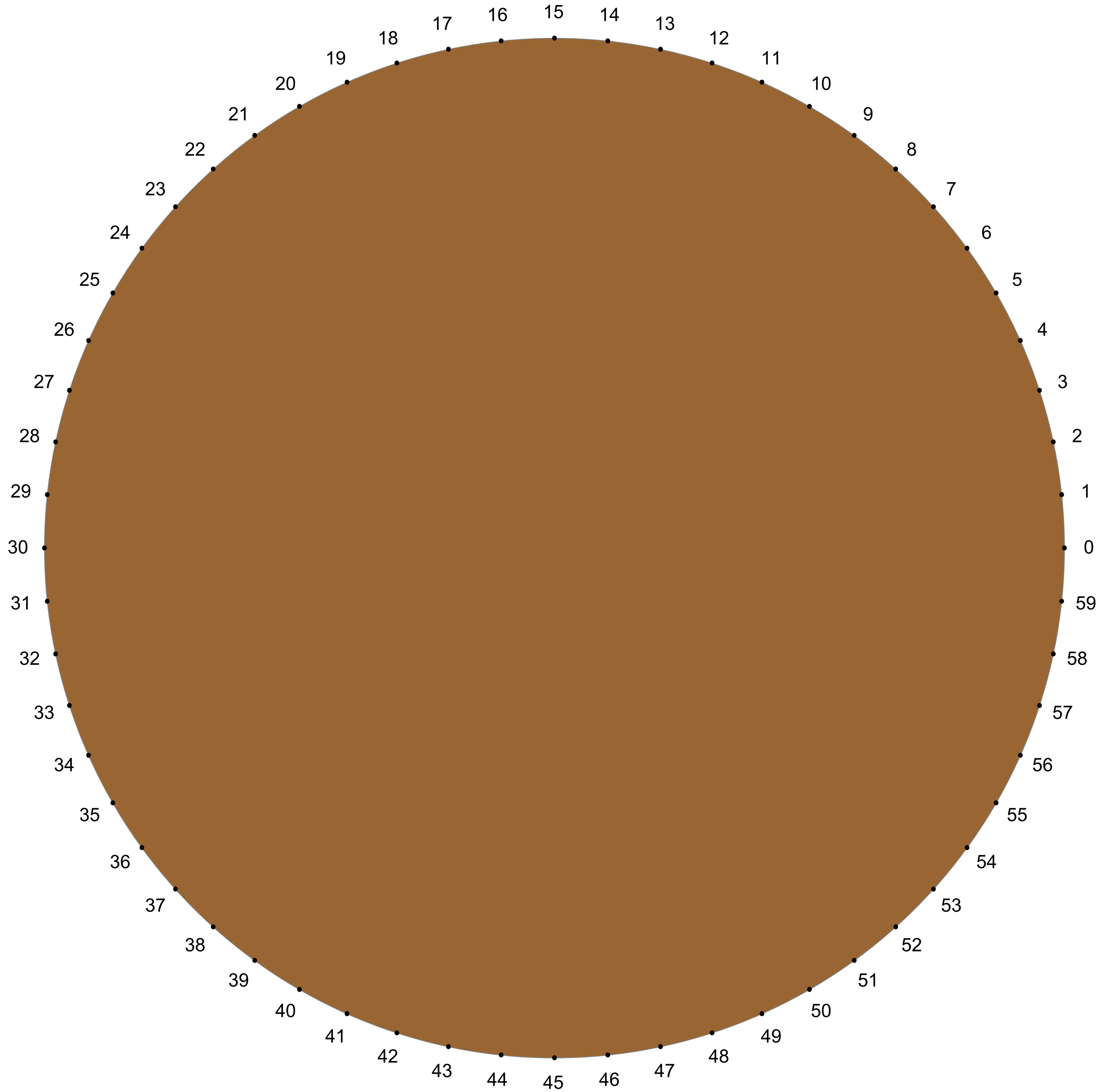




$e^{i(t+\epsilon)}$
e^{it}
$z \xrightarrow{\epsilon \to 0} \frac{2}{3}e^{it} + \frac{1}{3}e^{2it}$
e^{i2t}
$e^{i(2t+2\epsilon)}$



$x = \big(\cos(2t) + 2\cos t\big)/3$
$y = \big(\sin(2t) + 2\sin t\big)/3$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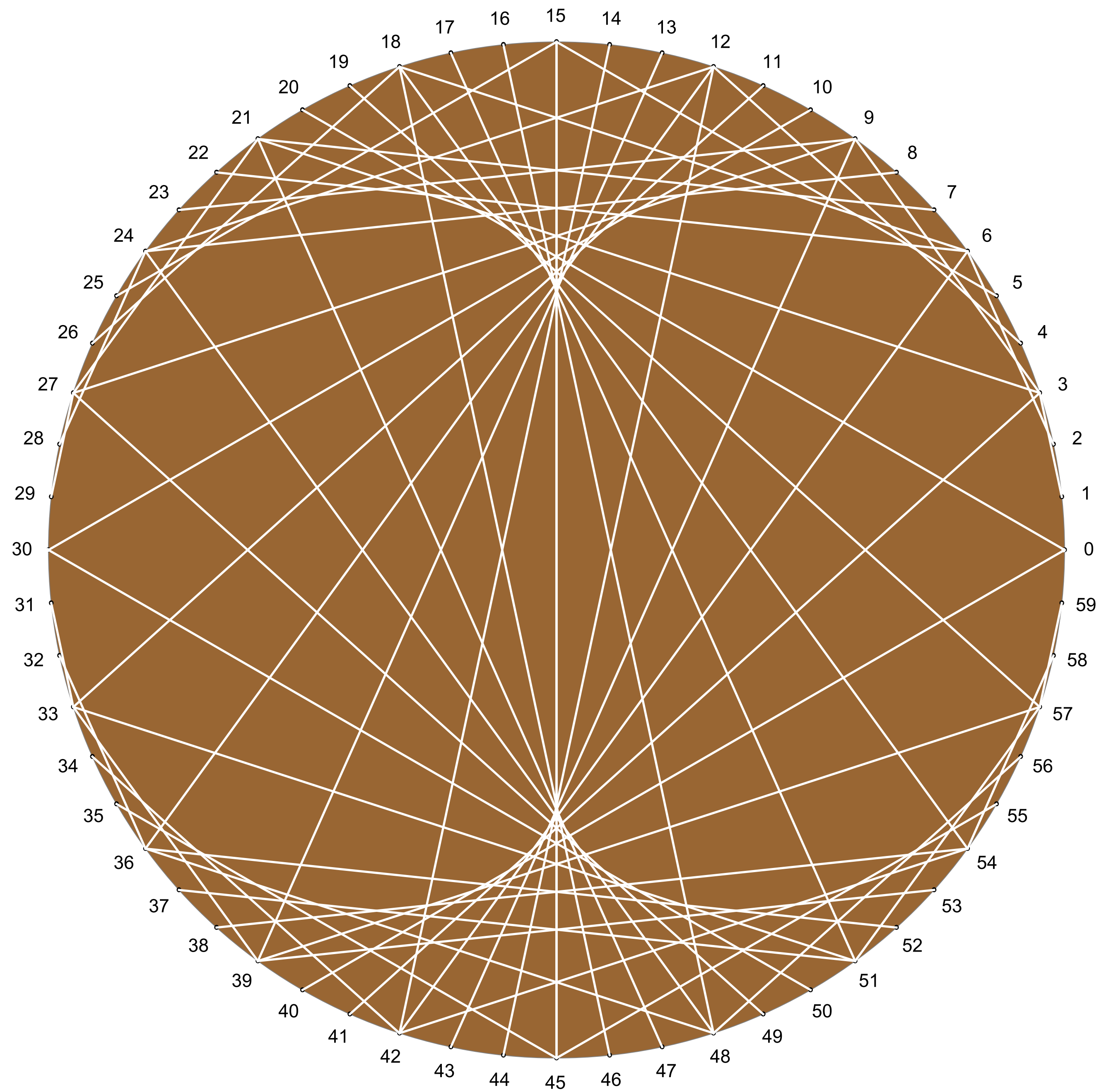
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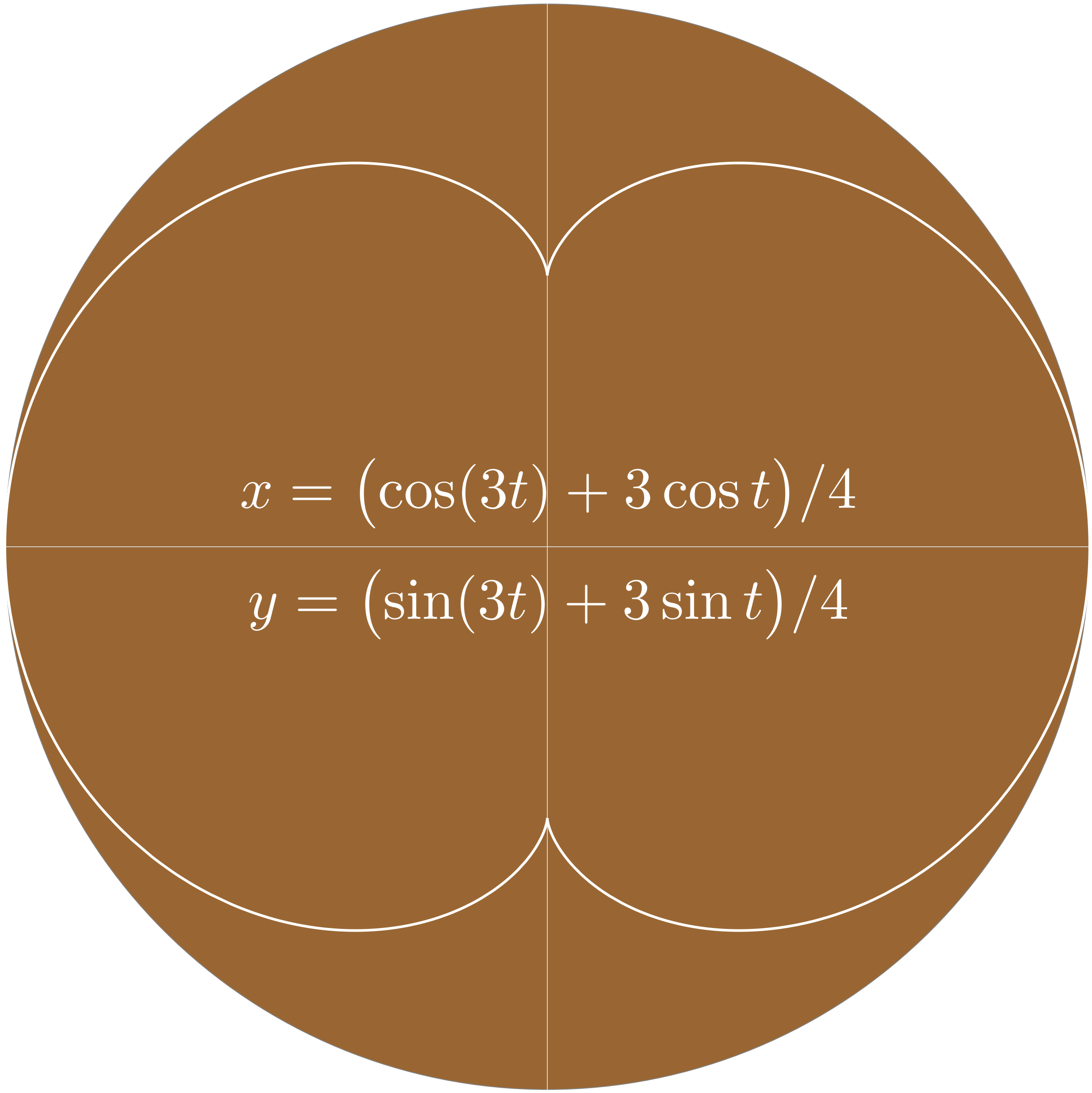
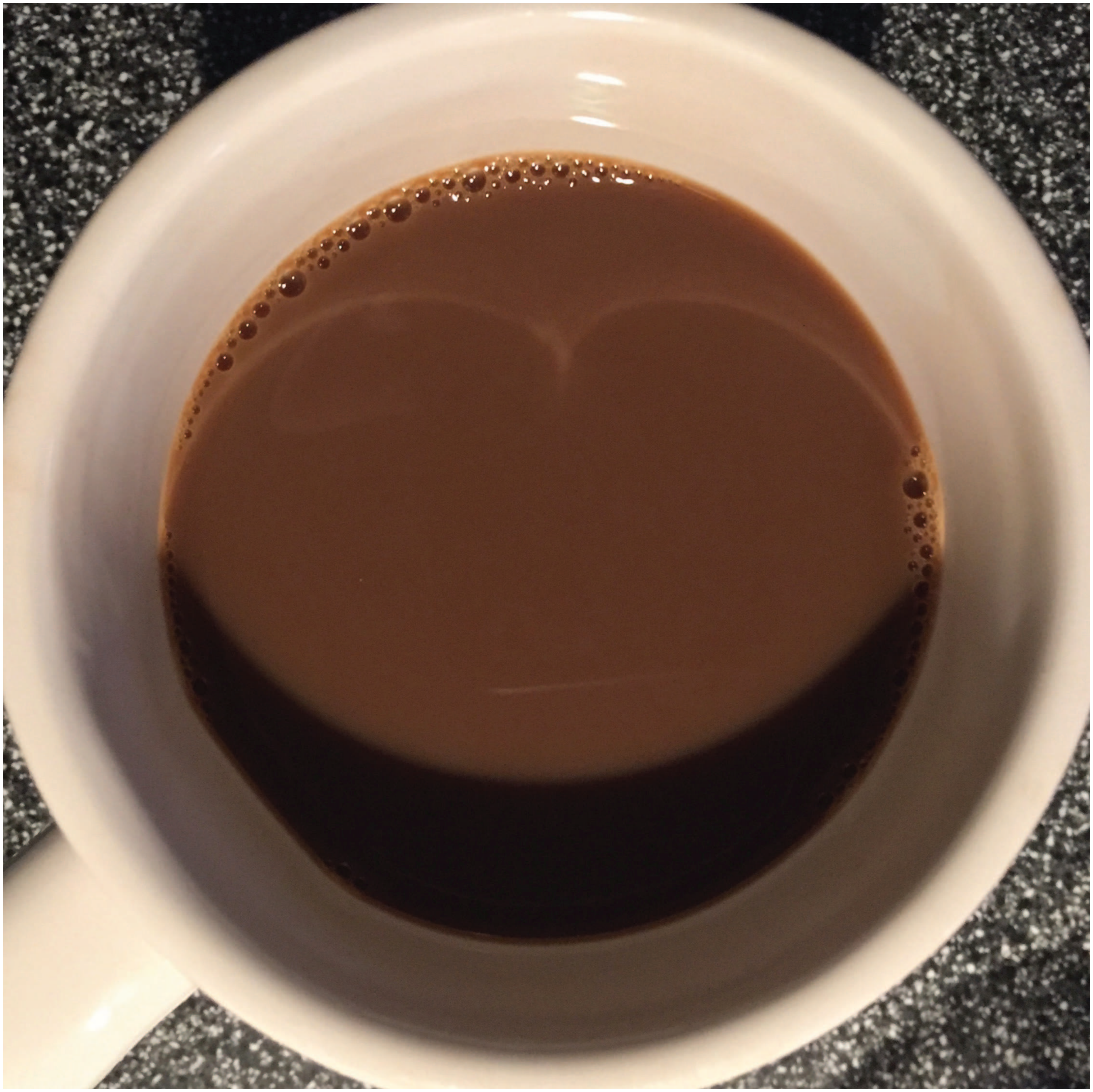
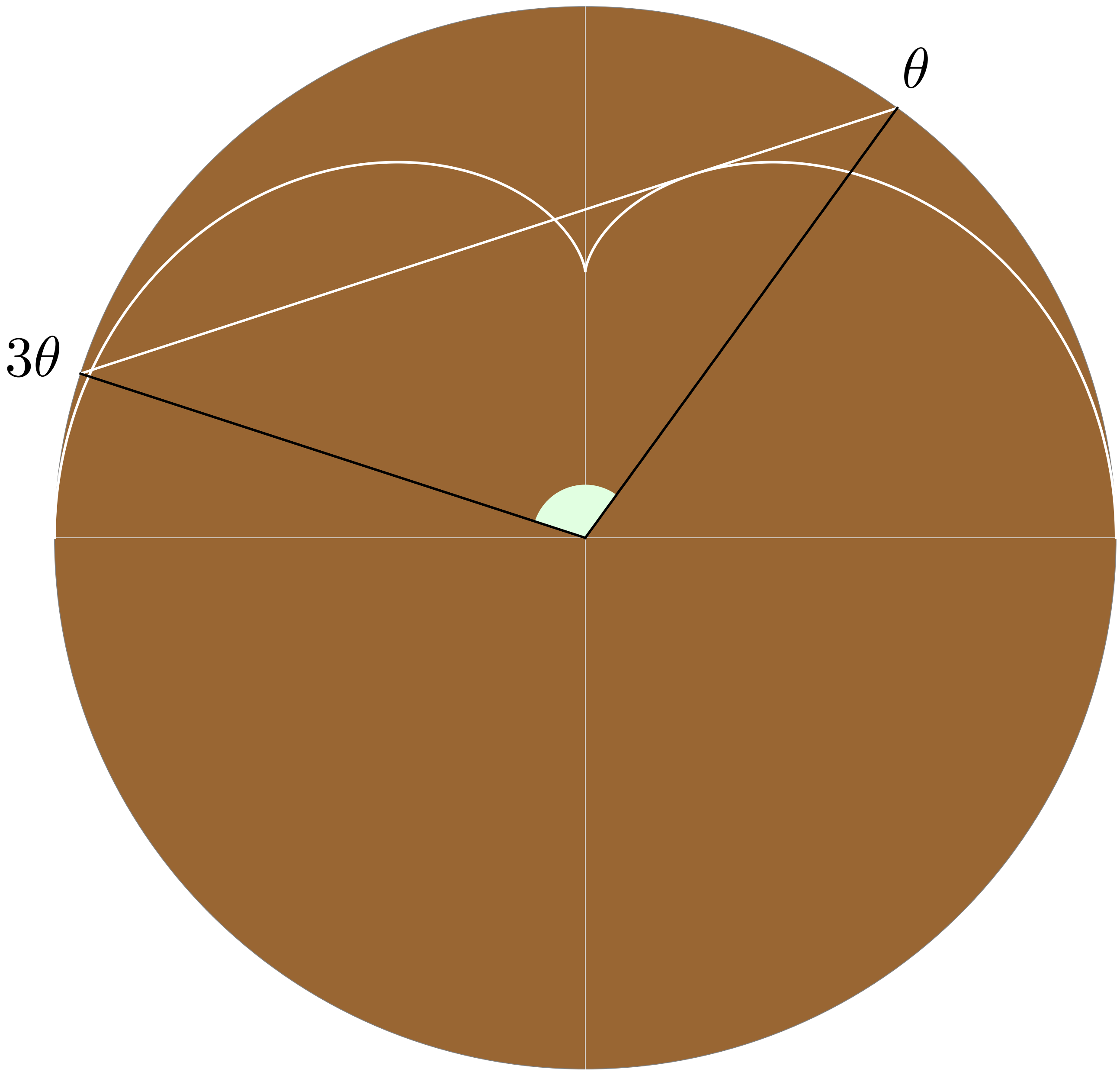


