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Math 110, Spring 2004 Notes

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Preliminaries

1.1 The spectral theorem for symmetric matrices

Let A be a real $N \times N$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix}, \quad (1.1)$$

and

$$f = \begin{bmatrix} f^1 \\ f^2 \\ \vdots \\ f^N \end{bmatrix} \in \mathbb{R}^N$$

be a given vector. As usual we will let e_i denote the vector in $\mathbb{R}^N$ with all entries being zero except for the i^{th} which is taken to be one.

We will write

$$(u,v) := u \cdot v = \sum_{i=1}^N u_i v_i = u^{\text{tr}} v \text{ and}$$
$$|u|^2 = (u,u) = \sum_{i=1}^N u_i^2 = u^{\text{tr}} u.$$

Recall that $\{v_i\}_{i=1}^N \subset \mathbb{R}^N$ is said to be an orthonormal basis if

$$(v_i, v_j) = \delta_{ij} := \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}. \quad (1.2)$$

The following proposition and its infinite dimensional analogue will be the basis for much of this course.

Proposition 1.1. *If $\{v_i\}_{i=1}^N \subset \mathbb{R}^N$ satisfies Eq. (1.2) then $\{v_i\}_{i=1}^N$ is a basis for $\mathbb{R}^N$ and if $u \in \mathbb{R}^N$ we have*

$$u = \sum_{i=1}^N (u, v_i) v_i. \quad (1.3)$$

Proof. Suppose that $u = \sum_{i=1}^N a_i v_i$ for some $a_i \in \mathbb{R}$. Then

$$(u, v_j) = \left(\sum_{i=1}^N a_i v_i, v_j \right) = \sum_{i=1}^N a_i (v_i, v_j) = \sum_{i=1}^N a_i \delta_{ij} = a_j.$$

In particular if $u = 0$ we learn that $a_j = (u, v_j) = 0$ and we have shown that $\{v_i\}_{i=1}^N$ is a **linearly independent set.** Since $\dim(\mathbb{R}^N) = N$, it now follows that $\{v_i\}_{i=1}^N$ is a basis for $\mathbb{R}^N$ and hence every $u \in \mathbb{R}^N$ may be written in the form $u = \sum_{i=1}^N a_i v_i$. By what we have just proved, we must have $a_i = (u, v_i)$, i.e. Eq. (1.3) is valid. ■

Definition 1.2. *A matrix A as in Eq. (1.1) is symmetric $A = A^{\text{tr}}$, i.e. if $a_{ij} = a_{ji}$ for all i, j.*

The following characterization of a symmetric matrix will be more useful for our purposes.

Lemma 1.3. *If A is a real $N \times N$ matrix then, for all $u, v \in \mathbb{R}^N$,*

$$(Au, v) = \left(u, A^{\text{tr}} v\right). \quad (1.4)$$

Moreover A is symmetric iff

$$(Au, v) = (u, Av) \text{ for all } u, v \in \mathbb{R}^N. \quad (1.5)$$

Proof. Eq. (1.4) is a consequence of the following matrix manipulations

$$(Au, v) = (Au)^{\text{tr}} v = u^{\text{tr}} A^{\text{tr}} v = \left(u, A^{\text{tr}} v\right)$$

which are based on the fact that $(AB)^{\text{tr}} = B^{\text{tr}} A^{\text{tr}}$. Hence if A is symmetric, then Eq. (1.5) holds. Conversely, if Eq. (1.5) holds, by taking $u = e_i$ and $v = e_j$ in Eq. (1.5) we learn that

$$a_{ji} = \left(\begin{bmatrix} a_{1i} \\ \vdots \\ a_{N,i} \end{bmatrix}, e_j \right) = (Ae_i, e_j) = (e_i, Ae_j) = \left(e_i, \begin{bmatrix} a_{1j} \\ \vdots \\ a_{N,j} \end{bmatrix} \right) = a_{ij}.$$

■

Corollary 1.4. *Suppose that $A = A^{\mathrm{tr}}$ and $v, w \in \mathbb{R}^N$ are eigenvectors of A with eigenvalues λ and μ respectively. If $\mu \neq \lambda$ then v and w are orthogonal, i.e. $(v, w) = 0$.*

Proof. If $Av = \lambda v$ and $Aw = \mu w$ with $\lambda \neq \mu$ then

$$\lambda (v, w) = (\lambda v, w) = (Av, w) = (v, Aw) = (v, \mu w) = \mu (v, w)$$

or equivalently, $(\lambda - \mu)(v, w) = 0$. Sine $\lambda \neq \mu$, we must conclude that $(v, w) = 0$. ■

The following important theorem from linear algebra gives us a method for guaranteeing that a matrix is diagonalizable. Again much of this course is based on an infinite dimensional generalization of this theorem.

Theorem 1.5 (Spectral Theorem). *If A in Eq. (1.1) is a symmetric matrix, then A has an orthonormal basis of eigenvectors, $\{v_1, \dots, v_N\}$ and the corresponding eigenvalues, $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ are all real.*

Example 1.6. Suppose that

$$A := \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \tag{1.6}$$

then

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \det \begin{bmatrix} \frac{1}{2} - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} - \lambda \end{bmatrix} \\ &= \left(\frac{1}{2} - \lambda\right)^2 - \frac{9}{4} \end{aligned}$$

which we set equal to zero to learn

$$\left(\frac{1}{2} - \lambda\right)^2 = \frac{9}{4}$$

or equivalently, $\left(\lambda - \frac{1}{2}\right) = \pm\frac{3}{2}$ and hence A has eigenvalues,

$$\lambda_1 = -1 \text{ and } \lambda_2 = 2.$$

Since

$$A + I := \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \cong \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and

$$A - 2I = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \cong \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

we learn that

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longleftrightarrow \lambda_1 = -1$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \longleftrightarrow \lambda_2 = 2.$$

Notice that $(v_1, v_2) = 0$ as is guaranteed by Corollary 1.4. The normalized eigenvectors are given by $2^{-1/2} v_1$ and $2^{-\frac{1}{2}} v_2$. Consequently if $f \in \mathbb{R}^2$, we have

$$\begin{aligned} f &= \left(2^{-1/2} v_1, f\right) 2^{-1/2} v_1 + \left(2^{-1/2} v_2, f\right) 2^{-1/2} v_2 \\ &= \frac{1}{2} (v_1, f) v_1 + \frac{1}{2} (v_2, f) v_2. \end{aligned} \tag{1.7}$$

Remark 1.7. As above, it often happens that naturally we find a orthogonal but **not** orthonormal basis $\{v_i\}_{i=1}^N$ for $\mathbb{R}^N$, i.e. $(v_i, v_j) = 0$ if $i \neq j$ but $(v_i, v_i) \neq 1$. We can still easily expand in terms of these vectors. Indeed, $\left\{|v_i|^{-1} v_i\right\}_{i=1}^N$ is an orthonormal basis for $\mathbb{R}^N$ and therefore if $f \in \mathbb{R}^N$ we have

$$f = \sum_{i=1}^N \left(f, |v_i|^{-1} v_i\right) |v_i|^{-1} v_i = \sum_{i=1}^N \frac{(f, v_i)}{|v_i|^2} v_i.$$

Example 1.8. Working as above, one shows the symmetric matrix,

$$A := \begin{bmatrix} 1 & 7 & -2 \\ 7 & 1 & -2 \\ -2 & -2 & 10 \end{bmatrix}, \tag{1.8}$$

has characteristic polynomial given by

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= -\left(\lambda^3 - 12\lambda^2 - 36\lambda + 432\right) \\ &= -(\lambda - 6)(\lambda - 12)(\lambda + 6). \end{aligned}$$

Thus the eigenvalues of A are given by $\lambda_1 = -6$, $\lambda_2 = 6$ and $\lambda_3 = 12$ and the corresponding eigenvectors are

$$v_1 := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow -6,\ v_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow 6,\ v_3 := \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \leftrightarrow 12.$$

Again notice that $\{v_1, v_2, v_3\}$ is an orthogonal set as is guaranteed by Corollary 1.4. Relative to this basis we have the expansion

$$\begin{aligned} f &= (f, v_1) \frac{v_1}{|v_1|^2} + (f, v_2) \frac{v_2}{|v_2|^2} + (f, v_3) \frac{v_3}{|v_3|^2} \\ &= \frac{1}{2} (f, v_1) v_1 + \frac{1}{3} (f, v_2) v_2 + \frac{1}{6} (f, v_2) v_3. \end{aligned}$$

For example if $f = (1,2,3)^{\text{tr}}$, then

$$f = \frac{1}{2}v_1 + 2v_2 + \frac{1}{2}v_3. \tag{1.9}$$

Exercise 1.1. Verify that the vectors $\{v_i\}_{i=1}^3$ are eigenvectors of A in Eq. (1.8) which have the stated eigenvalues. **Hint:** you are only asked to verify not solve from scratch.

Exercise 1.2. Find eigenvectors $\{v_i\}_{i=1}^3$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^3$ for the symmetric matrix,

$$A := \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Make sure you **choose** them to be orthogonal. Also express the following vectors,

$$f = (1,0,2)^{\text{tr}} \text{ and } g = (0,1,2)^{\text{tr}} \text{ and } h = (-1,1,0)^{\text{tr}},$$

as linear combinations of the $\{v_i\}_{i=1}^3$ that you have found.

Exercise 1.3. Suppose that A is a $N \times N$ symmetric matrix and $\{v_i\}_{i=1}^N$ is a basis of eigenvectors of A with corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$. Suppose $f \in \mathbb{R}^N$ has been decomposed as

$$f = \sum_{i=1}^N a_i v_i.$$

Show:

1. $A^n f = \sum_{i=1}^N a_i \lambda_i^n v_i$.
2. More generally, suppose that

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$$

is a polynomial in λ, then

$$p(A) f = \sum_{i=1}^N a_i p(\lambda_i) v_i$$

and in particular $p(A) v = p(\lambda) v$ is $Av = \lambda v$.

1.2 Cylindrical and Spherical Coordinates

Our goal in this section is to work out the Laplacian in cylindrical and spherical coordinates. We will need these results later in the course. Our method is to make use of the following two observations:

1. If $\{u_i\}_{i=1}^3$ is **any** orthonormal basis for $\mathbb{R}^3$ then

$$\nabla f \cdot \nabla g = \sum_{i=1}^3 (\nabla f, u_i)(\nabla g, u_i) = \sum_{i=1}^3 \partial_{u_i} f \partial_{u_i} g$$

and

2. if g has compact support in a region Ω, then by integration by parts

$$\int_\Omega \Delta f g dV = -\int_\Omega \nabla f \cdot \nabla g dV. \tag{1.10}$$

The following theorem is a far reaching generalization of Eq. (1.10).

Theorem 1.9 (Divergence Theorem). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded region with smooth boundary, $n : \partial\Omega \to \mathbb{R}^n$ be the unit outward pointing normal to Ω. If $Z \in C^1(\bar{\Omega}, \mathbb{R}^n)$, then*

$$\int_{\partial\Omega} Z(x) \cdot n(x) d\sigma(x) = \int_\Omega \nabla \cdot Z(x)\ dx. \tag{1.11}$$

Corollary 1.10 (Integration by parts). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded region with smooth boundary, $n : \partial\Omega \to \mathbb{R}^n$ be the unit outward pointing normal to Ω. If $Z \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $f \in f \in C^1(\bar{\Omega}, \mathbb{R})$, then*

$$\int_\Omega f(x) \nabla \cdot Z(x)\ dx = -\int_\Omega \nabla f(x) \cdot Z(x)\ dx + \int_{\partial\Omega} f(x) Z(x) \cdot n(x) d\sigma(x). \tag{1.12}$$

Also if $g \in C^2(\bar{\Omega}, \mathbb{R})$, then

$$\int_\Omega f(x) \Delta g(x)\ dx = -\int_\Omega \nabla f(x) \cdot \nabla g(x)\ dx + \int_{\partial\Omega} f(x) \nabla g(x) \cdot n(x) d\sigma(x). \tag{1.13}$$

Proof. Eq. (1.12) follows by applying Theorem 1.9 with Z replaced by fZ making use of the fact that

$$\nabla \cdot (fZ) = \nabla f \cdot Z + f \nabla \cdot Z.$$

Eq. (1.13) follows from Eq. (1.12) by taking $Z = \nabla g$. ■

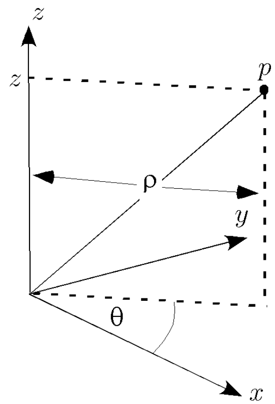


Fig. 1.1. Cylindrical and polar coordinates.