$x = \big(\cos(3t) + 3\cos t\big)/4$
$y = \big(\sin(3t) + 3\sin t\big)/4$





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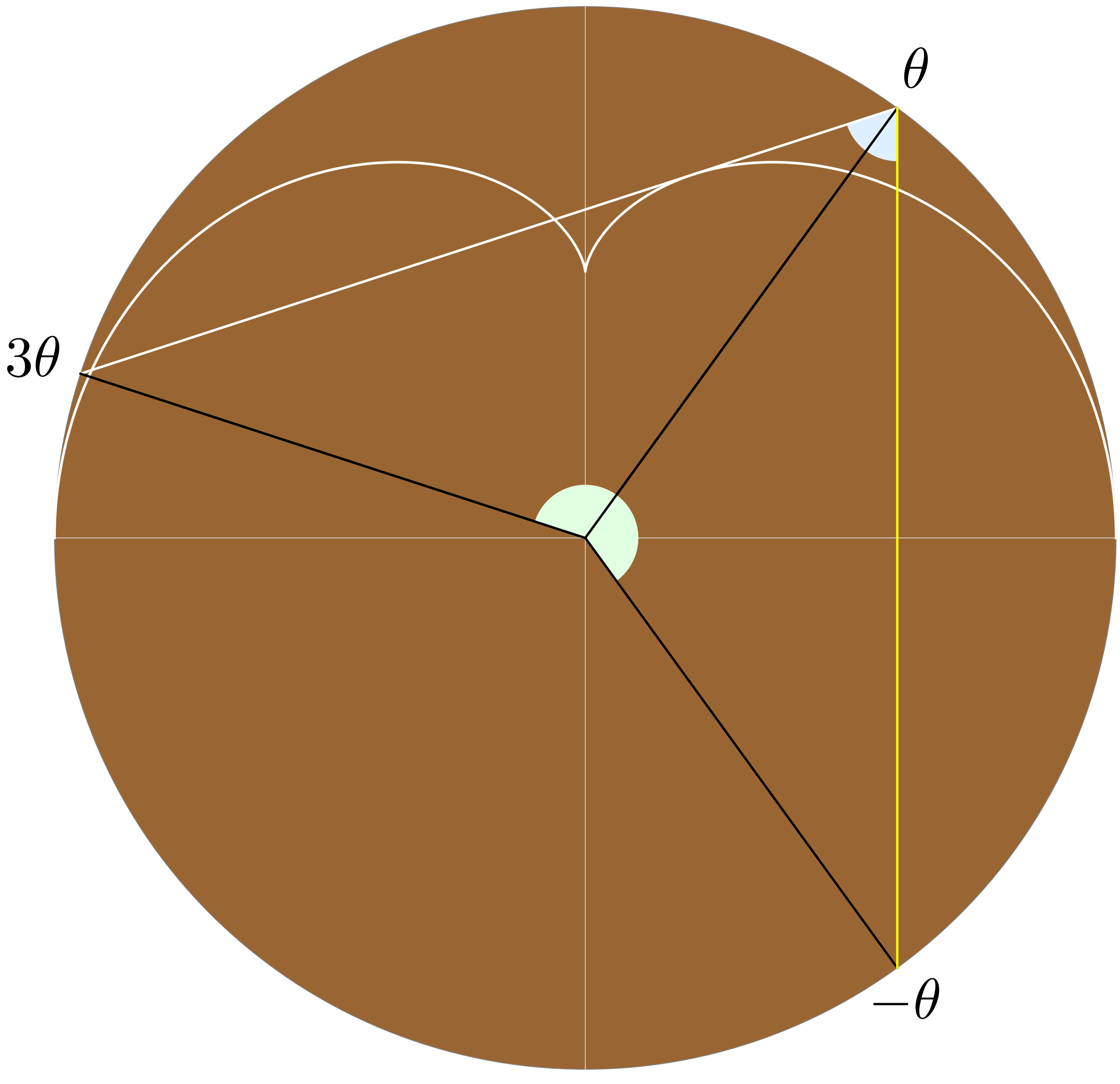
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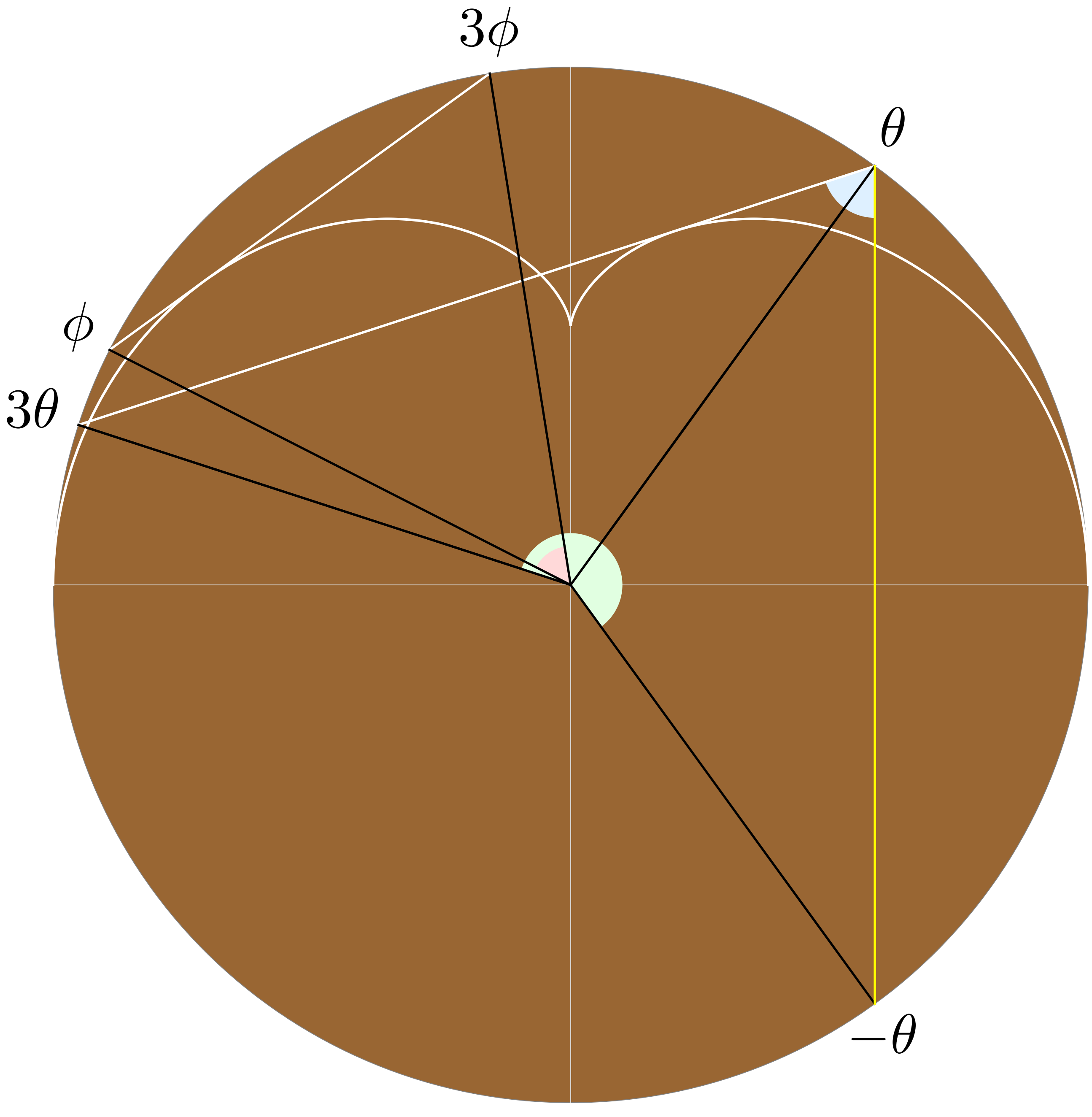

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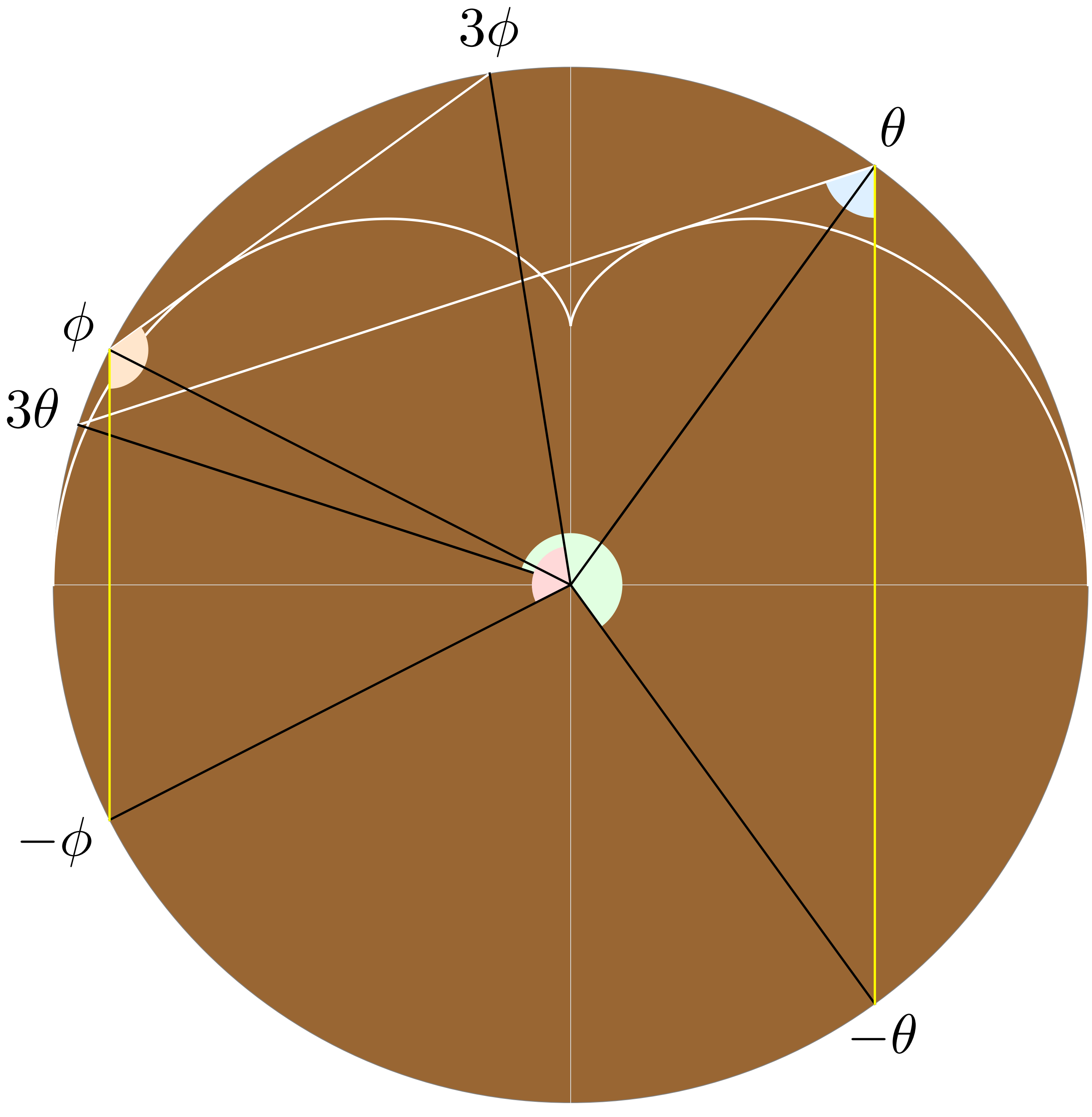