1.2.1 Cylindrical coordinates

Recall that cylindrical coordinates, see Figures 1.1, are determined by

$$(x, y, z) = \mathbf{R}(\rho, \theta, z) \equiv (\rho\cos\theta, \rho\sin\theta, z).$$

In these coordinates we have $dV = r^2 \sin\varphi dr d\theta d\varphi$.

$$dV = \rho d\rho d\theta dz.$$

Proposition 1.11 (Laplacian in Cylindrical Coordinates). *The Laplacian in cylindrical coordinates is given by*

$$\Delta f = \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\theta^2 f + \partial_z^2 f. \tag{1.14}$$

Proof. We further observe that

$$\begin{aligned}\mathbf{R}_\rho(\rho, \theta, z) &= (\cos\theta, \sin\theta, 0)\\ \mathbf{R}_\theta(\rho, \theta, z) &= (-\rho\sin\theta, \rho\cos\theta, 0)\\ \mathbf{R}_z(\rho, \theta, z) &= (0, 0, 1)\end{aligned}$$

so that

$$\left\{\mathbf{R}_\rho(\rho, \theta, z), \rho^{-1}\mathbf{R}_\theta(\rho, \theta, z), \mathbf{R}_z(\rho, \theta, z)\right\}$$

is an orthonormal basis for $\mathbb{R}^3$. Therefore,

$$\begin{aligned}(\nabla f, \nabla g) &= (\nabla f, \mathbf{R}_\rho)(\nabla g, \mathbf{R}_\rho) + \left(\nabla f, \rho^{-1}\mathbf{R}_\theta\right)\left(\nabla g, \rho^{-1}\mathbf{R}_\theta\right) + (\nabla f, \mathbf{R}_z)(\nabla g, \mathbf{R}_z)\\ &= \frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}.\end{aligned}$$

If g has compact support in a region Ω, then by integration by parts,

$$\begin{aligned}\int_\Omega \Delta f g dV &= -\int_\Omega \nabla f\cdot\nabla g dV\\ &= -\int_\Omega\left[\frac{\partial f}{\partial\rho}\frac{\partial g}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right]\cdot\rho d\rho d\theta dz\\ &= \int_\Omega\left[\partial_\rho\left(\rho\partial_\rho f\right) + \frac{1}{\rho}\partial_\theta^2 f + \rho\partial_z^2 f\right] g\cdot d\rho d\theta dz\\ &= \int_\Omega\left[\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\theta^2 f + \partial_z^2 f\right] g\rho d\rho d\theta dz\\ &= \int_\Omega\left[\frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\theta^2 f + \partial_z^2 f\right] g dV.\end{aligned}$$

Since this formula holds for arbitrary g with small support, we conclude that

$$\Delta f = \frac{1}{\rho}\partial_\rho\left(\rho\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\theta^2 f + \partial_z^2 f.$$

■

1.2.2 Spherical coordinates

We will now work out the Laplacian in spherical coordinates by a similar method. Recall that spherical coordinates, see Figures 1.2, are determined by

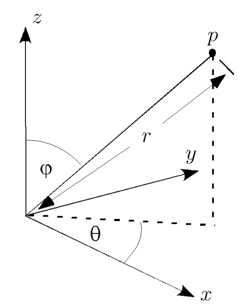


Fig. 1.2. Defining spherical coordinates of a point in $\mathbb{R}^3$.

$$(x, y, z) = \mathbf{R}(r, \theta, \varphi) \equiv (r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi).$$

In these coordinates systems we have

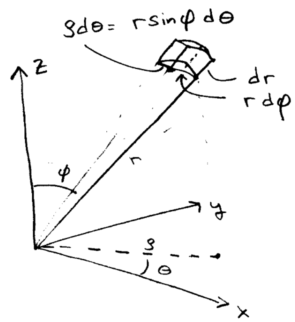


Fig. 1.3. A picture proof that $dxdydz = r^2 \sin\phi dr d\theta d\phi$, where $r^2 \sin\phi dr d\theta d\phi$ should be viewed as $(r \sin\phi d\theta)(r d\phi)dr$.

$$dV = r^2 \sin\varphi dr d\theta d\varphi.$$

See Figure 1.3.

Proposition 1.12 (Laplacian in spherical coordinates). *The Laplacian in spherical coordinates is given by*

$$\Delta f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\varphi}\partial_\varphi(\sin\varphi\partial_\varphi f) + \frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f. \tag{1.15}$$

Proof. Since

$$\begin{aligned}
\mathbf{R}_r(r,\theta,\varphi) &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\\
\mathbf{R}_\theta(r,\theta,\varphi) &= (-r\sin\varphi\sin\theta, r\sin\varphi\cos\theta, 0)\\
\mathbf{R}_\varphi(r,\theta,\varphi) &= (r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, -r\sin\varphi)
\end{aligned}$$

it is easily verified that

$$\left\{\mathbf{R}_r(r,\theta,\varphi), \frac{1}{r\sin\varphi}\mathbf{R}_\theta(\rho,\theta,z), \frac{1}{r}\mathbf{R}_\varphi(\rho,\theta,z)\right\}$$

is an orthonormal basis for $\mathbb{R}^3$. Therefore

$$\begin{aligned}
(\nabla f, \nabla g) =& (\nabla f, \mathbf{R}_r)(\nabla g, \mathbf{R}_r) + \frac{1}{r^2\sin^2\varphi}(\nabla f, \mathbf{R}_\theta)(\nabla g, \mathbf{R}_\theta)\\
&+ \frac{1}{r^2}(\nabla f, \mathbf{R}_\varphi)(\nabla g, \mathbf{R}_\varphi)\\
=& \frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + \frac{1}{r^2\sin^2\varphi}\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta} + \frac{1}{r^2}\frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\varphi}.
\end{aligned}$$

If g has compact support in a region Ω, then

$$\begin{aligned}
\int_\Omega \Delta f g dV &= -\int_\Omega \nabla f\cdot\nabla g dV\\
&= -\int_\Omega \left[\frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + \frac{1}{r^2\sin^2\varphi}\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta} + \frac{1}{r^2}\frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\varphi}\right]\cdot r^2\sin\varphi dr d\varphi d\theta\\
&= -\int_\Omega \left[r^2\sin\varphi\frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + \frac{1}{\sin\varphi}\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta} + \sin\varphi\frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\varphi}\right]\cdot dr d\varphi d\theta\\
&= \int_\Omega \left[\partial_r\left(r^2\partial_r g\right)\sin\varphi + \frac{1}{\sin\varphi}\partial_\theta^2 f + \partial_\varphi\left(\sin\varphi\cdot\partial_\varphi f\right)\right] g\cdot dr d\varphi d\theta\\
&= \int_\Omega \left[\frac{1}{r^2}\partial_r\left(r^2\partial_r g\right) + \frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f + \frac{1}{r^2\sin\varphi}\partial_\varphi\left(\sin\varphi\partial_\varphi f\right)\right] g\cdot r^2\sin\varphi dr d\varphi d\theta\\
&= \int_\Omega \left[\frac{1}{r^2}\partial_r\left(r^2\partial_r g\right) + \frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f + \frac{1}{r^2\sin\varphi}\partial_\varphi\left(\sin\varphi\partial_\varphi f\right)\right] g\cdot dV.
\end{aligned}$$

Since this formula holds for arbitrary g we conclude that

$$\Delta f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\varphi}\partial_\varphi(\sin\varphi\partial_\varphi f) + \frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f.$$

∎

1.2.3 Exercises

In the following two exercises, I am using the conventions in the Lecture notes and not the book.

Exercise 1.4. Compute Δf where f is given in cylindrical coordinates as:

$$f = \rho^3\cos\theta + z\rho$$

Exercise 1.5. Compute Δf where f is given in spherical coordinates as:

$$f = r^{-1} + \cos\theta\sin\varphi.$$

2

PDE Examples

2.1 The Wave Equation

Example 2.1 (Wave Equation for a String). Suppose that we have a stretched string supported at $x = 0$ and $x = L$ and $y = 0$. Suppose that the string only undergoes vertical motion (pretty bad assumption). Let $u(t, x)$ and $T(t, x)$ denote the height and tension respectively of the string at (t, x), $\delta(x)$ denote the density in equilibrium and T_0 be the equilibrium string tension. Let $J =$

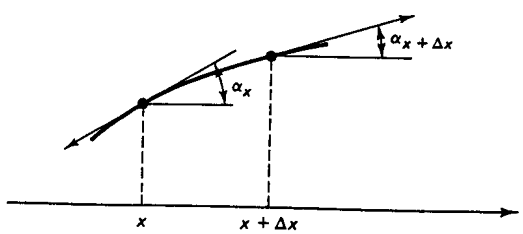


Fig. 2.1. A piece of displace string

$[x, x + \Delta x] \subset [0, L]$, then

$$P_J(t) := \int_J u_t(t, x)\delta(x)dx$$

is the momentum of the piece of string above J. (Notice that $\delta(x)dx$ is the weight of the string above x.) Newton's equations state

$$\frac{dP_J(t)}{dt} = \int_J u_{tt}(t, x)\delta(x)dx = \text{Force on String}.$$

Since the string is to only undergo vertical motion we require

$$T(t, x + \Delta x)\cos(\alpha_{x+\Delta x}) - T(t, x)\cos(\alpha_x) = 0$$

for all Δx and therefore that $T(t, x)\cos(\alpha_x) =: H$ for some constant H, i.e. the horizontal component of the tension is constant. Looking at Figure 2.2, the

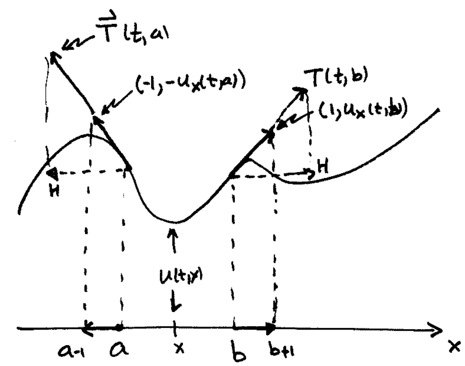


Fig. 2.2. Computing the net vertical force due to tension on the part of the string above $[a, b]$.

tension on the piece of string above $J = [a, b]$ at the right endpoint b must be given by $H(1, u_x(t, b))$ while the tension at the left endpoint, a, must be given by $-H(-1, -u_x(t, a))$. So the net tension force on the string above J is

$$H[u_x(t, b) - u_x(t, a)] = H\int_a^b u_{xx}(t, x)\, dx.$$

Finally there may be a component due to gravity and air resistance, say

$$\text{gravity} = -g\int_a^b \delta(x)dx \text{ and}$$

$$\text{air resistance} = -\int_a^b k(x)u_t(t, x)dx.$$

So Newton's equations become

$$\int_a^b u_{tt}(t, x)\delta(x)dx = \int_a^b [Hu_{xx}(t, x) - g\delta(x) - k(x)u_t(t, x)]\, dx.$$

Differentiating this equation in b at $b = x$ then shows

$$u_{tt}(t,x)\delta(x) = u_{xx}(t,x) - g\delta(x) - k(x)u_t(t,x)$$

or equivalently that

$$u_{tt}(t,x) = \frac{1}{\delta(x)}u_{xx}(t,x) - g - \frac{k(x)}{\delta(x)}u_t(t,x). \tag{2.1}$$

Example 2.2 (Wave equation. for a drum head). Suppose that $u(t,x)$ represents the height at time t of a drum head over a point $x \in \Omega$ — Ω being the base of the drum head, see Figure 2.3. As for the string we will make the simplifying

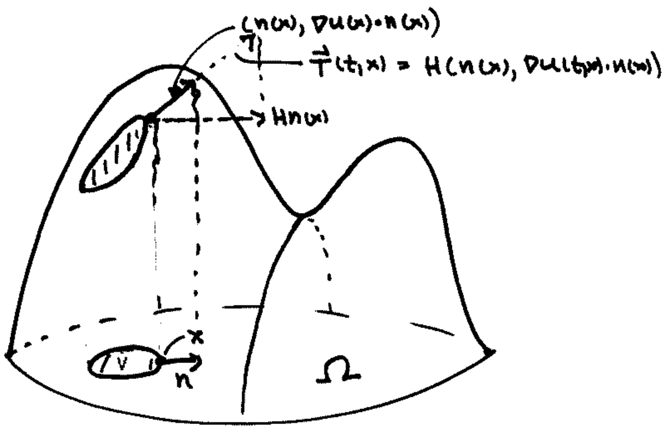


Fig. 2.3. A deformed membrane attached to a "wire" base. We are also compute the tension density on a region of the membrane above a region V in the plane.

assumption that the membrane only moves vertically or equivalently that the horizontal component of tension/unit-length is a constant value, H.

Let $V \subset \Omega$ be a test region and consider the membrane which lie above V as in Figure 2.3. Then

$$P_V(t) := \int_V u_t(t,x)\delta(x)dx$$

is the momentum of the piece of string above V where $\delta(x)dx$ is the weight of the membrane above x. Newton's equations state

$$\frac{dP_V(t)}{dt} = \int_V u_{tt}(t,x)\delta(x)dx = \text{Force on membrane.}$$

To find the vertical force on the membrane above V, let $x \in \partial V$, then

$$(n(x), \nabla u(t,x)\cdot n(x)) = \frac{d}{ds}|_0 (x + sn(x), u(t, x + sn(x))$$

is a vector orthogonal to the boundary of the region above V and by assumption the tension/unit-length at x is $H(n(x), \nabla u(t,x)\cdot n(x))$. Thus the vertical component of the force on the membrane above V is given by

$$H\int_{\partial V} \nabla u(t,x)\cdot n(x)\, d\ell(x) = H\int_V \nabla\cdot\nabla u(t,x)\,dx = H\int_V \Delta u(t,x)\,dx.$$

Finally there may be a component due to gravity and air resistance, say

$$\text{gravity} = -\int_V g\delta(x)dx \text{ and}$$
$$\text{air resistance} = -\int_V k(x)u_t(t,x)dx.$$

So Newton's equations become

$$\int_V u_{tt}(t,x)\delta(x)dx = \int_V \left[H\Delta u(t,x) - g\delta(x) - k(x)u_t(t,x)\right]dx.$$

Since V is arbitrary, this implies

$$\delta(x)u_{tt}(t,x) = H\Delta u(t,x) - g\delta(x) - k(x)u_t(t,x)$$

or equivalently that

$$u_{tt}(t,x) = \frac{H}{\delta(x)}\Delta u(t,x) - g - \frac{k(x)}{\delta(x)}u_t(t,x). \tag{2.2}$$

Example 2.3 (Wave equation for a metal bar). Suppose that have a metal wire which we is going to be deformed and then released. We would like to find the equation that the displacement $u(t,x)$ of the section of the bar originally at location x must solve, see Figure 2.4 below.