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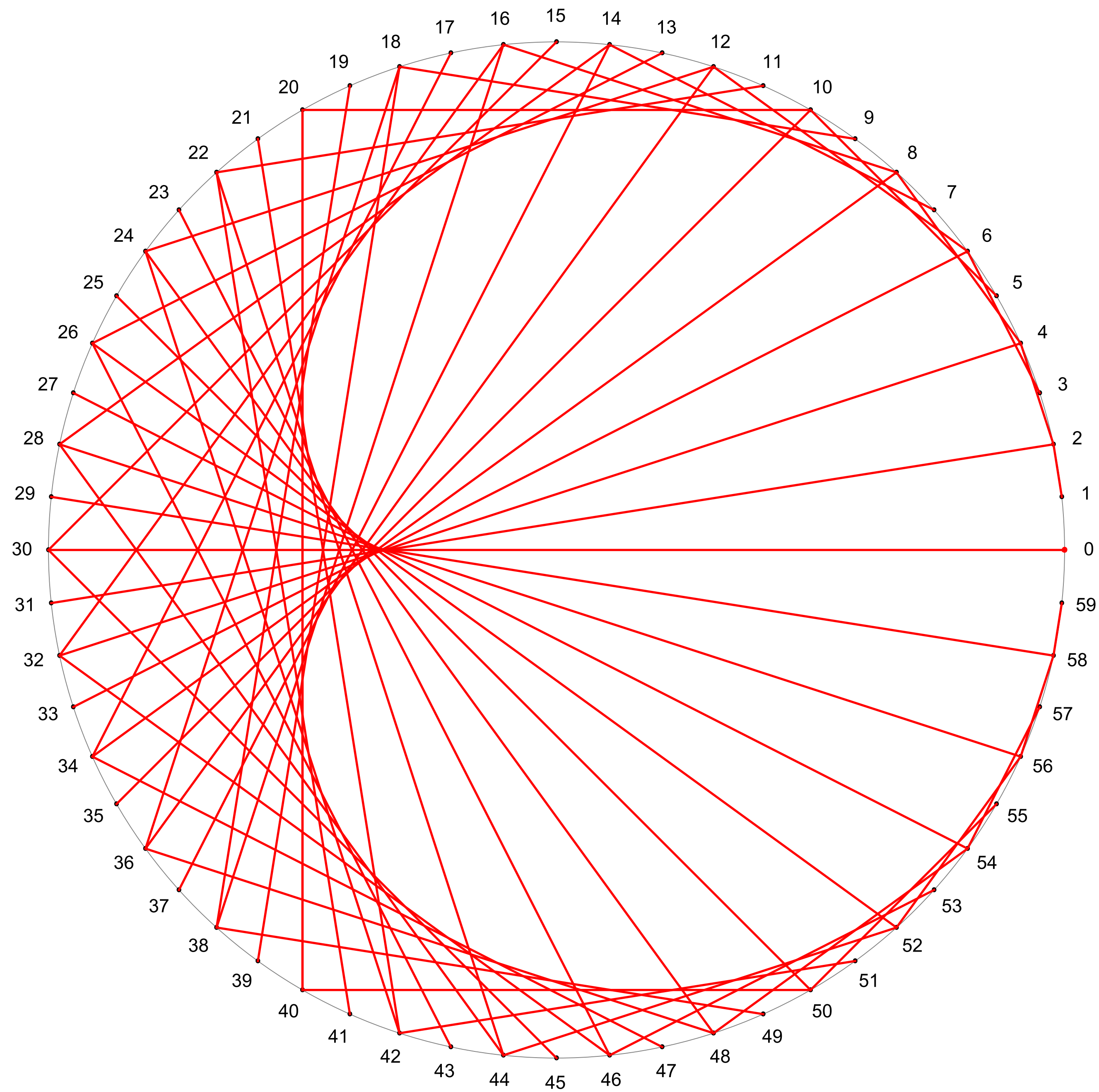
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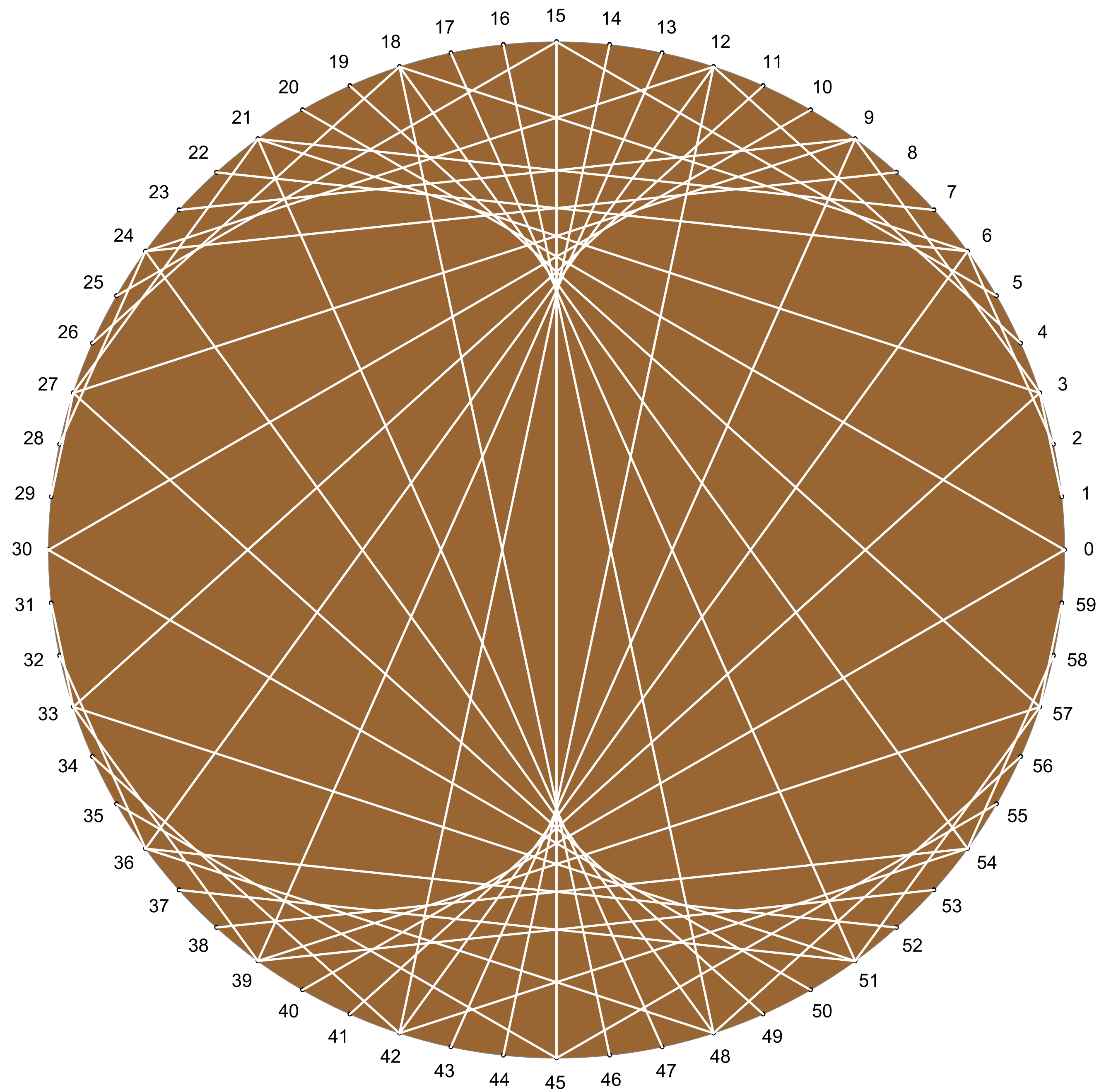
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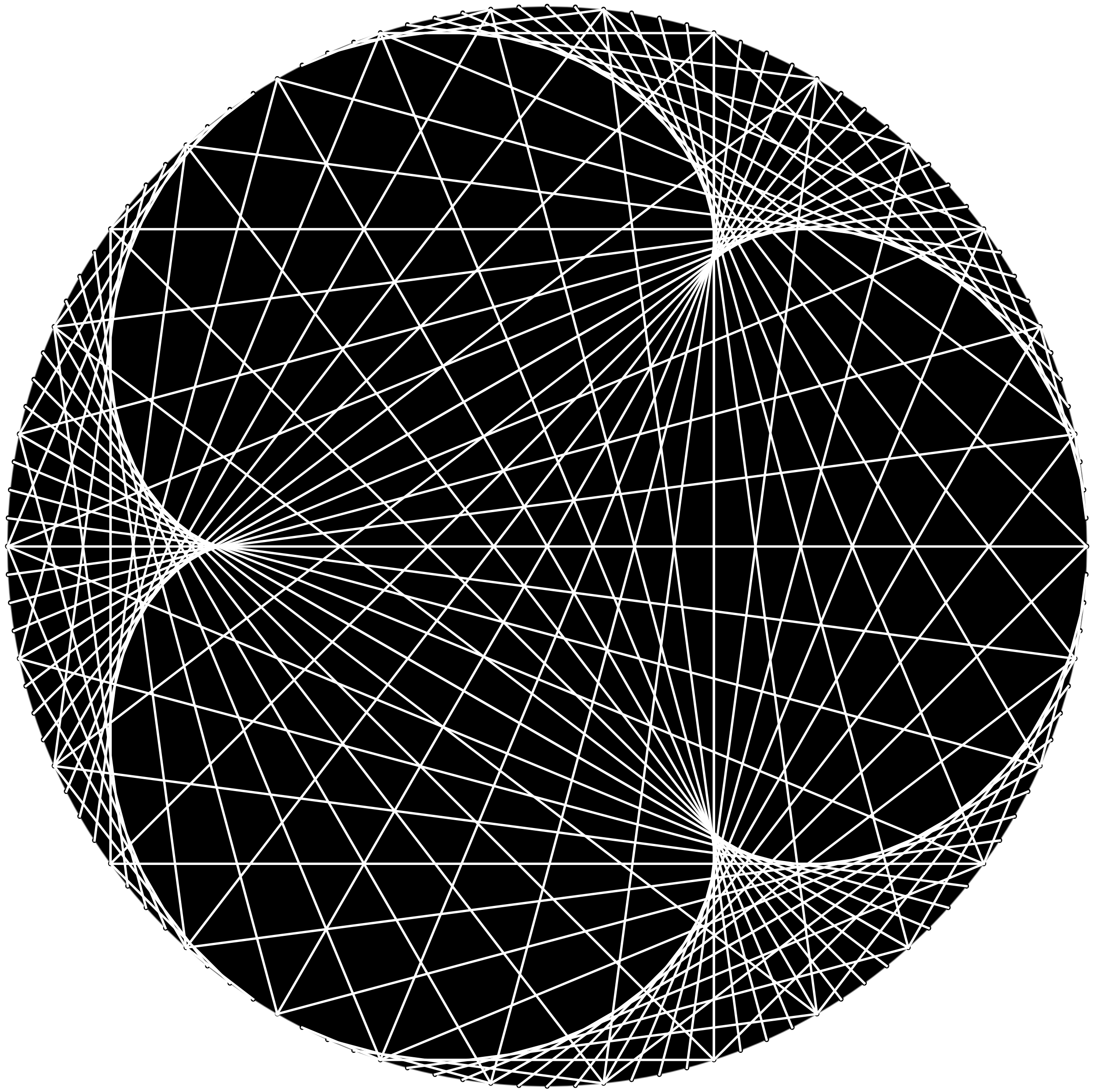
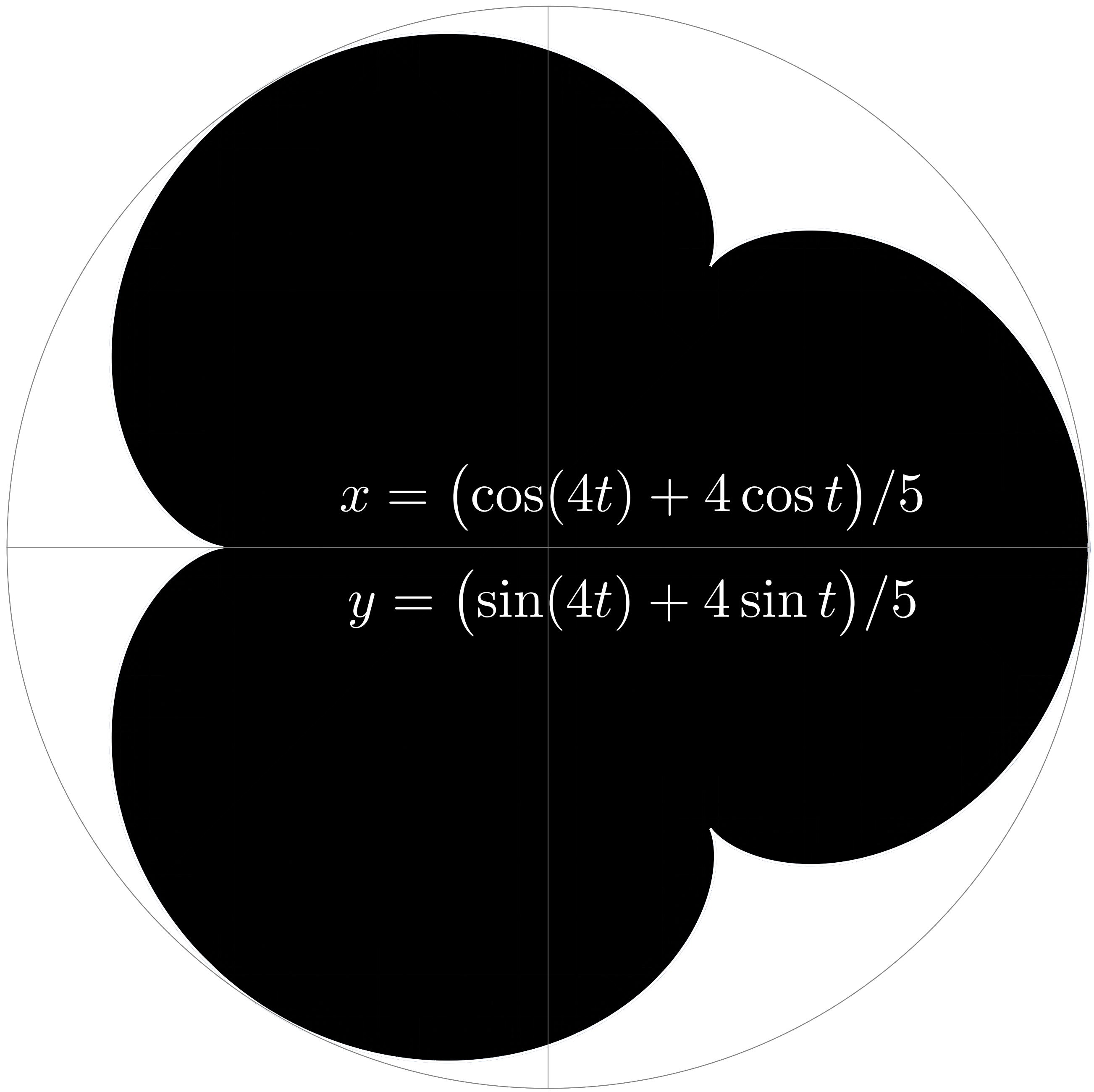
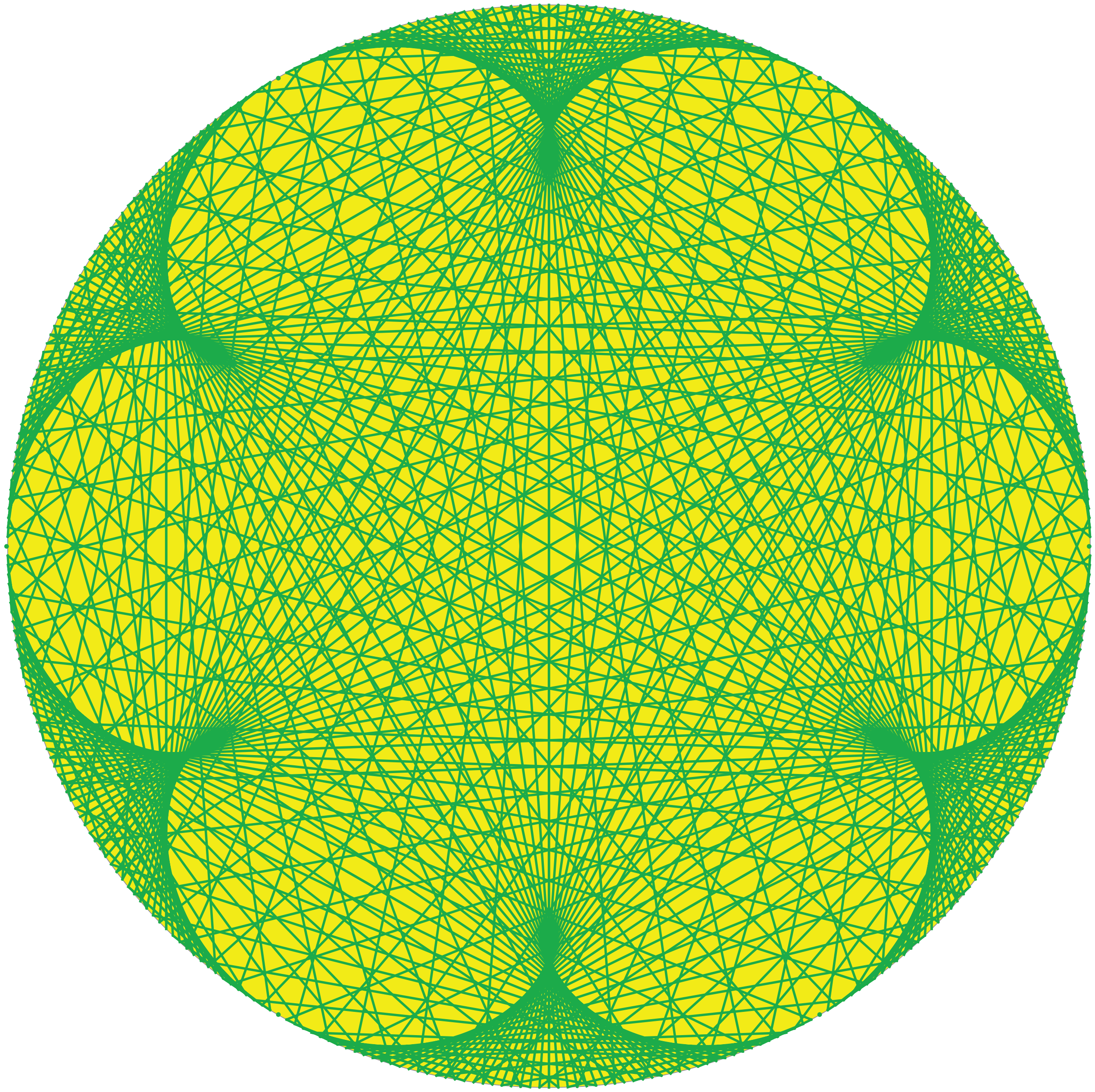
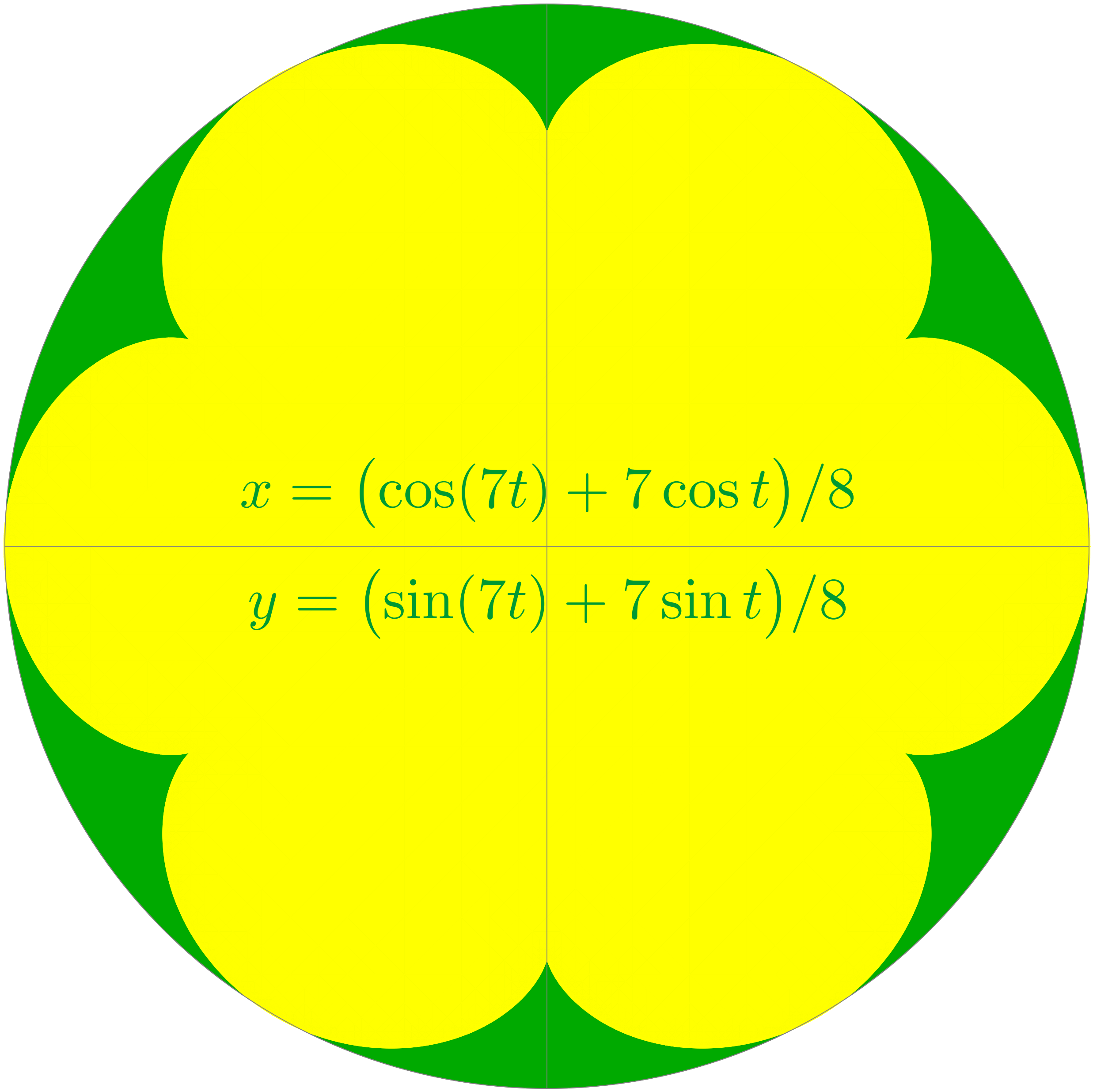