To do this will write down Newton's equation of motion. First off, the longitudinal force **on** the left face of the section which was originally between x and $x + \Delta x$ is approximately,

$$-AE\frac{u(t, x+\Delta x) - u(t,x)}{\Delta x},$$

where E is Young's modulus of elasticity and A is the area of the bar. (The minus represents the fact that we must pull to the left to get the current configuration in the figure.) Letting $\Delta x \to 0$, we find the force of the section that was originally at x is given by $-AEu_x(t,x)$. Now suppose that Δx is not necessarily small. Then we have the momentum of the region of the bar originally between x and $x + \Delta x$ is given by

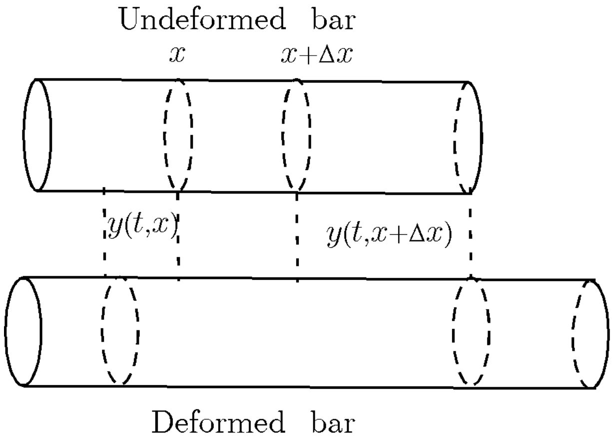


Fig. 2.4. The picture represents an elastic bar in its un-deformed state and then in a deformed state. The quantity $y(t,x)$ represents the displacement of the section that was originally at location x in the un-deformed bar. In the above figure $y(t,x) < 0$ which $y(t, x+\Delta x) > 0$.

$$\int_x^{x+\Delta x} u_t(t,x)\,\delta(x)\,dx$$

where $\delta(x)$ is the linear mass density. Therefore,

$$\begin{aligned}\text{Mass} \times \text{acceleration} &= \frac{d}{dt}\int_x^{x+\Delta x} u_t(t,x)\,\delta(x)\,dx = \int_x^{x+\Delta x} u_{tt}(t,x)\,\delta(x)\,dx \\ &= \text{the net force on this section of the bar} \\ &= -AEu_x(t,x) + AEu_x(t,x+\Delta x)\end{aligned}$$

where $-AEu_x(t,x)$ is the force on left end and $AEu_x(t,x+\Delta x)$ is the force on the right end. Hence we have

$$\int_x^{x+\Delta x} u_{tt}(t,x)\,\delta(x)\,dx = AEu_x(t,x+\Delta x) - AEu_x(t,x)$$

which upon differentiating in Δx at $\Delta x = 0$ shows

$$u_{tt}(t,x)\,\delta(x) = AEu_{xx}(t,x).$$

2.1.1 d'Alembert's solution to the 1-dimensional wave equation

Here we are going to try to find solutions to the wave equation, $y_{tt} = a^2 y_{xx}$. Since this equation may be written as

$$\left(\partial_t^2 - a^2\partial_x^2\right) y = 0 \tag{2.3}$$

and

$$\left(\partial_t^2 - a^2\partial_x^2\right) = \left(\partial_t - a\partial_x\right)\left(\partial_t + a\partial_x\right)$$

we are lead to consider the wave equation in the new variables,

$$u = x + at \text{ and } v = x - at.$$

In these variables we have

$$\begin{aligned}\partial_t &= \frac{\partial u}{\partial t}\partial_u + \frac{\partial v}{\partial t}\partial_v = a\partial_u - a\partial_v \text{ and} \\ \partial_x &= \frac{\partial u}{\partial x}\partial_u + \frac{\partial v}{\partial x}\partial_v = \partial_u + \partial_v\end{aligned}$$

from which it follows that

$$\partial_t - a\partial_x = -2a\partial_v \text{ and } \partial_t + a\partial_x = 2a\partial_u$$

and hence the wave equation in (u,v) – coordinates becomes,

$$0 = \left(\partial_t - a\partial_x\right)\left(\partial_t + a\partial_x\right) y = -2a\partial_v 2a\partial_u y = -4a^2 y_{uv},$$

i.e. $y_{uv} = 0$. Integrating this equation in v shows $y_u = F(u)$ and then integrating in u shows

$$y = \int F(u)\,du + \psi(v) = \varphi(u) + \psi(v).$$

Thus we have shown if y solves the wave equation then

$$y(t,x) = \varphi(x+at) + \psi(x-at) \tag{2.4}$$

for some functions φ and ψ.

Exercise 2.1. Show that if $y(t,x)$ has the form given in Eq. (2.4) with φ and ψ being twice continuously differentiable functions, then y solves the wave Eq. (2.3).

To get a unique solution to Eq. (2.3) we must introduce some initial conditions. For example, let us further assume that

$$y(0,x) = f(x) \text{ and } y_t(0,x) = 0.$$

This then implies that

$$\begin{aligned}f(x) &= \varphi(x) + \psi(x) \text{ and} \\ 0 &= a\varphi'(x) - a\psi'(x),\end{aligned}$$

The latter equation shows that $\psi(x) = \varphi(x) + C$ and using this in the first equation implies that

$$f(x) = 2\varphi(x) + C$$

or

$$\varphi(x) = \frac{1}{2}(f(x) - C) \text{ and } \psi(x) = \frac{1}{2}(f(x) + C).$$

Thus we have found the solution to be given by

$$y(t,x) = \frac{1}{2}\{f(x+at) + f(x-at)\}.$$

In the homework you are asked to generalize this result to prove the following theorem.

Theorem 2.4 (d'Alembert's solution). *If $f(x)$ is twice continuously differentiable and $g(x)$ is continuously differentiable for $x \in \mathbb{R}$, then the unique solution to*

$$y_{tt} = a^2 y_{xx} \text{ with} \tag{2.5}$$

$$y(0,x) = f(x) \text{ and } y_t(0,x) = g(x) \tag{2.6}$$

is given by

$$y(t,x) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds. \tag{2.7}$$

Example 2.5. Here we wish to solve for $x \geq 0$ and $t \geq 0$,

$$\partial_t^2 y = \partial_x^2 y \text{ with } y(0,x) = f(x) \text{ and } \dot{y}(0,x) = 0 \text{ with } y(t,0) = 0.$$

As before we know that $y(t,x) = \varphi(x+t) + \psi(x-t)$. We must now implement all of the boundary conditions,

$$\begin{aligned} f(x) = y(0,x) &= \varphi(x) + \psi(x) \\ 0 = \dot{y}(0,x) &= \varphi'(x) - \psi'(x) \text{ and} \\ 0 = y(t,0) &= \varphi(t) + \psi(-t). \end{aligned}$$

This suggests that we define $\psi(-t) := -\varphi(t)$ for $t > 0$, and also that

$$\begin{aligned} \varphi(x) &= \psi(x) + C \\ f(x) &= 2\psi(x) + C \end{aligned}$$

or

$$\begin{aligned} \psi(x) &= \frac{1}{2}(f(x) - C) \\ \varphi(x) &= \frac{1}{2}(f(x) + C). \end{aligned}$$

Thus our answer is given by

$$y(t,x) = \frac{1}{2}[f(x+t) + f(x-t)]$$

where by above,

$$\frac{1}{2}(f(-x) - C) = \psi(-x) = -\varphi(x) = -\frac{1}{2}(f(x) + C)$$

and thus

$$f(-x) := -f(x).$$

Thus we have

$$y(t,x) = \frac{1}{2}[f(x+t) + f(x-t)]$$

where f is extend to all of $\mathbb{R}$ to be an odd function.

2.2 Heat Equations

Example 2.6 (Heat or Diffusion Equation in 1-dimension). Let us consider the temperature in a rod Ω. We will let

1. $\delta(x)$ denote the linear density of the rod
2. $c(x)$ denote the heat capacity of the rod per unit mass at x
3. $\kappa(x)$ be the thermal conductivity of the rod at x. By Newton's Law of cooling, the heat flow from left to right in the rod at location x should be approximately equal to

$$\frac{\kappa(x)}{\Delta}(u(x) - u(x+\Delta)). \tag{2.8}$$

Notice the Δ appearing in the denominator represents the fact that the thicker the insulation in your house the less heat transfer that you have. Passing to the limit in Eq. (2.8) then gives Fourier's law, namely the heat flow from left to right in the rod at location x is given by

$$-\kappa(x)\, u'(x). \tag{2.9}$$

(In the book, it is **typically assumed** that $\delta(x) = \delta$, $\kappa(x) = K$ and $c(x) = \sigma$ are all constant.)

4. $u(t,x)$ be the temperature of the rod at time t and location x.
5. $H(t,x)$ represent heat source at x and time t. For example we may be passing a current through the wire and the resistance of the wire is both spatially and time dependent. Alternatively we may be heating the wire with an external source.

Let $B=[a,b]$ be a sub-region of the rod, see Figure 2.5. Then

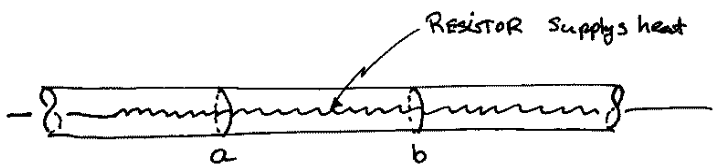


Fig. 2.5. Part of a rod with a test region $B=[a,b]$ being examined.

$$E(t)=\int_a^b u(t,x)\,\delta(x)\,c(x)\,dx$$

represents the heat energy in B at time t. Hence

$$\dot{E}(t)=\int_a^b u_t(t,x)\,\delta(x)\,c(x)\,dx$$

is the rate of change of heat energy in B. This may alternatively be computed as the rate at which heat enters the system which is given by

$$\begin{aligned}\dot{E}(t)&=\int_a^b H(t,x)\,dx+(\kappa(b)\,u_x(t,b)-\kappa(a)\,u_x(t,a))\\&=\int_a^b\left[H(t,x)+\frac{d}{dx}(\kappa(x)\,u_x(t,x))\right]dx.\end{aligned}$$

Hence we conclude that

$$\int_a^b u_t(t,x)\,\delta(x)\,c(x)\,dx=\int_a^b\left[H(t,x)+\frac{d}{dx}(\kappa(x)\,u_x(t,x))\right]dx$$

for all sub-intervals Ω in the rod and therefore (again just differentiate in b) that

$$\delta(x)\,c(x)\,u_t(t,x)=\frac{d}{dx}(\kappa(x)\,u_x(t,x))+H(t,x). \tag{2.10}$$

This equation may be written as

$$u_t(t,x)=Lu(t,x)+h(t,x)$$

where

$$Lf(x):=\frac{1}{p(x)}\frac{d}{dx}\left(\kappa(x)\frac{d}{dx}f(x)\right),$$

$$p(x)=\delta(x)\,c(x)\ \text{ and }\ h(t,x):=\frac{H(t,x)}{p(x)}.$$

If we further assume that the rod in **not** perfectly insulated along its length and the ambient temperature is not constant, we may end up with another terms in computing $\dot{E}(t)$ of the form

$$\int_a^b Q(x)\,[u(t,x)-T(x)]\,dx$$

and we would then arrive at a heat equation of the form

$$u_t(t,x)=Lu(t,x)+h(t,x)$$

where

$$Lf(x):=\frac{1}{p(x)}\frac{d}{dx}\left(\kappa(x)\frac{d}{dx}f(x)\right)+\frac{1}{p(x)}q(x)\,f(x) \tag{2.11}$$

for some function $q(x)$ and a modified function $h(t,x)$.

Example 2.7 (Heat or Diffusion Equation in d - dimensions). Suppose that $\Omega\subset\mathbb{R}^d$ is a region of space filled with a material, $\delta(x)$ is the density of the material at $x\in\Omega$ and $c(x)$ is the heat capacity. Let $u(t,x)$ denote the temperature at time $t\in[0,\infty)$ at the spatial point $x\in\Omega$. Now suppose that $B\subset\mathbb{R}^d$ is a "little" volume in $\mathbb{R}^d$, ∂B is the boundary of B, and $E_B(t)$ is the heat energy contained in the volume B at time t. Then

$$E_B(t)=\int_B\delta(x)c(x)u(t,x)dx.$$

So on one hand,

$$\dot{E}_B(t)=\int_B\delta(x)c(x)\dot{u}(t,x)dx \tag{2.12}$$

while on the other hand,

$$\dot{E}_B(t)=\int_{\partial B}(\kappa(x)\nabla u(t,x)\cdot n(x))\,d\sigma(x), \tag{2.13}$$

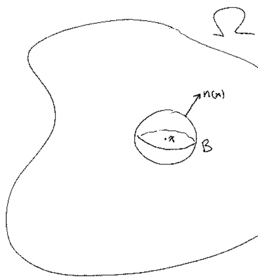


Fig. 2.6. A test volume B in Ω centered at x with outward pointing normal, n.

where $\kappa(x)$ is a $d\times d$–positive definite matrix representing the conduction properties of the material, $n(x)$ is the outward pointing normal to B at $x \in \partial B$, and $d\sigma$ denotes surface measure on ∂B.

In order to see that we have the sign correct in (2.13), suppose that $x \in \partial B$ and $\nabla u(x)\cdot n(x) > 0$, then the temperature for points near x outside of B are hotter than those points near x inside of B and hence contribute to a increase in the heat energy inside of B. (If we get the wrong sign, then the resulting equation will have the property that heat flows from cold to hot!)

Comparing Eqs. (2.12) to (2.13) after an application of the divergence theorem shows that

$$\int_B \delta(x)c(x)\dot{u}(t,x)dx = \int_B \nabla\cdot(\kappa(\cdot)\nabla u(t,\cdot))(x)\,dx. \tag{2.14}$$

Since this holds for all volumes $B \subset \Omega$, we conclude that the temperature functions should satisfy the following partial differential equation.

$$\delta(x)c(x)\dot{u}(t,x) = \nabla\cdot(\kappa(\cdot)\nabla u(t,\cdot))(x)\,. \tag{2.15}$$

or equivalently that

$$\dot{u}(t,x) = \frac{1}{\delta(x)c(x)}\nabla\cdot(\kappa(x)\nabla u(t,x)). \tag{2.16}$$

Setting $g^{ij}(x) := \kappa_{ij}(x)/(\delta(x)c(x))$ and

$$z^j(x) := \sum_{i=1}^d \partial(\kappa_{ij}(x)/(\delta(x)c(x)))/\partial x^i$$

the above equation may be written as:

$$\dot{u}(t,x) = Lu(t,x), \tag{2.17}$$

where

$$(Lf)(x) = \sum_{i,j} g^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}f(x) + \sum_j z^j(x)\frac{\partial}{\partial x^j}f(x). \tag{2.18}$$

The operator L is a prototypical example of a second order "elliptic" differential operator.