$x = \big(\cos(4t) + 4\cos t\big)/5$
$y = \big(\sin(4t) + 4\sin t\big)/5$




$x = \left(\cos(7t) + 7\cos t\right)/8$
$y = \left(\sin(7t) + 7\sin t\right)/8$

An algebra problem

$$x = \frac{\cos(kt) + k\cos t}{k+1}$$

$$y = \frac{\sin(kt) + k\sin t}{k+1}$$

For each k, we would like to find a real-valued function $g(x, y)$ that is negative exactly on the (compact) region bounded by this curve.

An algebra problem

$$x = \frac{\cos(kt) + k \cos t}{k+1}$$
$$y = \frac{\sin(kt) + k \sin t}{k+1}$$

For each k, we would like to find a real-valued function $g(x, y)$ that is negative exactly on the (compact) region bounded by this curve.

If $g(x, y)$ is continuous, *e.g.*, polynomial, then it will vanish on the curve.

An algebra problem

$$x = \frac{\cos(kt) + k\cos t}{k+1}$$
$$y = \frac{\sin(kt) + k\sin t}{k+1}$$

For each k, we would like to find a real-valued function $g(x, y)$ that is negative exactly on the (compact) region bounded by this curve.

If $g(x, y)$ is continuous, *e.g.*, polynomial, then it will vanish on the curve.

The simplest case is $k = 1$, *i.e.*, a circle:

$$x = \cos t$$
$$y = \sin t$$

The circle

Let $c = \cos t$. Then

$$
\begin{aligned}
x &= c \\
y &= \sin t
\end{aligned}
$$

The circle

Let $c = \cos t$. Then

$$x = c$$

$$y = \sin t$$

$$y^2 = \sin^2 t = 1 - c^2$$

The circle

Let $c = \cos t$. Then

$$x = c$$

$$y = \sin t$$

$$y^2 = \sin^2 t = 1 - c^2$$

$$x^2 = c^2$$

The circle

Let $c = \cos t$. Then

$$x = c$$

$$y = \sin t$$

$$y^2 = \sin^2 t = 1 - c^2$$

$$x^2 = c^2$$

So $g(x, y) = -1 + x^2 + y^2 = 0$ describes the circle, and $g(x, y) < 0$ on the interior of the disk.

The cardioid

$$\begin{aligned} 3x &= \cos(2t) + 2\cos t \\ &= \cos^2 t - \sin^2 t + 2\cos t \\ &= 2\cos^2 t - 1 + 2\cos t \\ &= -1 + 2\cos t + 2\cos^2 t \end{aligned}$$

The cardioid

$$
\begin{aligned}
3x &= \cos(2t) + 2\cos t \\
&= \cos^2 t - \sin^2 t + 2\cos t \\
&= 2\cos^2 t - 1 + 2\cos t \\
&= -1 + 2\cos t + 2\cos^2 t
\end{aligned}
$$

$$
\begin{aligned}
3y &= \sin(2t) + 2\sin t \\
&= 2\cos t\sin t + 2\sin t
\end{aligned}
$$

The cardioid

$$\begin{aligned}3x &= \cos(2t) + 2\cos t \\ &= \cos^2 t - \sin^2 t + 2\cos t \\ &= 2\cos^2 t - 1 + 2\cos t \\ &= -1 + 2\cos t + 2\cos^2 t\end{aligned}$$

$$\begin{aligned}3y &= \sin(2t) + 2\sin t \\ &= 2\cos t\sin t + 2\sin t\end{aligned}$$

$$\begin{aligned}9y^2 &= 4\cos^2 t\sin^2 t + 8\cos t\sin^2 t + 4\sin^2 t \\ &= 4\cos^2 t(1-\cos^2 t) + 8\cos t(1-\cos^2 t) + 4(1-\cos^2 t) \\ &= 4 + 8\cos t - 8\cos^3 t - 4\cos^4 t\end{aligned}$$

The cardioid

	1	c	c^2	c^3	c^4
1	1				
$3x$	-1	2	2		
$9x^2$	1	-4		8	4
$9y^2$	4	8		-8	-4

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6
1	1						
$3x$	-1	2	2				
$9x^2$	1	-4		8	4		
$9y^2$	4	8		-8	-4		
$27x^3$	-1	6	-6	-16	12	24	8
$27xy^2$	-4		24	24	-12	-24	-8

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6	c^7	c^8
1	1								
$3x$	-1	2	2						
$9x^2$	1	-4		8	4				
$9y^2$	4	8		-8	-4				
$27x^3$	-1	6	-6	-16	12	24	8		
$27xy^2$	-4		24	24	-12	-24	-8		
$81x^4$	1	-8	16	16	-56	-32	64	64	16
$81x^2y^2$	4	-8	-32	24	108	48	-64	-64	-16
$81y^4$	16	64	64	-64	-160	-64	64	64	16

The cardioid

	1	c	c^2	c^3	c^4	c^5	c^6	c^7	c^8
1	1								
$3x$	-1	2	2						
$9x^2$	1	-4		8	4				
$9y^2$	4	8		-8	-4				
$27x^3$	-1	6	-6	-16	12	24	8		
$27xy^2$	-4		24	24	-12	-24	-8		
$81x^4$	1	-8	16	16	-56	-32	64	64	16
$81x^2y^2$	4	-8	-32	24	108	48	-64	-64	-16
$81y^4$	16	64	64	-64	-160	-64	64	64	16

$$\Rightarrow \; -3 - 8 \cdot 3x - 6 \cdot 9(x^2 + y^2) + 81(x^4 + 2x^2y^2 + y^4) = 0$$

The cardioid

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$9x^2$	1	-4		8	4				
$9y^2$	4	8		-8	-4				
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$27xy^2$	-4		24	24	-12	-24	-8		
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$81y^4$	16	64	64	-64	-160	-64	64	64	16

$$\Rightarrow \; -3 - 8 \cdot 3x - 6 \cdot 9(x^2 + y^2) + 81(x^4 + 2x^2y^2 + y^4) = 0$$

So $g(x, y) = -1 - 8x - 18(x^2 + y^2) + 27(x^2 + y^2)^2 = 0$ describes the cardioid, and $g(x, y) < 0$, its interior.

Counting

We had to eventually find a linear combination of monomials in x and y that summed to 0, because up to degree d there are

$$\begin{aligned} \left(\frac{d}{2}+1\right)^2 \quad & \text{if } d \text{ even;} \\ \frac{d+1}{2}\left(\frac{d+1}{2}+1\right) \quad & \text{if } d \text{ odd;} \end{aligned}$$

monomials, but only $2d+1$ constraints on the coefficients, coming from the powers of c.

Fewer linear equations

Let $f_1(c)$ and $f_2(c)$ be polynomials of degrees m and n, respectively. For example,

$$f_1(c) = (1 + 3x) - 2c - 2c^2$$
$$f_2(c) = (4 - 9y^2) + 8c - 8c^3 - 4c^4,$$

in which case $m = 2$ and $n = 4$.

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in which case $m = 2$ and $n = 4$.

$f_1(c) = 0$ and $f_2(c) = 0$ share solutions iff they have a common factor, a polynomial $D(c)$ such that $f_i(c) = Q_i(c)D(c)$, whence

$$\frac{f_1(c)}{Q_1(c)} = \frac{f_2(c)}{Q_2(c)},$$

so $0 = Q_2(c)f_1(c) - Q_1(c)f_2(c)$.

Sylvester's matrix

That is, there are $m+n$ scalars $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{m-1}$ such that:

$$
\begin{aligned}
0 &= (a_0 + a_1 c + \cdots + a_{n-1} c^{n-1}) f_1(c) \\
&\quad + (b_0 + b_1 c + \cdots + b_{m-1} c^{m-1}) f_2(c) \\
&= \begin{bmatrix}
1+3x & & & & 4-9y^2 & \\
-2 & 1+3x & & & 8 & 4-9y^2 \\
-2 & -2 & 1+3x & & 0 & 8 \\
 & -2 & -2 & 1+3x & -8 & 0 \\
 & & -2 & -2 & -4 & -8 \\
 & & & -2 & & -4
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \end{bmatrix}
\end{aligned}
$$

Sylvester's matrix

That is, there are $m+n$ scalars $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{m-1}$ such that:

$$\begin{aligned} 0 &= (a_0 + a_1 c + \cdots + a_{n-1} c^{n-1}) f_1(c) \\ &\quad + (b_0 + b_1 c + \cdots + b_{m-1} c^{m-1}) f_2(c) \\ &= \begin{bmatrix} 1+3x & & & & 4-9y^2 & \\ -2 & 1+3x & & & 8 & 4-9y^2 \\ -2 & -2 & 1+3x & & 0 & 8 \\ & -2 & -2 & 1+3x & -8 & 0 \\ & & -2 & -2 & -4 & -8 \\ & & & -2 & & -4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \end{bmatrix} \end{aligned}$$

In general, this $(m+n) \times (m+n)$ matrix is called the Sylvester matrix of f_1 and f_2, $\mathrm{Syl}(f_1, f_2)$.

Sylvester's matrix

This homogeneous system of $m+n$ linear equations in $m+n$ variables has a nontrivial solution iff Sylvester's matrix is singular, *i.e.*, iff the resultant:

$$\operatorname{Res}(f_1, f_2) = \det \operatorname{Syl}(f_1, f_2) = 0.$$

Sylvester's matrix

This homogeneous system of $m+n$ linear equations in $m+n$ variables has a nontrivial solution iff Sylvester's matrix is singular, *i.e.*, iff the resultant:

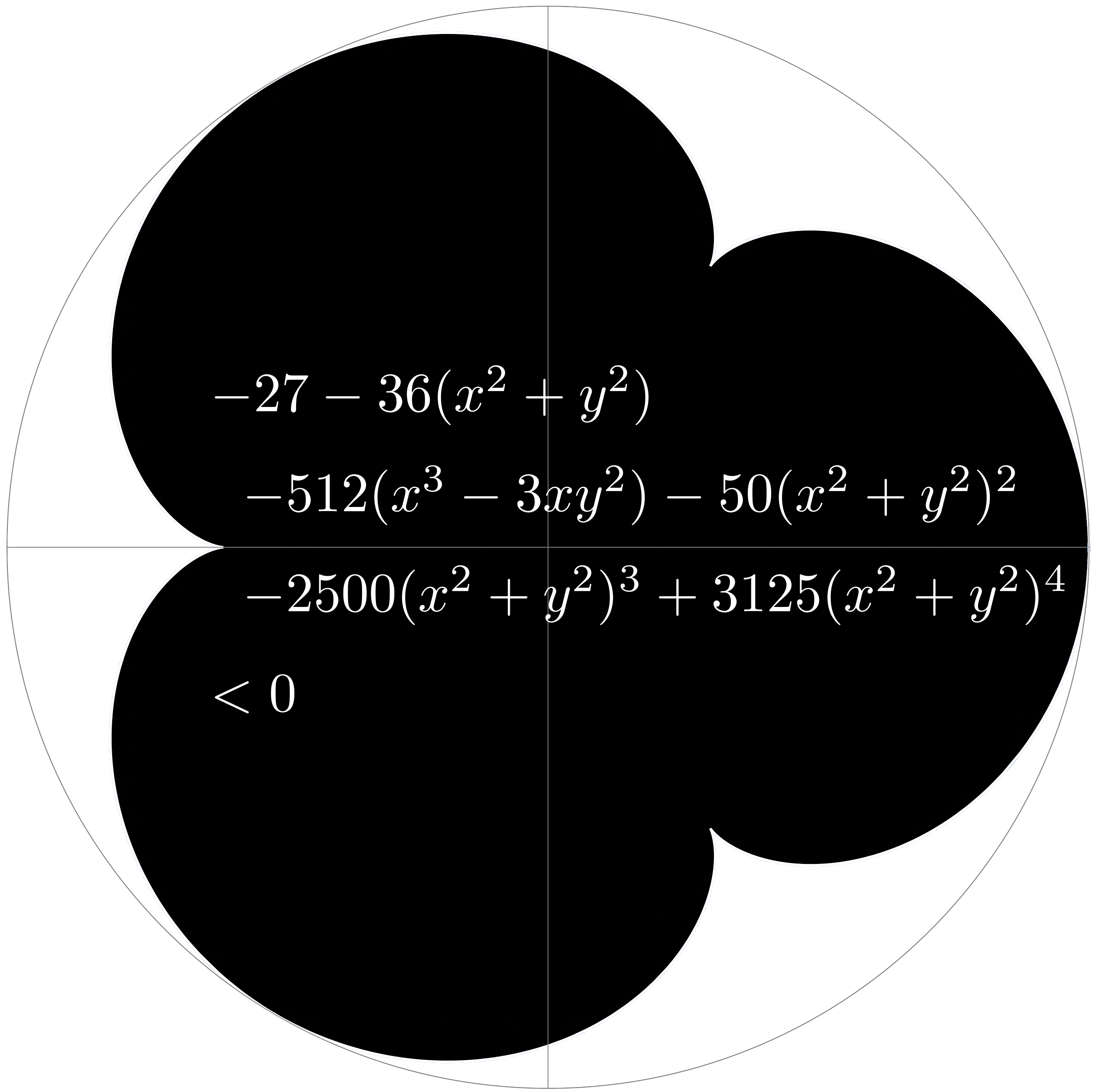
$$\operatorname{Res}(f_1, f_2) = \det \operatorname{Syl}(f_1, f_2) = 0.$$

For the f_i coming from the cardioid,

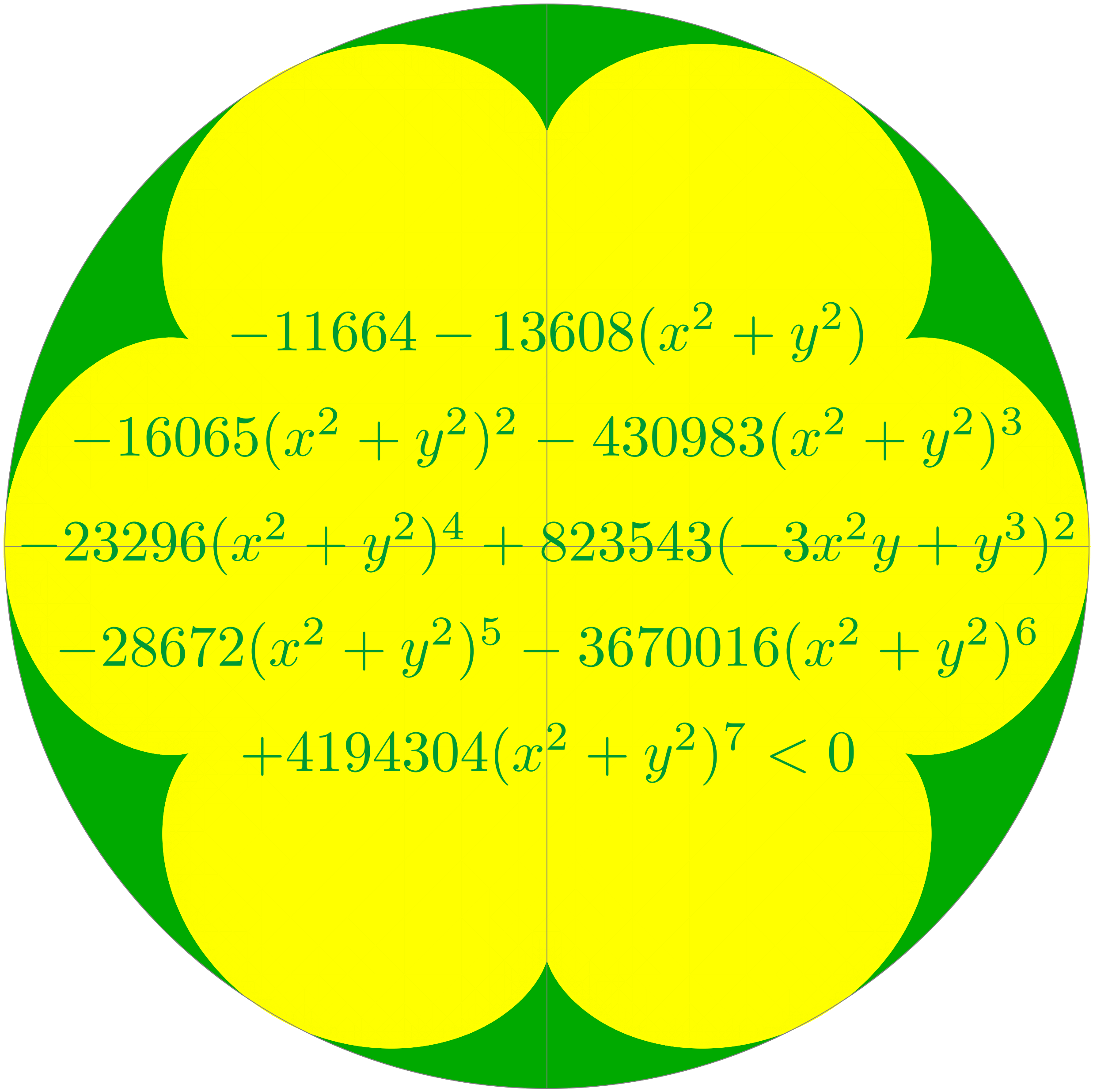
$$0 = \begin{vmatrix} 1+3x & & & & 4-9y^2 & \\ -2 & 1+3x & & & 8 & 4-9y^2 \\ -2 & -2 & 1+3x & & 0 & 8 \\ & -2 & -2 & 1+3x & -8 & 0 \\ & & -2 & -2 & -4 & -8 \\ & & & -2 & & -4 \end{vmatrix}$$

$$= -48 - 384x - 864(x^2+y^2) + 1296(x^4 + 2x^2y^2 + y^4)$$

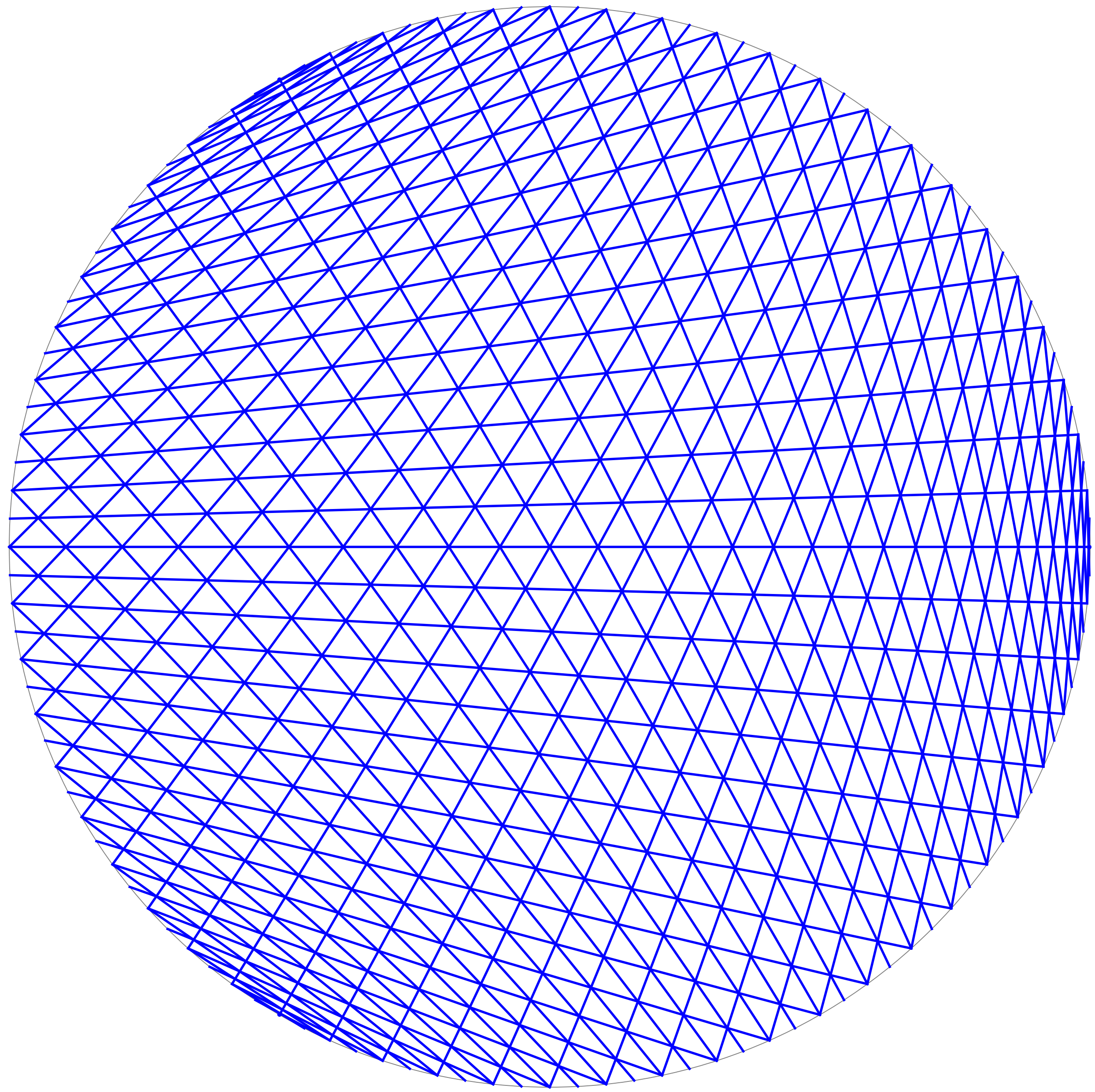
$$= 48\big(-1 - 8x - 18(x^2+y^2) + 27(x^2+y^2)^2\big).$$