Example 2.8 (Laplace and Poisson Equations). Laplace's Equation is of the form $Lu = 0$ and solutions may represent the steady state temperature distribution for the heat equation. Equations like $\Delta u = -\rho$ appear in electrostatics for example, where u is the electric potential and ρ is the charge distribution.

2.3 Other Equations

Example 2.9 (Shrodinger Equation and Quantum Mechanics).

$$i\frac{\partial}{\partial t}\psi(t,x) = -\frac{\Delta}{2}\psi(t,x) + V(x)\psi(t,x) \text{ with } \|\psi(\cdot,0)\|_2 = 1.$$

Interpretation,

$$\int_A |\psi(t,x)|^2\,dt = \text{ the probability of finding the particle in } A \text{ at time } t.$$

(Notice similarities to the heat equation.)

Example 2.10 (Maxwell Equations in Free Space).

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla\times\mathbf{B}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$
$$\nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{B} = 0.$$

Notice that

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times(\nabla\times\mathbf{E}) = \Delta\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) = \Delta\mathbf{E}$$

and similarly, $\frac{\partial^2\mathbf{B}}{\partial t^2} = \Delta\mathbf{B}$ so that all the components of the electromagnetic fields satisfy the wave equation.

Example 2.11 (Traffic Equation). Consider cars travelling on a straight road with coordinate $x \in \mathbb{R}$, let $u(t,x)$ denote the density of cars on the road at time t and location $x \in \mathbb{R}$, and $v(t,x)$ be the velocity of the cars at (t,x). Then for

$J = [a, b] \subset \mathbb{R}$, $N_J(t) := \int_a^b u(t, x)dx$ is the number of cars in the set J at time t. We must have

$$\int_a^b \dot{u}(t, x)dx = \dot{N}_J(t) = u(t, a)v(t, a) - u(t, b)v(t, b)$$
$$= -\int_a^b \frac{\partial}{\partial x}\left[u(t, x)v(t, x)\right] dx.$$

Since this holds for all intervals $[a, b]$, we must have

$$\dot{u}(t, x) = -\frac{\partial}{\partial x}\left[u(t, x)v(t, x)\right].$$

To make life more interesting, we may imagine that $v(t, x) = -F(u(t, x), u_x(t, x))$, in which case we get an equation of the form

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}G(u, u_x) \text{ where } G(u, u_x) = -u(t, x)F(u(t, x), u_x(t, x)).$$

A simple model might be that there is a constant maximum speed, v_m and maximum density u_m, and the traffic interpolates linearly between 0 (when $u = u_m$) to v_m when $(u = 0)$, i.e. $v = v_m(1 - u/u_m)$ in which case we get

$$\frac{\partial}{\partial t}u = -v_m\frac{\partial}{\partial x}\left(u(1 - u/u_m)\right).$$

Example 2.12 (Burger's Equation). Suppose we have a stream of particles travelling on $\mathbb{R}$, each of which has its own constant velocity and let $u(t, x)$ denote the velocity of the particle at x at time t. Let $x(t)$ denote the trajectory of the particle which is at x_0 at time t_0. We have $C = \dot{x}(t) = u(t, x(t))$. Differentiating this equation in t at $t = t_0$ implies

$$0 = \left[u_t(t, x(t)) + u_x(t, x(t))\dot{x}(t)\right]|_{t=t_0} = u_t(t_0, x_0) + u_x(t_0, x_0)u(t_0, x_0)$$

which leads to Burger's equation

$$0 = u_t + u\ u_x.$$

Example 2.13 (Minimal surface Equation). Let $D \subset \mathbb{R}^2$ be a bounded region with reasonable boundary, $u_0 : \partial D \to \mathbb{R}$ be a given function. We wish to find the function $u : D \to \mathbb{R}$ such that $u = u_0$ on ∂D and the graph of u, $\Gamma(u)$ has least area. Recall that the area of $\Gamma(u)$ is given by

$$A(u) = Area(\Gamma(u)) = \int_D \sqrt{1 + |\nabla u|^2}dx.$$

Assuming u is a minimizer, let $v \in C^1(D)$ such that $v = 0$ on ∂D, then

$$0 = \frac{d}{ds}|_0 A(u + sv) = \frac{d}{ds}|_0 \int_D \sqrt{1 + |\nabla(u + sv)|^2}dx$$
$$= \int_D \frac{d}{ds}|_0 \sqrt{1 + |\nabla(u + sv)|^2}dx$$
$$= \int_D \frac{1}{\sqrt{1 + |\nabla u|^2}}\nabla u \cdot \nabla v\ dx$$
$$= -\int_D \nabla \cdot \left(\frac{1}{\sqrt{1 + |\nabla u|^2}}\nabla u\right) v\ dx$$

from which it follows that

$$\nabla \cdot \left(\frac{1}{\sqrt{1 + |\nabla u|^2}}\nabla u\right) = 0.$$

Example 2.14 (Navier – Stokes). Here $u(t, x)$ denotes the velocity of a fluid ad (t, x), $p(t, x)$ is the pressure. The Navier – Stokes equations state,

$$\frac{\partial u}{\partial t} + \partial_u u = \nu \Delta u - \nabla p + f \text{ with } u(0, x) = u_0(x) \tag{2.19}$$
$$\nabla \cdot u = 0 \text{ (incompressibility)} \tag{2.20}$$

where f are the components of a given external force and u_0 is a given divergence free vector field, ν is the viscosity constant. The Euler equations are found by taking $\nu = 0$. Equation (2.19) is Newton's law of motion again, $F = ma$. See **http://www.claymath.org** for more information on this Million Dollar problem.

Linear ODE

3.1 First order linear ODE

We would like to solve the ordinary differential equation

$$\dot{u}(t) = Au(t) \text{ with} \tag{3.1}$$
$$u(0) = f. \tag{3.2}$$

The method of **separation of variables** or **eigenvector expansions** proposed to begin by looking for solutions of the form $u(t) = T(t)v$ to Eq. (3.1). Here v is a fixed vector in $\in \mathbb{R}^N$ and $T(t)$ is some unknown function of t. Substituting $u(t) = T(t)v$ into Eq. (3.1) gives

$$\dot{T}(t)v = T(t)Av$$

or equivalently that

$$Av = \frac{\dot{T}(t)}{T(t)}v.$$

Since the left side of this equation is independent of t we must have

$$\frac{\dot{T}(t)}{T(t)} = \lambda \tag{3.3}$$

for some $\lambda \in \mathbb{R}$. The solution to Eq. (3.3) is of course $T(t) = e^{t\lambda}T(0)$ and therefore we have shown the following lemma.

Lemma 3.1. *If $u(t) = T(t)v$ solves Eq. (3.1), then v is an* ***eigenvector*** *of A and if λ is the corresponding eigenvalue (i.e. $Av = \lambda v$) then*

$$u(t) = e^{\lambda t}T(0)v.$$

Conversely if $Av = \lambda v$ then $u(t) = e^{\lambda t}v$ solves Eq. (3.1).

Proposition 3.2 (Principle of superposition). *If $u(t)$ and $v(t)$ solves Eq. (3.1) then so does $u(t) + cv(t)$ for any $c \in \mathbb{R}$.*

Proof. This is a simple consequence of the fact that matrix multiplication and differentiation are linear operations. In detail,

$$\begin{aligned}\frac{d}{dt}(u(t) + cv(t)) &= \dot{u}(t) + c\dot{v}(t) = Au(t) + cAv(t) \\ &= A(u(t) + cv(t)).\end{aligned}$$

■

Consequently if $Av_i = \lambda_i v_i$ for $i = 1, 2, \ldots, k$, then

$$u(t) = \sum_i e^{t\lambda_i}v_i.$$

solves Eq. (3.1).

Theorem 3.3. *Suppose the matrix A is diagonalizable, i.e. there exists a basis $\{v_i\}_{i=1}^N$ for $\mathbb{R}^N$ consisting of eigenvectors A. Then to any $f \in \mathbb{R}^N$ there is a unique solution, $u(t)$, to Eqs. (3.1) and (3.2). Moreover, if we expand f in terms of the basis $\{v_i\}_{i=1}^N$ as*

$$f = \sum_{i=1}^N a_i v_i,$$

then the unique solution to Eqs. (3.1) and (3.2) is given by

$$u(t) = \sum_{i=1}^N a_i e^{t\lambda_i}v_i. \tag{3.4}$$

Proof. The fact that Eq. (3.4) solves Eqs. (3.1) and (3.2) follows from the principle of superposition and the fact that $e^{t\lambda_i} = 1$ when $t = 0$.

Conversely, suppose that u solves Eqs. (3.1) and (3.2), then

$$u(t) = \sum_{i=1}^N a_i(t)v_i \tag{3.5}$$

for some functions $a_i(t)$ with $a_i(0) = a_i$. Now on one hand

$$\dot{u}(t) = \sum_{i=1}^N \dot{a}_i(t)v_i$$

while on the other hand

$$\dot{u}(t) = Au(t) = A\sum_{i=1}^{N} a_i(t) v_i = \sum_{i=1}^{N} a_i(t) A v_i = \sum_{i=1}^{N} a_i(t) \lambda_i v_i.$$

Subtracting these two equation shows

$$0 = \sum_{i=1}^{N} \dot{a}_i(t) v_i - \sum_{i=1}^{N} a_i(t) \lambda_i v_i = \sum_{i=1}^{N} (\dot{a}_i(t) - a_i(t)\lambda_i) v_i.$$

Since $\{v_i\}_{i=1}^N$ is a basis for $\mathbb{R}^N$ it follows, for all i, that

$$\dot{a}_i(t) = a_i(t)\lambda_i \text{ with } a_i(0) = 1$$

and therefore, $a_i(t) = e^{t\lambda_i} a_i$. Putting this result back into Eq. (3.5) gives Eq. (3.4). ∎

Definition 3.4. *If $f \in \mathbb{R}^N$ we will write $e^{tA}f$ for the solution, $u(t)$, to Eqs. (3.1) and (3.2).*

Fact 3.5 *Eqs. (3.1) and (3.2) have a unique solution independent as to whether A has a basis of eigenvectors or not. Moreover we may compute e^{tA} using the **matrix power series** expansion,*

$$e^{tA} f = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f. \tag{3.6}$$

Notice that formula in Eq. (3.6) is consistent with our previous results. For example if $v \in \mathbb{R}^N$ and $Av = \lambda v$, then $A^n v = \lambda^n v$ and therefore,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n v = \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^n v = e^{t\lambda} v = e^{tA} v.$$

More generally, if $u = \sum_{i=1}^k a_i v_i$ with $Av_i = \lambda_i v_i$, then

$$e^{tA} u = \sum_{i=1}^{k} a_i e^{t\lambda_i} v_i = \sum_{i=1}^{k} a_i e^{tA_i} v_i.$$

Remark 3.6. As the notation suggests, it is true that

$$e^{tA} e^{sA} = e^{(t+s)A}$$

as you are asked to prove in Exercise 3.1 below. **However**, it is **not** generally true that

$$e^{(A+B)} = e^A e^B = e^B e^A,$$

see Proposition 3.8 below.

Example 3.7. Let us find e^{tA} when

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The eigenvalues of A are given as the roots of the **characteristic polynomial**,

$$p(\lambda) = \det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda).$$

These roots are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. As usual we find the corresponding eigenvectors as solutions to the equation $(A - \lambda I) u = 0$. The result is, eigenvectors:

$$v_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow 1,\ v_2 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow 2,\ v_3 := \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \leftrightarrow 3.$$

Since

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

it follows that the columns of e^{tA} are given by

$$e^{tA}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$e^{tA}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e^{tA}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - e^{tA}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$e^{tA}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^{tA}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2e^{tA}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + e^{tA}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$= e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and therefore,

$$e^{tA}=\begin{bmatrix} e^t & -e^t+e^{2t} & e^t-2e^{2t}+e^{3t} \\ 0 & e^{2t} & -2e^{2t}+2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Proposition 3.8. *Let A and B be two $N\times N$ matrices. Then the following are equivalent:*

1. $0=[A,B]:=AB-BA.$
2. $e^{tA}B=Be^{tA}$ *for all* $t\in\mathbb{R}$,
3. $e^{tA}e^{sB}=e^{sB}e^{tA}$ *for all* $s,t\in\mathbb{R}$.

Moreover if $[A,B]=0$ *then* $e^{(A+B)}=e^Ae^B$ *and in particular*

$$e^{tA}e^{sA}=e^{(t+s)A} \text{ for all } s,t\in\mathbb{R}. \tag{3.7}$$

Proof. If $[A,B]=0$, then

$$\frac{d}{dt}e^{tA}Be^{-tA}=e^{tA}\left[A,B\right]e^{-tA}=0$$

and therefore, $e^{tA}B=Be^{tA}$ for all $t\in\mathbb{R}$. It now follows that

$$\frac{d}{ds}e^{tA}e^{sB}=e^{tA}Be^{sB}=Be^{tA}e^{sB}$$
$$\frac{d}{ds}e^{sB}e^{tA}=Be^{sB}e^{tA}$$

and so by uniqueness of solutions to these ODE we conclude $e^{tA}e^{sB}=e^{sB}e^{tA}$ for all $s,t\in\mathbb{R}$. If $e^{tA}e^{sB}=e^{sB}e^{tA}$ for all $s,t\in\mathbb{R}$ then

$$AB=\frac{d}{dt}|_0e^{tA}B=\frac{d}{dt}|_0\frac{d}{ds}|_0e^{tA}e^{sB}=\frac{d}{dt}|_0\frac{d}{ds}|_0e^{sB}e^{tA}=\frac{d}{dt}|_0Be^{tA}=BA.$$

For the last assertion, let $T\left(t\right):=e^{tA}e^{tB}$, then

$$\begin{aligned}\frac{d}{dt}T\left(t\right)&=Ae^{tA}e^{tB}+e^{tA}Be^{tB}=Ae^{tA}e^{tB}+Be^{tA}e^{tB}\\&=\left(A+B\right)T\left(t\right) \text{ with } T\left(0\right)=I.\end{aligned}$$

So again by uniqueness of solutions,

$$e^{tA}e^{tB}=T\left(t\right)=e^{t(A+B)}.$$

■

3.2 Solving for e^{tA} using the Spectral Theorem

Example 3.9. Let

$$A:=\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

as in Example 1.6 with eigenvectors/eigenvalues given by

$$v_1=\begin{bmatrix}1\\1\end{bmatrix}\longleftrightarrow\lambda_1=-1$$

and

$$v_2=\begin{bmatrix}1\\-1\end{bmatrix}\longleftrightarrow\lambda_2=2.$$

Recall that

$$f=\frac{1}{2}\left(v_1,f\right)v_1+\frac{1}{2}\left(v_2,f\right)v_2$$

and hence

$$\begin{aligned}e^{tA}f&=\frac{1}{2}\left(v_1,f\right)e^{tA}v_1+\frac{1}{2}\left(v_2,f\right)e^{tA}v_2\\&=\frac{1}{2}\left(v_1,f\right)e^{-t}v_1+\frac{1}{2}\left(v_2,f\right)e^{2t}v_2.\end{aligned} \tag{3.8}$$

Taking $f=e_1$ and $f=e_2$ then implies

$$\begin{aligned}e^{tA}e_1&=\frac{1}{2}e^{tA}v_1+\frac{1}{2}e^{tA}v_2=\frac{1}{2}e^{-t}v_1+\frac{1}{2}e^{2t}v_2\\&=\frac{1}{2}e^{-t}\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}\\&=\frac{1}{2}\begin{bmatrix}e^{-t}+e^{2t}\\e^{-t}-e^{2t}\end{bmatrix}.\end{aligned}$$

and

$$\begin{aligned}e^{tA}e_2&=\frac{1}{2}e^{tA}v_1-\frac{1}{2}e^{tA}v_2=\frac{1}{2}e^{-t}v_1-\frac{1}{2}e^{2t}v_2\\&=\frac{1}{2}e^{-t}\begin{bmatrix}1\\1\end{bmatrix}-\frac{1}{2}e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}\\&=\frac{1}{2}\begin{bmatrix}e^{-t}-e^{2t}\\e^{-t}+e^{2t}\end{bmatrix}.\end{aligned}$$

Thus we may conclude that

$$e^{tA}=\left[e^{tA}e_1\ e^{tA}e_2\right]=\frac{1}{2}\begin{bmatrix}e^{-t}+e^{2t} & e^{-t}-e^{2t}\\e^{-t}-e^{2t} & e^{-t}+e^{2t}\end{bmatrix}.$$

Alternatively, from Eq. (3.8)

$$e^{tA}f = \frac{1}{2}(v_1, f)\, e^{-t} v_1 v_1^{\mathrm{tr}} f + \frac{1}{2} e^{2t} v_2 v_2^{\mathrm{tr}} f$$

and therefore

$$\begin{aligned} e^{tA} &= \frac{1}{2}(v_1, f)\, e^{-t} v_1 v_1^{\mathrm{tr}} + \frac{1}{2} e^{2t} v_2 v_2^{\mathrm{tr}} \\ &= \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{2t} & e^{-t} - e^{2t} \\ e^{-t} - e^{2t} & e^{-t} + e^{2t} \end{bmatrix}. \end{aligned}$$

Example 3.10. The matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

has eigenvectors/eigenvalues given by

$$v_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftrightarrow -1 \text{ and } v_2 := \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftrightarrow -3$$

As usual, if $f \in \mathbb{R}^2$ then

$$f = \frac{1}{2}(v_1, f)\, v_1 + \frac{1}{2}(v_2, f)\, v_2.$$

It then follows that

$$\begin{aligned} e^{tA}f &= \frac{1}{2}(v_1, f)\, e^{tA} v_1 + \frac{1}{2}(v_2, f)\, e^{tA} v_2 \\ &= \frac{1}{2}(v_1, f)\, e^{-t} v_1 + \frac{1}{2}(v_2, f)\, e^{-3t} v_2. \end{aligned}$$

Taking $f = e_1$ and then $f = e_2$ gives

$$\begin{aligned} e^{tA}e_1 &= \frac{1}{2}(v_1, e_1)\, e^{-t} v_1 + \frac{1}{2}(v_2, e_1)\, e^{-3t} v_2 \\ &= \frac{1}{2} e^{-t} v_1 - \frac{1}{2} e^{-3t} v_2 \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} \\ e^{-t} - e^{-3t} \end{bmatrix} \end{aligned}$$

and similarly

$$e^{tA}e_2 = \frac{1}{2} \begin{bmatrix} e^{-t} - e^{-3t} \\ e^{-t} + e^{-3t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} e^{tA} &= \begin{bmatrix} e^{tA}e_1 & e^{tA}e_2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix}. \end{aligned}$$

Example 3.11. Continuing the notation and using the results of Example 1.8,

$$A := \begin{bmatrix} 1 & 7 & -2 \\ 7 & 1 & -2 \\ -2 & -2 & 10 \end{bmatrix}$$

with eigenvectors/eigenvalues given by

$$v_1 := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow -6,\ v_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow 6,\ v_3 := \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \leftrightarrow 12.$$

and

$$f = \frac{1}{2}(f, v_1)\, v_1 + \frac{1}{3}(f, v_2)\, v_2 + \frac{1}{6}(f, v_2)\, v_3.$$

For example if $f = (1, 2, 3)^{\mathrm{tr}}$, then

$$f = \frac{1}{2} v_1 + 2 v_2 + \frac{1}{2} v_3$$

and hence

$$\begin{aligned} e^{tA}(1, 2, 3)^{\mathrm{tr}} &= \frac{1}{2} e^{tA} v_1 + 2 e^{tA} v_2 + \frac{1}{2} e^{tA} v_3 \\ &= \frac{1}{2} e^{-6t} v_1 + 2 e^{6t} v_2 + \frac{1}{2} e^{12t} v_3. \end{aligned}$$

A straightforward but tedious computation shows

$$e^{tA} = \frac{1}{6} \begin{bmatrix} 3e^{-6t} + 2e^{6t} + e^{12t} & -3e^{-6t} + 2e^{6t} + e^{12t} & 2e^{6t} - 2e^{12t} \\ -3e^{-6t} + 2e^{6t} + e^{12t} & 3e^{-6t} + 2e^{6t} + e^{12t} & 2e^{6t} - 2e^{12t} \\ 2e^{6t} - 2e^{12t} & 2e^{6t} - 2e^{12t} & 2e^{6t} + 4e^{12t} \end{bmatrix}.$$

This can alternatively be done using a computer algebra package, which is what I did.

3.3 Second Order Linear ODE

We would like to solve the ordinary differential equation

$$\ddot{u}(t) = Au(t) \text{ with} \tag{3.9}$$
$$u(0) = f \text{ and } \dot{u}(0) = g \tag{3.10}$$

for some $f, g \in \mathbb{R}^N$ and A a $N \times N$ matrix. Again we might begin by trying to find solutions to Eq. (3.9) by considering functions of the form $u(t) = T(t)v$. In order for $u(t) = T(t)v$ to be a solution we must have

$$\ddot{T}(t)v = T(t)Av$$

and working as above we concluded that there must exists λ such that

$$\ddot{T}(t) = \lambda T(t) \text{ and } Av = \lambda v.$$

The general solution to the equation

$$\ddot{T}(t) = \lambda T(t)$$

is

$$T(t) = c_\lambda(t)T(0) + s_\lambda(t)\dot{T}(0)$$

where

$$c_\lambda(t) := \begin{cases} \cos\sqrt{-\lambda}t \text{ if } \lambda \le 0 \\ \cosh\sqrt{\lambda}t \text{ if } \lambda \ge 0 \end{cases}$$

and

$$s_\lambda(t) := \begin{cases} \frac{\sin\sqrt{-\lambda}t}{\sqrt{-\lambda}} & \text{if } \lambda < 0 \\ t & \text{if } \lambda = 0 \\ \frac{\sinh\sqrt{\lambda}t}{\sqrt{\lambda}} & \text{if } \lambda > 0. \end{cases}$$

Theorem 3.12. *Suppose the matrix A is diagonalizable, i.e. there exists a basis $\{v_i\}_{i=1}^N$ for $\mathbb{R}^N$ consisting of eigenvectors A with corresponding eigenvalues $\{\lambda_i\}_{i=1}^N \subset \mathbb{R}$. Then for any $f, g \in \mathbb{R}^N$ there is a unique solution, $u(t)$, to Eqs. (3.9) and (3.10). Moreover, if we expand f and g in terms of the basis $\{v_i\}_{i=1}^N$ as*

$$f = \sum_{i=1}^N a_i v_i \text{ and } g = \sum_{i=1}^N b_i v_i$$

then the unique solution to Eqs. (3.9) and (3.10) is given by

$$u(t) = \sum_{i=1}^N \left[a_i c_{\lambda_i}(t) + b_i s_{\lambda_i}(t)\right] v_i. \tag{3.11}$$

Proof. It is easy to check that u defined as in Eq. (3.11) solves Eqs. (3.9) and (3.10) which proves the existence of solutions. The uniqueness of solutions may also be proved similarly to what was done in Theorem 3.3. Indeed, suppose that

$$u(t) = \sum_{i=1}^N \alpha_i(t) v_i$$

then the equation, $\ddot{u} = Au$, is equivalent to

$$\sum_{i=1}^N \ddot{\alpha}_i(t) v_i = \ddot{u}(t) = Au(t) = \sum_{i=1}^N \alpha_i(t) Av_i = \sum_{i=1}^N \alpha_i(t) \lambda_i v_i$$

and since $\{v_i\}_{i=1}^N$ is a basis for $\mathbb{R}^N$ we must have

$$\ddot{\alpha}_i(t) = \lambda_i \alpha_i(t) \text{ for all } i. \tag{3.12}$$

Moreover,

$$\sum_{i=1}^N a_i v_i = f = u(0) = \sum_{i=1}^N \alpha_i(0) v_i \text{ and}$$
$$\sum_{i=1}^N b_i v_i = g = \dot{u}(0) = \sum_{i=1}^N \dot{\alpha}_i(0) v_i$$

implies that

$$\alpha_i(0) = a_i \text{ and } \dot{\alpha}_i(0) = b_i \text{ for all } i. \tag{3.13}$$

This completes the proof, since the unique solution to Eqs. (3.12) and (3.13) is given by

$$\alpha_i(t) = a_i c_{\lambda_i}(t) + b_i s_{\lambda_i}(t).$$

■

Notation 3.13 *From now on, let us agree that*

$$\cos\sqrt{-\lambda}t := \cosh\sqrt{\lambda}t \textit{ if } \lambda \le 0$$
$$\frac{\sin\sqrt{-\lambda}t}{\sqrt{-\lambda}} := \frac{\sinh\sqrt{\lambda}t}{\sqrt{\lambda}} \textit{ if } \lambda < 0 \textit{ and}$$
$$\frac{\sin\sqrt{-\lambda}t}{\sqrt{-\lambda}} := t \textit{ if } \lambda = 0.$$

With the above notation it is natural to write the general solution Eqs. (3.9) and (3.10) as

$$u(t) = \left(\cos\sqrt{-A}t\right) f + \frac{\sin\sqrt{-A}t}{\sqrt{-A}} g$$

with the understanding that

1. $\cos\sqrt{-A}t$ and $\frac{\sin\sqrt{-A}t}{\sqrt{-A}}$ are linear (i.e. matrices) and
2. if $Av = \lambda v$ then

$$\left(\cos\sqrt{-A}t\right)v := \left(\cos\sqrt{-\lambda}t\right)v \text{ and}$$
$$\frac{\sin\sqrt{-A}t}{\sqrt{-A}}v := \frac{\sin\sqrt{-\lambda}t}{\sqrt{-\lambda}}v.$$

Example 3.14. Continuing the notation and using the results of Example 1.8,

$$A := \begin{bmatrix} 1 & 7 & -2 \\ 7 & 1 & -2 \\ -2 & -2 & 10 \end{bmatrix}$$

with eigenvectors/eigenvalues given by

$$v_1 := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow -6,\ v_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow 6,\ v_3 := \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \leftrightarrow 12.$$

We will solve,

$$\ddot{u}(t) = Au(t) \text{ with}$$
$$u(0) = f = (1,2,3)^{\mathrm{tr}} \text{ and } \dot{u}(0) = g = (1,-1,1)^{\mathrm{tr}}.$$

As above we have

$$f = \frac{1}{2}(f,v_1)v_1 + \frac{1}{3}(f,v_2)v_2 + \frac{1}{6}(f,v_2)v_3$$
$$= \frac{1}{2}v_1 + 2v_2 + \frac{1}{2}v_3$$

and

$$g = 0v_1 + \frac{1}{3}v_2 + \frac{2}{6}(f,v_2)v_3 = \frac{1}{3}(2v_2 + v_3).$$

Therefore,

$$\cos\left(\sqrt{-A}t\right)f = \frac{1}{2}\cos\left(\sqrt{3}t\right)v_1 + 2\cosh\left(\sqrt{3}t\right)v_2 + \frac{1}{2}\cosh\left(\sqrt{12}t\right)v_3$$

and

$$\frac{\sin\left(\sqrt{-A}t\right)}{\sqrt{-A}}g = \frac{1}{3}\left(2\frac{\sinh\left(\sqrt{3}t\right)}{\sqrt{3}}v_2 + \frac{\sinh\left(\sqrt{12}t\right)}{\sqrt{12}}v_3\right)$$

and the solution is given by

$$u(t) = \cos\left(\sqrt{-A}t\right)f + \frac{\sin\left(\sqrt{-A}t\right)}{\sqrt{-A}}g$$
$$= \frac{1}{2}\cos\left(\sqrt{3}t\right)v_1$$
$$+ \left[2\cosh\left(\sqrt{3}t\right) + \frac{2}{3}\frac{\sinh\left(\sqrt{3}t\right)}{\sqrt{3}}\right]v_2$$
$$+ \left[\frac{1}{2}\cosh\left(\sqrt{12}t\right) + \frac{1}{3}\frac{\sinh\left(\sqrt{12}t\right)}{\sqrt{12}}\right]v_3.$$

3.4 ODE Exercises

Exercise 3.1. Here you are asked to give another proof of Eq. (3.7). Let A be an $N \times N$, matrix, $f \in \mathbb{R}^N$ and $s,t \in \mathbb{R}$. Show

$$e^{tA}e^{sA}f = e^{(t+s)A}f.$$

Outline: Let $u(t) := e^{tA}e^{sA}f$ and $v(t) = e^{(t+s)A}f$ and show both u and v solve the differential equation,

$$\dot{w}(t) = Aw(t) \text{ with } w(0) = e^{sA}f$$

and then use uniqueness of solutions of this equation (see Fact 3.5) to conclude that $u(t) = v(t)$.