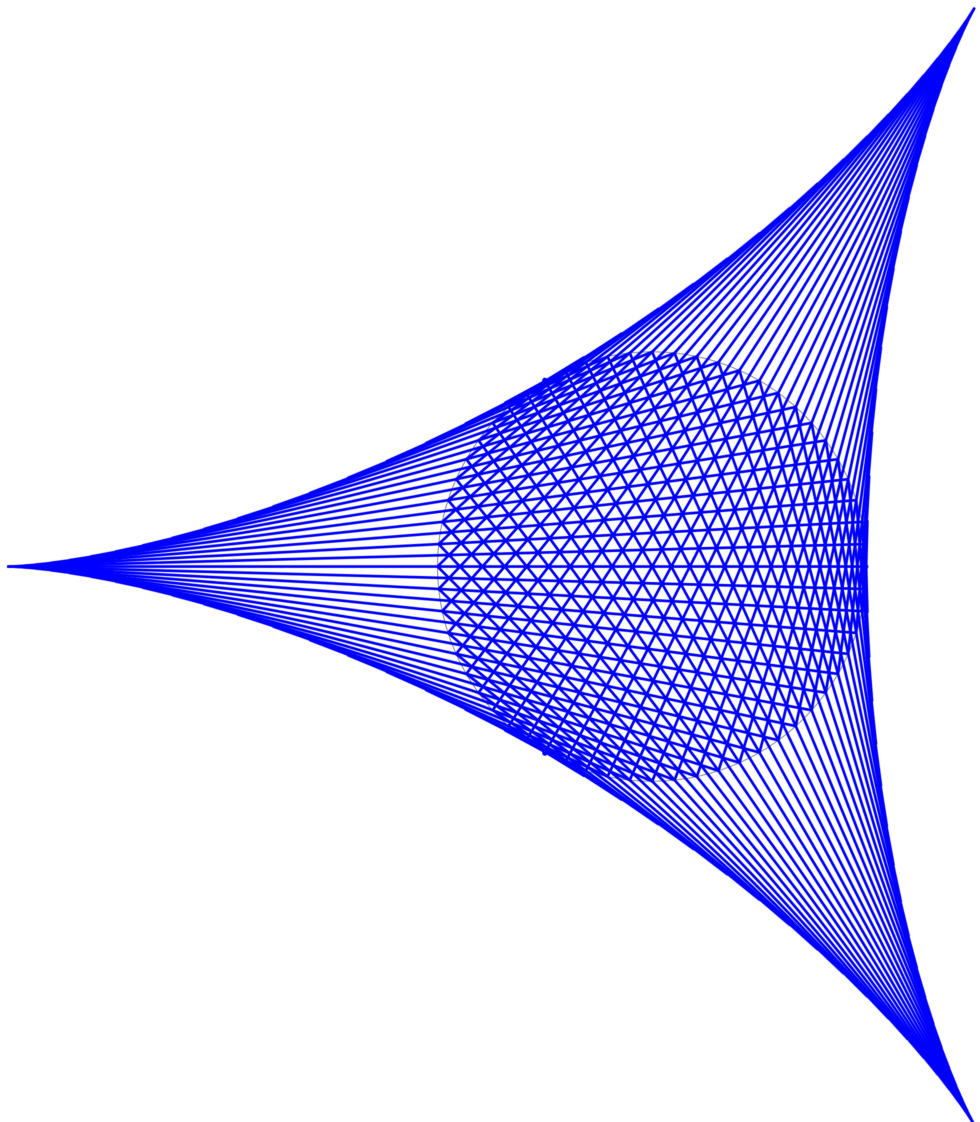


$-27 - 36(x^2 + y^2)$
$-512(x^3 - 3xy^2) - 50(x^2 + y^2)^2$
$-2500(x^2 + y^2)^3 + 3125(x^2 + y^2)^4$
< 0



$$-11664 - 13608(x^2+y^2) - 16065(x^2+y^2)^2 - 430983(x^2+y^2)^3 - 23296(x^2+y^2)^4 + 823543(-3x^2y+y^3)^2 - 28672(x^2+y^2)^5 - 3670016(x^2+y^2)^6 + 4194304(x^2+y^2)^7 < 0$$





Tricuspid

When $k = -2$:

$$x = \frac{\cos(-2t) - 2\cos t}{-2+1}$$

$$y = \frac{\sin(-2t) - 2\sin t}{-2+1}$$

Tricuspid

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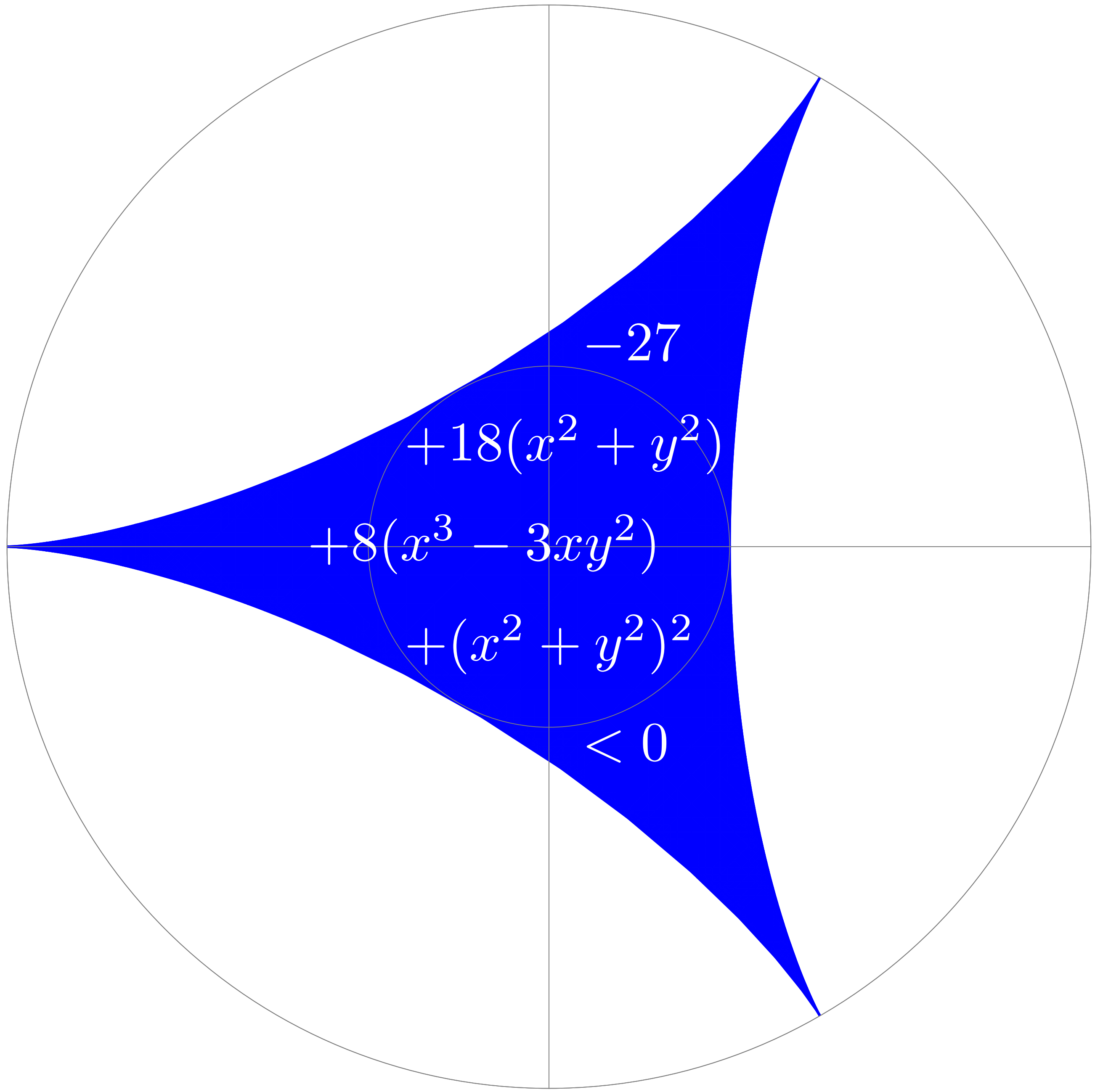
$$x = \frac{\cos(-2t) - 2\cos t}{-2+1}$$

$$y = \frac{\sin(-2t) - 2\sin t}{-2+1}$$

Eliminating t gives the equation for the tricuspid:

$$-27 + 18(x^2 + y^2) + 8(x^3 - 3xy^2) + (x^2 + y^2)^2 = 0.$$


-27
$+18(x^2 + y^2)$
$+8(x^3 - 3xy^2)$
$+(x^2 + y^2)^2$
< 0

Tricuspid

When $k = -2$:

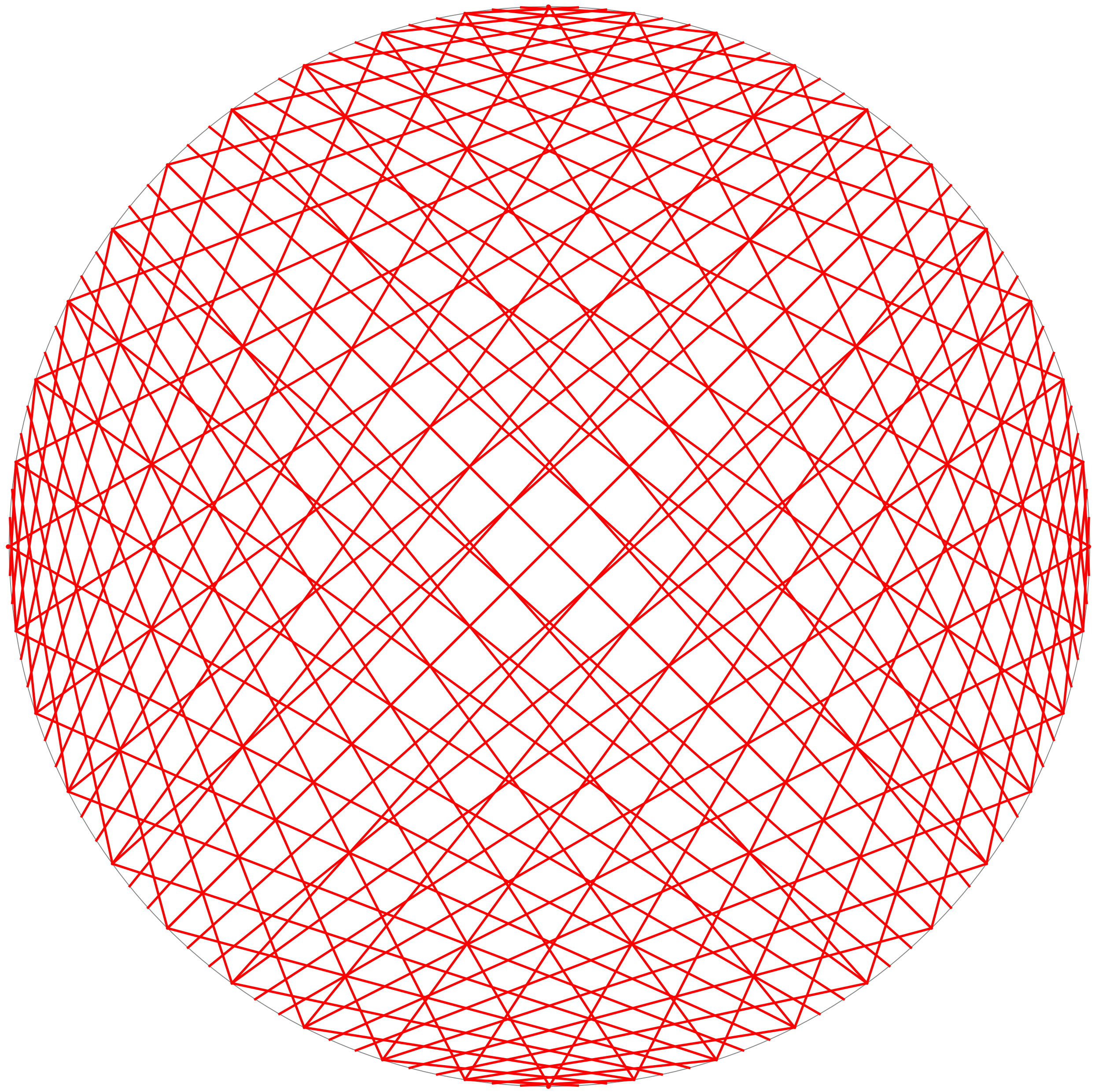
$$x = \frac{\cos(-2t) - 2\cos t}{-2+1}$$
$$y = \frac{\sin(-2t) - 2\sin t}{-2+1}$$

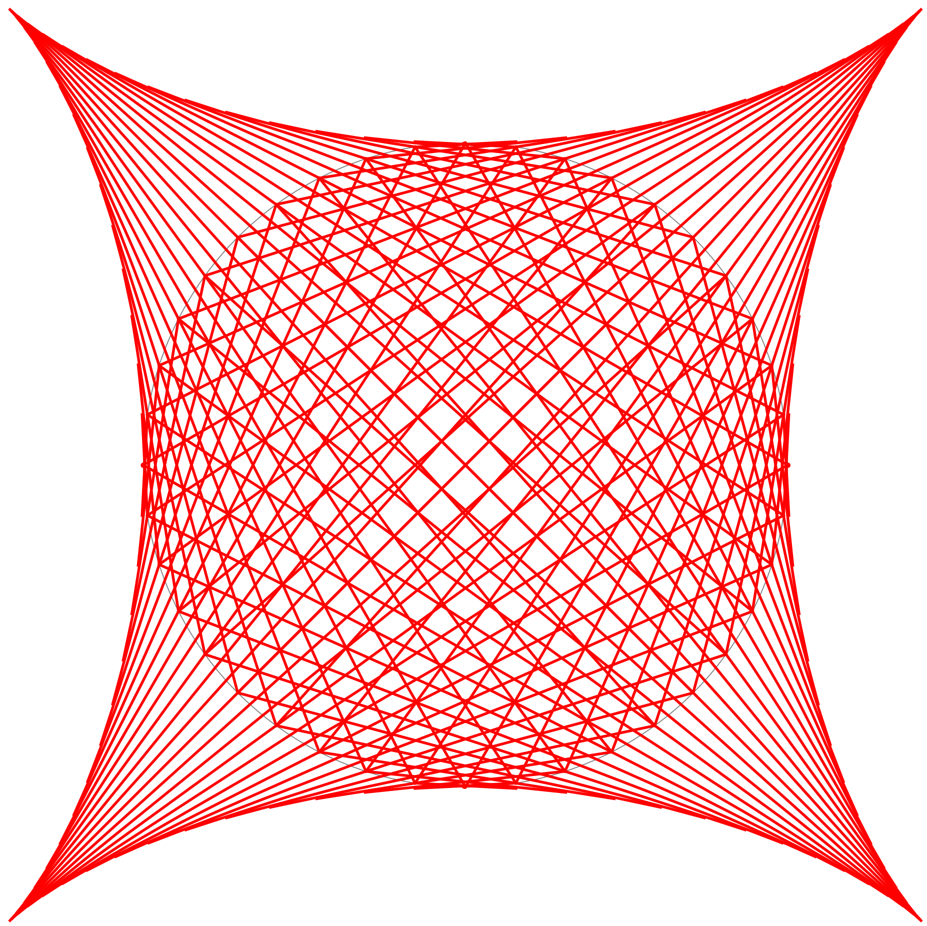
Eliminating t gives the equation for the tricuspid:

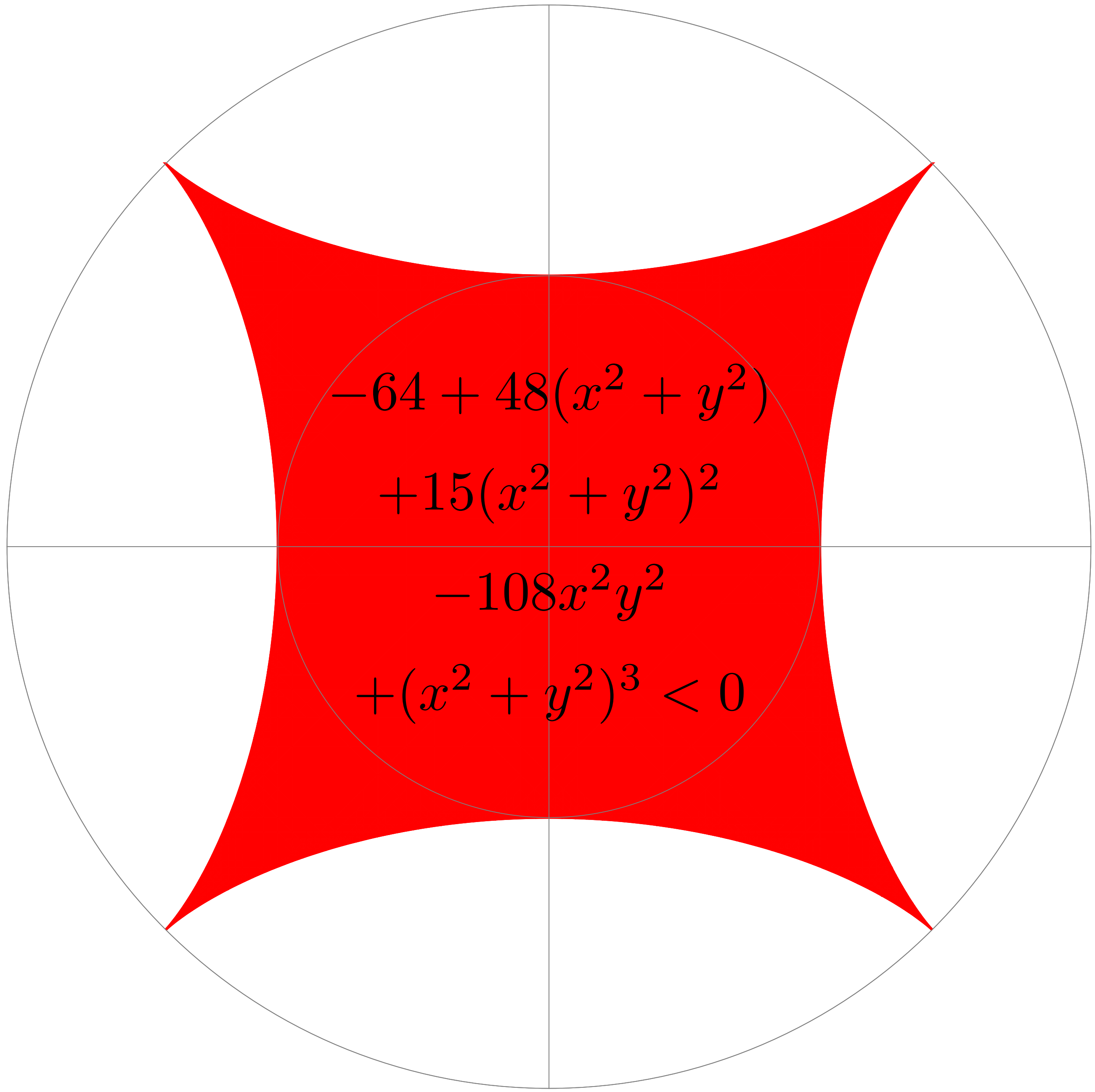
$$-27 + 18(x^2 + y^2) + 8(x^3 - 3xy^2) + (x^2 + y^2)^2 = 0.$$

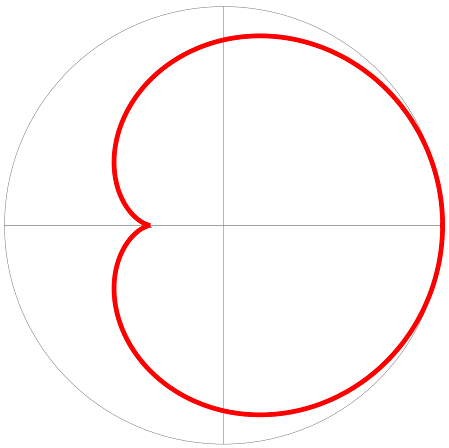
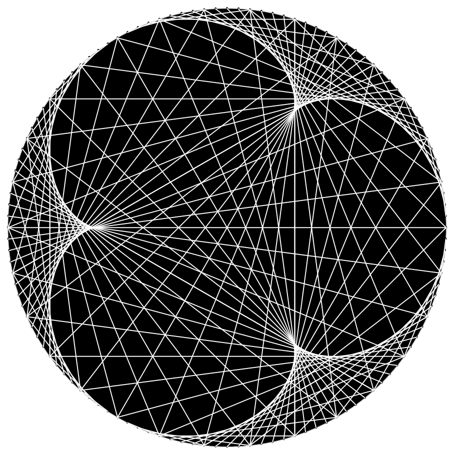
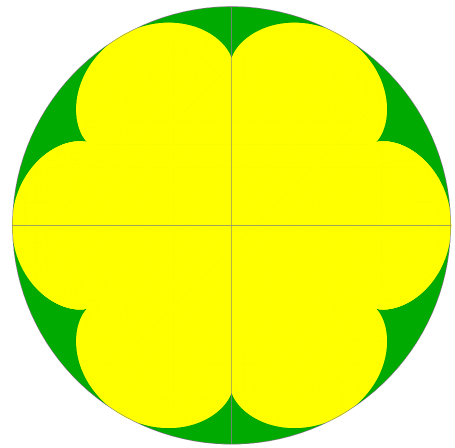
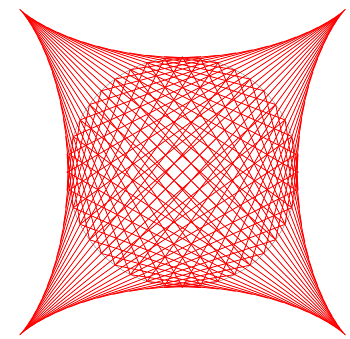
Compare with the equation for the cardioid:

$$-1 - 8x - 18(x^2 + y^2) + 27(x^2 + y^2)^2 = 0.$$






$-64 + 48(x^2 + y^2)$
$+15(x^2 + y^2)^2$
$-108x^2y^2$
$+(x^2 + y^2)^3 < 0$

HAPPY VALENTINE’S DAY!

Inspirations

Ann Marielson's mathematical art, http://www.aisonart.co.uk/.

elementary mathematical problem solving,
http://math.ucsd.edu/~dmeyer/teaching/elementary.html.

Grant W. Allen, *Semi-Algebraic Entanglement Consistency Relations: Fundamental and Dynamical*, UCSD Physics Ph.D. thesis (2017).

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Ludwig Uhland, *Des Goldschmieds Töchterlein* (1815).

George Gordon, Lord Byron, *Childe Harold's Pilgrimage*, Canto the Fourth (1818).

James Joseph Sylvester, *The Laws of Verse, or Principles of Versification Exemplified in Metrical Translations: together with an Annotated Reprint of the Inaugural Presidential Address to the Mathematical and Physical Section of the British Association at Exeter* (1870).

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Luigi Cremona, "*Sur l'hypocycloïde à trois rebroussements*", *Journal für Mathematik* **LXIV** (1865) 101–123.

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