Exercise 3.2. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show

$$e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

using the following three methods.

1. Showing

$$\frac{d}{dt}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = A\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and

$$\left.\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right|_{t=0} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

2. by explicitly summing the series
$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$
3. Show $\frac{d^2}{dt^2} e^{tA} = -e^{tA}$ and then solve this equation using $e^{tA}|_{t=0} = I$ and $\frac{d}{dt}|_0 e^{tA} = A$.

Exercise 3.3. Combine Exercises 3.1 and 3.2 to give a proof of the trigonometric identities:
$$\cos(s+t) = \cos s \cos t - \sin s \sin t \tag{3.14}$$
and
$$\sin(s+t) = \cos s \sin t + \cos t \sin s. \tag{3.15}$$

Exercise 3.4. Let $a, b, c \in \mathbb{R}$ and
$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$
Show
$$e^{tA} = \begin{pmatrix} 1 & at & bt + \frac{1}{2}act^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{pmatrix}$$
by summing the matrix power series. Also find $e^{t(\lambda I + A)}$ where $\lambda \in \mathbb{R}$ and I is the 3×3 identity matrix.

Exercise 3.5. Let
$$A := \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$
and
$$f = (1, 0, 2)^{\mathrm{tr}} \text{ and } g = (0, 1, 2)^{\mathrm{tr}}.$$
be as in Exercise 1.2. Solve the following equations
$$\dot{u}(t) = Au(t) \text{ with } u(0) = f \text{ and}$$
$$\ddot{u}(t) = Au(t) \text{ with } u(0) = f \text{ and } \dot{u}(0) = g.$$
Write your solutions in the form
$$u(t) = \sum_{i=1}^{3} a_i(t) v_i$$
where the functions a_i are to be determined.

Hint: Recall from Exercise 1.2 (you should have shown) that
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow 0, \; v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow -3, \; v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \leftrightarrow -3.$$
is an orthogonal basis of eigenvectors (with corresponding eigenvalues) for A.

Linear Operators and Separation of Variables

Definition 4.1. *A **linear combination** of the vectors* $\{v_i\}_{i=1}^n \subset \mathbb{R}^3$ *(or* $\{v_i\}_{i=1}^n \subset V$ *with* V *being any vector space) is a vector of the form,*

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n,$$

with $\{c_i\}_{i=1}^n$ *being real (or complex) constants.*

We are going to be interested in the case that the vector space V consists of a class of functions on some domain, $\Omega \subset \mathbb{R}^n$. If u_1 and u_2 are functions on $\Omega \subset \mathbb{R}^n$, and $c_1, c_2 \in \mathbb{R}$ we write $(c_1 u_1 + c_2 u_2) =: u$ for the function, $u : \Omega \to \mathbb{R}$ such that

$$u(x) = c_1 u_1(x) + c_2 u_2(x) \text{ for all } x \in \Omega.$$

For example we may consider, $u_1 + u_2$, $u_1 + 3u_2$, and $\mathbf{0} = 0u_1 + 0u_2$.

Definition 4.2. *A **linear space of functions**, V, is a class of functions with common domain so that if* u_1, u_2 *are in the class then so is* $c_1 u_1 + c_2 u_2$ *for all* $c_1, c_2 \in \mathbb{R}$, *i.e. the space of functions* V *is closed under taking linear combinations.*

Example 4.3.

$$\mathcal{D} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable on } \mathbb{R}\}$$

or

$$\mathcal{C} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous on } \mathbb{R}\}.$$

Consider operator, $L : \mathcal{D} \to \{\text{all functions on } \mathbb{R}\}$ defined by $Lf = f'$. This operator is linear, namely, we have

$$L(c_1 f_1 + c_2 f_2) = (c_1 f_1 + c_2 f_2)' = c_1 f_1' + c_2 f_2' = c_1 L(f_1) + c_2 L(f_2).$$

It is interesting to note that L does not map $\mathcal{D}$ to $\mathcal{C}$. For example, let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

then

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = 0 & x = 0 \end{cases}$$

so that $\varphi \in \mathcal{D}$ however $f' \notin \mathcal{C}$.

Definition 4.4. *A **Linear operator**, is a mapping, L, of one linear space of functions to another such that*

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

for all u_1, u_2 *in the domain function space and* $c_1, c_2 \in \mathbb{R}$.

An induction argument shows the linearity condition in Definition 4.4 implies

$$L\left(\sum_{i=1}^n c_i u_i\right) = \sum_{i=1}^N c_i L(u_i)$$

for all u_i in the domain function space and $c_i \in \mathbb{R}$.

Example 4.5. Let Ω be some open subset of $\mathbb{R}^2$, for example $\Omega = \mathbb{R}^2$ or $\Omega = \left\{x \in \mathbb{R}^2 | x_1^2 + x_2^2 < 5\right\}$ and let $\mathcal{D}$ denote those functions $u : \Omega \to \mathbb{R}$ such that u and all of its partial derivatives up to order two exist and are continuous. (In the future we denote this class of functions by $C^2(\Omega)$.) Then the following are example of linear operators taking $\mathcal{D}$ to the class of continuous functions on Ω :

1. $Lu = \frac{\partial^2 u}{\partial x^2}$
2. $Lu = \frac{\partial^2 u}{\partial x \, \partial y}$
3. $(Lu)(x, y) = x \frac{\partial u}{\partial y}(x, y) + y \frac{\partial u}{\partial x}(x, y)$.

Whereas, the following operator is an example of a non-linear operator; $Lu = \frac{\partial}{\partial x} u + u^2$. To see this operator is not linear, notice that

$$\begin{aligned} L(u_1 + u_2) &= \frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial x} u_2 + (u_1 + u_2)^2 \\ &\neq \frac{\partial}{\partial x} u_1 + u_1^2 + \frac{\partial}{\partial x} u_2 + u_2^2 = Lu_1 + Lu_2. \end{aligned}$$

For example, let $u_1 = \frac{1}{x}$ and $u_2 = \frac{1}{x+5}$ (also see Exercise 13.9), then

$$\begin{aligned} L(u_1) &= -\frac{1}{x^2} + \frac{1}{x^2} = 0 \\ L(u_2) &= -\frac{1}{(x+5)^2} + \frac{1}{(x+5)^2} = 0 \end{aligned}$$

while

$$L(u_1+u_2) = L\left(\frac{1}{x}+\frac{1}{x+5}\right)$$
$$= \frac{-1}{x^2} - \frac{1}{(x+5)^2} + \left(\frac{1}{x}+\frac{1}{x+5}\right)^2$$
$$= -\frac{1}{x^2} - \frac{1}{(x+5)^2} + \frac{1}{x^2} + \frac{1}{(x+5)^2} + \frac{2}{x+5)} = \frac{2}{x(x+5)} \neq 0.$$

Definition 4.6. *If L, M are two linear operators on the same class of functions we define $L+M$*

$$(L+M)(u) := Lu + Mu.$$

If M is a linear operator on the range-space of L we also define LM by $((LM)u) = L(Mu)$.

These new operators are still linear, for example,

$$(LM)(c_1u_1+c_2u_2) = L(c_1M(u_1)+c_2M(u_2))$$
$$= c_1L(M(u_1)) + c_2L(M(u_2)))$$
$$= c_1(LM)(u_1) + c_2(LM)(c_2).$$

However, it is in general not true (see Exercise 13.2) that $LM = ML$, in fact LM may be defined while ML is not defined. For example, let $Lu = x^2u$ and $Mu = \frac{\partial}{\partial x}u$ then taking $u = e^{xy}$ we find

$$LM(u) = L\left(\frac{\partial}{\partial x}e^{xy}\right) = L(y\ e^{xy}) = x^2ye^{xy}$$

while

$$M(Lu) = \frac{\partial}{\partial x}(x^2e^{xy}) = 2xe^{xy} + yx^2e^{xy} \neq LM(u).$$

In general, in this class we will be interested in linear differential operators of the form

$$Lu = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$$

where A, B, C, D, E, F are functions of x and y. The homogeneous partial differential equations, $Lu = 0$ is shorthand notation for u solving the equation,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0.$$

Lemma 4.7 (Principle of Superposition). *If L is a linear differential operator as above and u_1 and u_2 solve the homogeneous partial differential equations, $Lu_1 = 0$ and $Lu_2 = 0$, then any linear combination, $c_1u_1 + c_2u_2$ also satisfies the same equation, namely,*

$$L(c_1u_1 + c_2u_2) = 0.$$

Example 4.8 (Homogeneous Wave Equation). The wave equation $u_{tt} = a^2u_{xx}$, is equivalent to writing $Lu = 0$ where $L = \frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}$. So if $u_1, \dots, u_M$ are solutions to the wave equation, (i.e., $Lu_n = 0$), then any linear combination,

$$c_1u_1 + \cdots + c_nu_M,$$

is another solution as well. (See Exercises 13.4, 13.6, 13.8 for more on this and the issue of boundary conditions.) To be more explicit let us notice that

1. Show that $u_n(t,x) := \sin(nx)\sin(ant)$ for $n \in \mathbb{N}$ all solve the equation, $Lu = 0$. Indeed,
$$Lu_n = \frac{\partial^2}{\partial t^2}[\sin(nx)\sin(ant)] - a^2\frac{\partial^2}{\partial x^2}[\sin(nx)\sin(ant)]$$
$$= \frac{\partial}{\partial t}(\sin(nx)an\cos(ant)) - a^2\frac{\partial}{\partial x}(n\ \cos(nx)\sin(ant))$$
$$= -a^2n^2\sin(nt)\sin(ant) - a^2[-n^2\sin(nx)\sin(ant)] = 0.$$
2. So by the superposition principle,
$$u(x,y) = \sum_{n=1}^{N} c_n\sin(nx)\sin(ant)$$
with $c_1, \dots, c_M \in \mathbb{R}$ also satisfies the wave equation. (Later we will allow for infinite linear combinations and we will then choose the constants, c_i, so that certain boundary conditions are satisfied.)

4.1 Introduction to Fourier Series

In this section, I would like to explain how certain functions like $\sin nx$ and $\cos nx$ are going to appear in our study of partial differential equations. Suppose L is the differential operator, $L = \frac{d^2}{dx^2}$, acting on functions on $\Omega = [a,b]$. Define the **inner product,**

$$(f,g) := \int_{\Omega=[a,b]} f(x)\,g(x)\,dx,$$

for functions $f, g : \Omega \to \mathbb{R}$. Two integration by parts now shows,

$$(Lf,g) = \int_\Omega f''(x)\cdot g(x)\,dx = -\int_\Omega f'(x)\cdot g'(x)\,dx + f'(x)\,g(x)\,|_a^b$$
$$= \int_\Omega f(x)\cdot g''(x)\,dx + [f'(x)\,g(x) - f(x)\cdot g'(x)]\,|_a^b$$
$$= (f,Lg) + [f'(x)\,g(x) - f(x)\cdot g'(x)]\,|_a^b.$$

There are now a number of **boundary conditions** that may be imposed on f and g so that boundary terms in the previous equation are zero. For example we may assume $f,g\in D_{per}$ where D_{per} denotes those twice continuously differentiable functions such that $f(b)=f(a)$ and $f'(b)=f'(a)$. Or we might assume $f,g\in D_{\text{Dirichlet}}$ or $f,g\in D_{\text{Neumann}}$ where $D_{\text{Dirichlet}}$ (D_{Neumann}) consists of those twice continuously differentiable functions such that $f(a)=0=f(b)$ ($f'(a)=0=f'(b)$). In any of these cases, we will have

$$(Lf,g)=(f,Lg)$$

and so in analogy with the Spectral Theorem 1.5 we should expect that L has an orthonormal basis of eigenvectors. Let us find these eigenvectors in a few examples. Before doing this it is useful to record a few integrals.

Lemma 4.9. *Let n be a positive integer, then*

$$\int_0^\pi \sin^2 nx dx=\int_0^\pi \cos^2 nx dx=\frac{\pi}{2}, \tag{4.1}$$

$$\int_{-\pi}^\pi \sin^2 nx dx=\int_{-\pi}^\pi \cos^2 nx dx=\pi, \tag{4.2}$$

and

$$\int_{-\pi}^\pi \sin nx \cos nx dx=0. \tag{4.3}$$

Proof. Recall that

$$\cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1.$$

Therefore, taking $\theta=nx$ and integrating we find,

$$\begin{aligned}0&=\frac{1}{2n}\sin 2nx|_0^\pi=\int_0^\pi \cos 2nx dx\\&=\int_0^\pi\left[1-2\sin^2 nx\right]dx=\int_0^\pi\left[2\cos^2 nx-1\right]dx\\&=\pi-2\int_0^\pi \sin^2 nx dx=2\int_0^\pi\cos^2 nx dx-\pi\end{aligned}$$

which gives Eq. (4.1). Similarly, replacing $\int_0^\pi$ by $\int_{-\pi}^\pi$ above shows Eq. (4.2) is valid as well. Finally,

$$\int_{-\pi}^\pi \sin nx\cos nx dx=\frac{1}{2n}\sin^2 nx|_{-\pi}^\pi=0-0=0.$$

■

Example 4.10 (Fourier Series). Let $a=-\pi$ and $b=\pi$, $L=\frac{d^2}{dx^2}$ and $D=D_{per}$ so that

$$(f,g)=\int_{-\pi}^\pi f(x)g(x)dx \tag{4.4}$$

and

$$(Lf,g)=-(f',g')=-\int_{-\pi}^\pi f'(x)g'(x)dx=(f,Lg).$$

Thus if $Lf=\lambda f$ with $f\in D$ we have

$$\lambda(f,f)=(Lf,f)=-\int_{-\pi}^\pi\left[f'(x)\right]^2dx\le 0$$

from which it follows that $\lambda\le 0$. So we need only look for negative eigenvalues. If $\lambda=0$ the eigenvalue equation becomes $f''=0$ and hence $f(x)=Ax+B$. We will only have $f\in D$ if $A=0$ and therefore let $f_0=1$.

We may now suppose that $\lambda=-\omega^2<0$ in which case the eigenvalue equation becomes

$$f''=-\omega^2 f$$

which has

$$f(x)=A\cos\omega x+B\sin\omega x$$

as the general solution. We still must enforce the boundary values. For example $f(\pi)=f(-\pi)$ implies

$$A\cos(-\omega\pi)+B\sin(-\omega\pi)=A\cos\omega\pi+B\sin\omega\pi$$

or $B\sin\omega\pi=0$. Similarly, $f'(\pi)=f'(-\pi)$ implies

$$-\omega A\sin(-\omega\pi)+\omega B\cos(-\omega\pi)=-\omega A\sin\omega\pi+\omega B\cos\omega\pi$$

that $A\sin\omega\pi=0$. Hence we either have $A=B=0$ (in which case $f\equiv 0$ which is not allowed) or $\sin\omega\pi=0$ from which it follows that $\omega=n\in\mathbb{Z}$. Hence we have

$$\beta:=\{\cos nx,\sin nx:n\in\mathbb{N}\}\cup\{1\}$$

as our possible eigenvectors. The eigenvalue associated to $\cos nx$ and $\sin nx$ is $\lambda_n=-n^2$. By Lemma 4.9, $(\cos nx,\sin nx)=0$, so that β is an orthogonal set and moreover,

$$\begin{aligned}(\cos nx,\cos nx)=(\sin nx,\sin nx)=\pi \text{ and}\\(1,1)=2\pi.\end{aligned}$$

Thus we expect that any reasonable function f on $[-\pi,\pi]$ may be written as

$$f(x) = \frac{1}{2\pi}(f,1)\,1 + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(f,\cos n(\cdot))\cos nx + (f,\sin n(\cdot))\sin nx\right]. \tag{4.5}$$

See Theorem 5.8, Theorem 5.17 and Theorem 6.2 below for more details on this point.

Example 4.11 (Fourier Sine Series / Dirichlet boundary conditions). Suppose $a = 0$ and $b = \pi$, $L = \frac{d^2}{dx^2}$ and $D = D_{\text{Dirichlet}}$ so if $f, g \in D$ then $f(0) = 0 = f(\pi)$. We now take

$$(f,g) = \int_0^{\pi} f(x)\,g(x)\,dx$$

and working as in the previous example we find

$$u_n(x) = \sin nx \text{ with } \lambda_n = -n^2 \text{ for } n \in \mathbb{N}.$$

By Lemma 4.9,

$$(\sin n(\cdot), \sin n(\cdot)) = \frac{\pi}{2}$$

and so by Theorem 6.2,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}(f,\sin n(\cdot))\sin nx$$

for any "reasonable" function f on $[0,\pi]$.

Example 4.12 (Fourier Cosine Series / Neumann boundary conditions.). Suppose $a = 0$ and $b = \pi$, $L = \frac{d^2}{dx^2}$ and $D = D_{\text{Neumann}}$ so if $f, g \in D$ then $f'(0) = 0 = f'(\pi)$.

Again we take

$$(f,g) = \int_0^{\pi} f(x)\,g(x)\,dx$$

and we find the eigenfunctions and eigenvalues to be

$$u_n(x) = \cos nx \text{ with } \lambda_n = -n^2 \text{ for } n \in \mathbb{N}\cup\{0\}.$$

By Lemma 4.9,

$$(\cos n(\cdot), \cos n(\cdot)) = \frac{\pi}{2} \text{ for } n \in \mathbb{N}$$
$$\text{and } (1,1) = \pi,$$

Thus Theorem 6.2 asserts that any "reasonable" function f on $[0,\pi]$ may be written as

$$f(x) = \frac{1}{\pi}(f,1)\,1 + \frac{2}{\pi}\sum_{n=1}^{\infty}(f,\cos n(\cdot))\cos nx.$$

4.2 Application / Separation of variables

Example 4.13. Use "separation of variables" to solve the heat equation,

$$u_t(t,x) = u_{xx}(t,x) \text{ with } u(t,0) = u(t,5) = 0 \tag{4.6}$$
$$\text{and } u(0,x) = f(x).$$

The technique is to first ignore the nonhomogeneous condition $u(0,x) = f(x)$ and look for any solutions to the Eq. (4.6) of the form

$$u(t,x) = T(t)\,X(x).$$

From this we get,

$$\dot{T}(t)\,X(x) = T(t)\,X''(x)$$

or equivalently that

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

where λ is a constant. Thus we require that

$$X''(x) = \lambda X(x) \text{ with } X(0) = 0 \text{ and } X(5) = 0.$$

The solutions to this Sturm-Liouville problem are given by

$$X_n(x) = \sin\frac{n\pi x}{5} \text{ with } \lambda = \lambda_n = -\left(\frac{n\pi}{5}\right)^2.$$

This then forces $T_n(t) = e^{-t\left(\frac{n\pi}{5}\right)^2}$. Thus we find that

$$u_n(t,x) = e^{-t\left(\frac{n\pi}{5}\right)^2}\sin\frac{n\pi x}{5}$$

are all solutions to Eq. (4.6). We then look for a general solution to our problem in the form

$$u(t,x) = \sum_{n=1}^{\infty} b_n u_n(t,x)$$

where we wish to choose the constants, b_n such that

$$f(x) = u(0,x) = \sum_{n=1}^{\infty} b_n u_n(0,x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{5}.$$

Letting

$$(f,g) = \int_0^5 f(x)\,g(x)\,dx$$

we have

$$\left(\sin\frac{n\pi x}{5},\sin\frac{m\pi x}{5}\right)=\delta_{mn}\frac{5}{2}$$

and therefore,

$$b_n=\frac{\left(f,\sin\frac{n\pi x}{5}\right)}{\left(\sin\frac{n\pi x}{5},\sin\frac{n\pi x}{5}\right)}=\frac{2}{5}\int_0^5 f\left(x\right)\sin\frac{n\pi x}{5}dx.$$

Example 4.14. Use separation of variables to solve Laplace's equation,

$$u_{xx}\left(x,y\right)+u_{yy}\left(x,y\right)=0 \text{ for } 0\le x\le 5 \text{ and } 0\le y\le 2$$

with boundary conditions,

$$\begin{aligned}&u\left(0,y\right)=u\left(5,y\right)=0,\\&u\left(x,0\right)=0 \text{ and } u\left(x,2\right)=f\left(x\right).\end{aligned}$$

To do this we will work as above and begin by ignoring the non-homogeneous boundary condition and solve the rest by separation of variables. So we write $u\left(x,y\right)=X\left(x\right)Y\left(x\right)$ and require that

$$\frac{X''}{X}+\frac{Y''}{Y}=0 \text{ with } X\left(0\right)=X\left(5\right)=0=Y\left(0\right).$$

As before we must have $X''=\lambda X$ and we know the solutions are given by

$$X_n\left(x\right)=\sin\frac{n\pi x}{5} \text{ with } \lambda=\lambda_n=-\left(\frac{n\pi}{5}\right)^2.$$

It them implies that

$$Y''\left(y\right)=\left(\frac{n\pi}{5}\right)^2 Y\left(y\right) \text{ with } Y\left(0\right)=0.$$

From this we concluded that

$$Y_n\left(y\right)=\sinh\frac{n\pi y}{5}$$

and we find that

$$u_n\left(x,y\right)=\sin\frac{n\pi x}{5}\sinh\frac{n\pi y}{5}$$

in this case. So working as above we try to find a solution of the form

$$u\left(x,y\right)=\sum_{n=1}^{\infty}b_n u_n\left(x,y\right).$$

All the boundary conditions are now satisfied except for

$$f\left(x\right)=u\left(x,2\right)=\sum_{n=1}^{\infty}b_n u_n\left(x,2\right)=\sum_{n=1}^{\infty}b_n\sinh\frac{2n\pi}{5}\sin\frac{n\pi x}{5}.$$

By the same logic as above we must have

$$b_n\sinh\frac{2n\pi}{5}=\frac{2}{5}\int_0^5 f\left(x\right)\sin\frac{n\pi x}{5}dx$$

and thus that

$$u\left(x,y\right)=\sum_{n=1}^{\infty}b_n\sinh\frac{2n\pi}{5}\sin\frac{n\pi x}{5}$$

with

$$b_n=\frac{2}{5\sinh\frac{2n\pi}{5}}\int_0^5 f\left(x\right)\sin\frac{n\pi x}{5}dx.$$

Example 4.15. Solve the wave equation, for $0\le x\le 5$ and $t\in\mathbb{R}$,

$$\begin{aligned}u_{tt}\left(t,x\right)=u_{x\ x}\left(t,x\right)&\text{ with } u\left(t,0\right)=u\left(t,5\right)=0\\ \text{and } u\left(0,x\right)=f\left(x\right)&\text{ and } u_t\left(0,x\right)=0.\end{aligned}$$

We could go through separation of variables here to answer this question, but this is getting tedious. I will just write down the answer as

$$u\left(t,x\right)=\cos\left(\sqrt{-\partial_x^2}t\right)f\left(x\right).$$

As we have seen,

$$f\left(x\right)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{5}$$

with

$$b_n=\frac{2}{5}\int_0^5 f\left(x\right)\sin\frac{n\pi x}{5}dx$$

and hence

$$\begin{aligned}u\left(t,x\right)&=\sum_{n=1}^{\infty}b_n\cos\left(\sqrt{-\partial_x^2}t\right)\sin\frac{n\pi x}{5}\\&=\sum_{n=1}^{\infty}b_n\cos\left(\frac{n\pi t}{5}\right)\sin\frac{n\pi x}{5}.\end{aligned}$$

It is interesting to notice that since

$$\sin\left(A+B\right)=\cos A\sin B+\sin A\cos B$$

we have

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

Thus we may write,

$$\sin\frac{n\pi x}{5}\cos\left(\frac{n\pi t}{5}\right)=\frac{1}{2}\left[\sin\left(\frac{n\pi(x+t)}{5}\right)+\sin\left(\frac{n\pi(x-t)}{5}\right)\right]$$

and thus

$$\begin{aligned}u(t,x)&=\frac{1}{2}\sum_{n=1}^{\infty}b_n\left[\sin\left(\frac{n\pi(x+t)}{5}\right)+\sin\left(\frac{n\pi(x-t)}{5}\right)\right]\\&=\frac{1}{2}\left[F(x+t)+F(x-t)\right]\end{aligned}$$

where

$$\begin{aligned}F(x)&=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{5}\right)\\&=\text{ the } 5-\text{ periodic extensions of } f(x).\end{aligned}$$

5

Orthogonal Function Expansions

5.1 Generalities about inner products on function spaces

Let Ω be a region in $\mathbb{R}^d$ (most of the time d will be one for us) and $p : \Omega \to (0, \infty)$ be a positive function. For functions $f, g : \Omega \to \mathbb{R}$ define

$$(f, g) := \int_{\Omega} f(x)\, g(x)\, p(x)\, dx.$$

This is an example of a **inner product**, i.e. something that behaves like the dot product on $\mathbb{R}^N$. For example we still have the following properties:

$$\begin{aligned}(f_1 + cf_2, g) &= (f_1, g) + c\,(f_2, g)\\ (f, g) &= (g, f)\\ \|f\|^2 := (f, f) &= 0 \text{ implies } f = 0.\end{aligned}$$

The following computation will be used frequently in this class:

$$\begin{aligned}\|f+g\|^2 &= (f+g, f+g) = \|f\|^2 + \|g\|^2 + (f, g) + (g, f)\\ &= \|f\|^2 + \|g\|^2 + 2(f, g). \end{aligned} \tag{5.1}$$

Definition 5.1. *Two functions $f, g : \Omega \to \mathbb{R}$ are **orthogonal** and we write $f \perp g$ iff $(f, g) = 0$. More generally, a collection of functions, $\{\varphi_i\}_{i=1}^n$, is an **orthogonal set** if $\varphi_i \perp \varphi_j$ (i.e. $(\varphi_i, \varphi_j) = 0$) for $i \neq j$. If we further have $\|\varphi_i\| = 1$ then we say $\{\varphi_i\}_{i=1}^n$ is an **orthonormal set.***

Exercise 5.1. Put in some exercise on orthogonal sets from the book here.

Theorem 5.2 (Schwarz Inequality). *For all $f, g : \Omega \to \mathbb{R}$,*

$$|(f, g)| \le \|f\| \|g\|$$

and equality holds iff f and g are linearly dependent.

Proof. If $g = 0$, the result holds trivially. So assume that $g \neq 0$ and observe; if $f = \alpha g$ for some $\alpha \in \mathbb{C}$, then $(f, g) = \alpha \|g\|^2$ and hence

$$|(f, g)| = |\alpha| \|g\|^2 = \|f\| \|g\|.$$

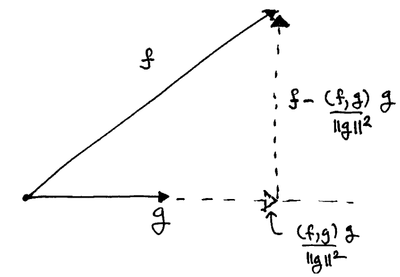


Fig. 5.1. The picture behind the proof of the Schwarz inequality.

Now suppose that $f \in H$ is arbitrary, let $h := f - \|g\|^{-2}(f, g)g$. (So z is the "orthogonal projection" of f onto g, see Figure 5.1.) Then

$$\begin{aligned}0 \le \|h\|^2 &= \left\| f - \frac{(f, g)}{\|g\|^2} g \right\|^2 = \|f\|^2 + \frac{(f, g)^2}{\|g\|^4}\|g\|^2 - 2\left(f, \frac{(f, g)}{\|g\|^2} g\right)\\ &= \|f\|^2 - \frac{(f, g)^2}{\|g\|^2}\end{aligned}$$

from which it follows that $0 \le \|g\|^2\|f\|^2 - (f, g)^2$ with equality iff $h = 0$ or equivalently iff $f = \|g\|^{-2}(f, g)g$. ■

Corollary 5.3 (Triangle inequality). *Let $f, g : \Omega \to \mathbb{R}$ be functions and $a \in \mathbb{R}$, then*

$$\|f + g\| \le \|f\| + \|g\| \text{ and} \tag{5.2}$$

$$\|af\| = |a| \|f\|. \tag{5.3}$$

Proof.

$$\begin{aligned}\|f+g\|^2 &= \|f\|^2 + \|g\|^2 + 2(f, g)\\ &\le \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| = (\|f\| + \|g\|)^2.\end{aligned}$$

Taking the square root of this inequality shows Eq. (5.2) holds. Taking the square root of the identity,

$$\|af\|^2 = \int_\Omega |a|^2 |f(x)|^2 \, dx = |a|^2 \int_\Omega |f(x)|^2 \, dx = |a|^2 \|f\|^2,$$

proves Eq. (5.3). ∎

Proposition 5.4 (Pythagorean's Theorem). *Suppose that $\{\varphi_i\}_{i=1}^n$ is an orthogonal set, then*

$$\left\| \sum_{i=1}^n \varphi_i \right\|^2 = \sum_{i=1}^n \|\varphi_i\|^2. \tag{5.4}$$

Proof. Let $s := \sum_{i=1}^n \varphi_i$, then

$$\|s\|^2 = (s, s) = \left(\sum_{i=1}^n \varphi_i, s \right) = \sum_{i=1}^n (\varphi_i, s)$$

and

$$(\varphi_i, s) = \left(\varphi_i, \sum_{j=1}^n \varphi_j \right) = \sum_{j=1}^n (\varphi_i, \varphi_j) = (\varphi_i, \varphi_i) = \|\varphi_i\|^2 .$$

The last two equations proves Eq. (5.4). ∎

Theorem 5.5 (Best Approximation Theorem). *Suppose $\{\varphi_i\}_{i=1}^n$ is an orthonormal set and $a_i \in \mathbb{R}$, then*

$$\left\| f - \sum_{i=1}^n a_i \varphi_i \right\|^2 = \left\| f - \sum_{i=1}^n (f, \varphi_i) \varphi_i \right\|^2 + \sum_{i=1}^n |(f, \varphi_i) - a_i|^2 \tag{5.5}$$

and therefore the best approximation to f by functions of the form $\sum_{i=1}^n a_i \varphi_i$ occurs when $a_i = (f, \varphi_i)$.

Proof. The function (vector),

$$h := f - \sum_{i=1}^n (f, \varphi_i) \varphi_i,$$

is orthogonal to $\{\varphi_i\}_{i=1}^n$ since

$$(h, \varphi_j) = \left(f - \sum_{i=1}^n (f, \varphi_i) \varphi_i, \varphi_j \right) = (f, \varphi_j) - \sum_{i=1}^n (f, \varphi_i) (\varphi_i, \varphi_j)$$
$$= (f, \varphi_j) - \sum_{i=1}^n (f, \varphi_i) \delta_{ij} = (f, \varphi_j) - (f, \varphi_j) = 0.$$

Since

$$f - \sum_{i=1}^n a_i \varphi_i = f - \sum_{i=1}^n (f, \varphi_i) \varphi_i + \sum_{i=1}^n [(f, \varphi_i) - a_i] \varphi_i$$
$$= h + \sum_{i=1}^n [(f, \varphi_i) - a_i] \varphi_i,$$

it follows by Pythagorean's Theorem, Proposition 5.4, that

$$\left\| f - \sum_{i=1}^n a_i \varphi_i \right\|^2 = \|h\|^2 + \sum_{i=1}^n \|[(f, \varphi_i) - a_i] \varphi_i\|^2$$
$$= \|h\|^2 + \sum_{i=1}^n |(f, \varphi_i) - a_i|^2$$
$$= \left\| f - \sum_{i=1}^n (f, \varphi_i) \varphi_i \right\|^2 + \sum_{i=1}^n |(f, \varphi_i) - a_i|^2 .$$

∎

Definition 5.6. *Let f be a function such that $\int_\Omega |f(x)|^2 p(x) \, dx < \infty$ and $\{\varphi_i\}_{i=1}^\infty$ be an orthonormal set, we will write $f \sim \sum_{i=1}^\infty (f, \varphi_i) \varphi_i$ to mean*

$$\lim_{n \to \infty} \int_\Omega \left| f(x) \sum_{i=1}^n (f, \varphi_i) \varphi_i(x) \right|^2 p(x) \, dx$$
$$= \lim_{n \to \infty} \left\| f - \sum_{i=1}^n (f, \varphi_i) \varphi_i \right\|^2 = 0.$$

*We say $\{\varphi_i\}_{i=1}^\infty$ is **complete** (or **closed** in the book's terminology) if $f \sim \sum_{i=1}^\infty (f, \varphi_i) \varphi_i$ whenever $\|f\|^2 < \infty$.*

Corollary 5.7 (Bessel's (In)equality). *Suppose $\{\varphi_i\}_{i=1}^n$ is an orthonormal set, then*

$$\sum_{i=1}^n |(f, \varphi_i)|^2 \le \|f\|^2 \text{ for all } f, \tag{5.6}$$

Moreover we get equality iff $f = \sum_{i=1}^n (f, \varphi_i) \varphi_i$. These statements remain true even when $n = \infty$ provided we interpret, $f = \sum_{i=1}^\infty (f, \varphi_i) \varphi_i$ to mean $f \sim \sum_{i=1}^\infty (f, \varphi_i) \varphi_i$. So we have $f \sim \sum_{i=1}^\infty (f, \varphi_i) \varphi_i$ iff Pythagorean's theorem holds, i.e. iff

$$\sum_{i=1}^\infty |(f, \varphi_i)|^2 = \|f\|^2.$$

Proof. Taking $a_i = 0$ in Eq. (5.5) shows

$$\|f\|^2 = \left\| f - \sum_{i=1}^{n} (f, \varphi_i) \varphi_i \right\|^2 + \sum_{i=1}^{n} |(f, \varphi_i)|^2$$

and hence that

$$\sum_{i=1}^{n} |(f, \varphi_i)|^2 = \|f\|^2 - \left\| f - \sum_{i=1}^{n} (f, \varphi_i) \varphi_i \right\|^2 \le \|f\|^2$$

with equality iff $f = \sum_{i=1}^{n} (f, \varphi_i) \varphi_i$. Letting $n \to \infty$ in the previous equation shows,

$$\sum_{i=1}^{\infty} |(f, \varphi_i)|^2 = \|f\|^2 - \lim_{n\to\infty} \left\| f - \sum_{i=1}^{n} (f, \varphi_i) \varphi_i \right\|^2 \le \|f\|^2$$

with equality iff

$$\lim_{n\to\infty} \left\| f - \sum_{i=1}^{n} (f, \varphi_i) \varphi_i \right\|^2 = 0,$$

i.e. iff $f \sim \sum_{i=1}^{\infty} (f, \varphi_i) \varphi_i$. ■

5.2 Convergence of the Fourier Series

For this section it will be convenient to define

$$(f, g) = \int_{-\pi}^{\pi} f(y) g(y) \, \frac{1}{\pi} dy$$

Recall from Example 4.10, if $f : \mathbb{R} \to \mathbb{R}$ is "reasonable" 2π - periodic function (i.e. $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$), we expect by analogy with the finite dimensional spectral theorem that

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \tag{5.7}$$

where

$$a_n := (f, \cos n(\cdot)) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy \text{ for } n = 0, 1, 2, \ldots$$

and

$$b_n := (f, \sin n(\cdot)) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy \text{ for } n = 1, 2, \ldots.$$

The following theorem gives a precise version of this statement.

Theorem 5.8 (Fourier Convergence Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π - periodic function which is piecewise continuous on $(-\pi, \pi)$. Then at points $x \in X$ where $f'(x\pm)$ exist we have*

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] = \frac{f(x+) + f(x-)}{2}.$$

Fact 5.9 *If $f : [-\pi, \pi] \to \mathbb{R}$ is any function such that $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$, we may still define*

$$f_N(x) = \frac{1}{2} a_0 + \sum_{n=1}^{N} [a_n \cos nx + b_n \sin nx]$$

with a_n and b_n as above. With this definition we will always have,

$$\frac{1}{\pi} \lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - f_N(x)|^2 dx = \lim_{N\to\infty} \|f - f_N\|^2 = 0,$$

i.e. that

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx].$$

5.3 Examples

Remark 5.10. We will use the following identities repeatedly.

$$\sin(A + B) = \cos A \sin B + \sin A \cos B, \tag{5.8}$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \tag{5.9}$$
$$\sin A \cos B = \frac{1}{2} (\sin(A + B) + \sin(A - B)) \tag{5.10}$$
$$\cos A \cos B = \frac{1}{2} (\cos(A + B) + \cos(A - B)) \tag{5.11}$$
$$\sin A \sin B = \frac{1}{2} (\cos(A - B) + \cos(A + B)). \tag{5.12}$$

Example 5.11. Suppose

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < 0 < x \end{cases},$$

then $a_n := (f, \cos n(\cdot)) = 0$ because f is odd while

$$b_n = (f, \sin n(\cdot)) = \frac{2}{\pi}\int_0^\pi \sin ny\ dy = -\frac{2}{\pi n}\cos ny|_0^\pi$$
$$= \frac{2}{\pi n}(1-\cos n\pi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd.}\end{cases}$$

Thus we conclude that

$$f(x) \sim \sum_{n \text{ odd}} \frac{4}{\pi n}\sin nx = \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)}\sin(2n-1)x.$$

The series converges for every $x \in \mathbb{R}$ and $x \in (-\pi,\pi)\setminus\{0\}$ it converges to $f(x)$ and at $x=0$ it converges to 0. By the way, by Bessel's equality we have

$$\sum_{n=1}^{\infty}\left(\frac{4}{\pi(2n-1)}\right)^2 = \|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} 1dx = 2$$

and hence we conclude that

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Here are some related graphs wherein $f_N(x) := \sum_{n=1:n \text{ odd}}^{N}\frac{4}{\pi n}\sin nx$.

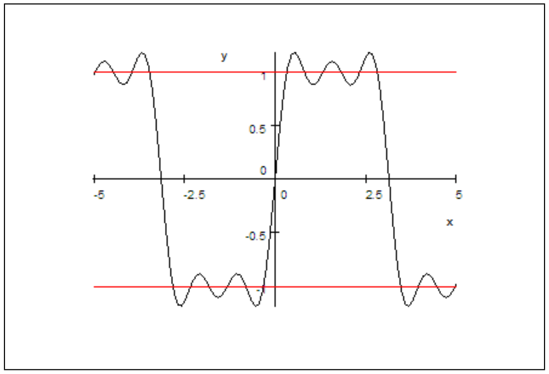


Fig. 5.2. A plot of $f_5(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$. which is approximating $f(x)$.

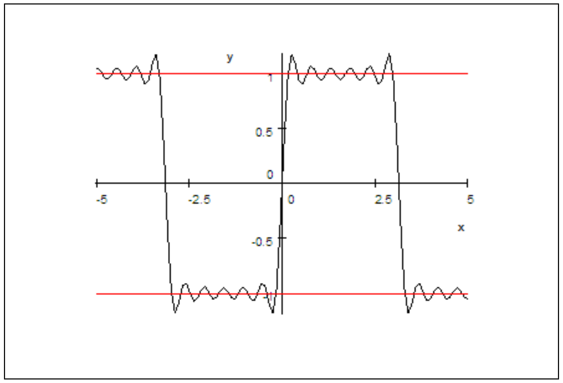


Fig. 5.3. A plot of $f_{11}(x)$.

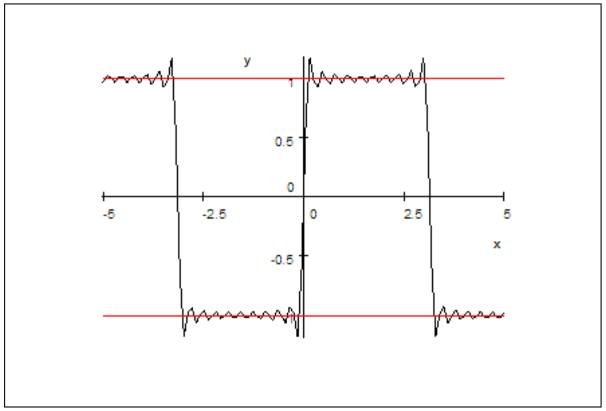


Fig. 5.4. A plot of $f_{19}(x)$. These last few picture illustrate what is known as the Gibb's effect; namely the convergence is not uniform. There is always the pesky little bump appearing near 0 (and $\pm\pi$).

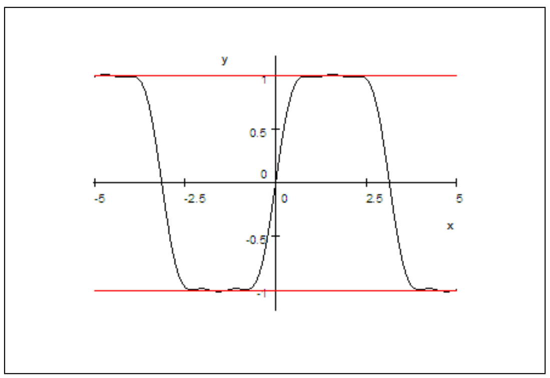


Fig. 5.5. A plot (at $t = 1/2$) of $\frac{4}{\pi}\left(e^{-t}\sin x + \frac{1}{3}e^{-9t}\sin 3x + \frac{1}{5}e^{-25t}\sin 5x\right)$ which is approximating the solution to the heat equation with periodic boundary condtions and with initial condition, $u(0,x) = f(x)$.

The following lemma will be useful in simplifying the computations in some of the examples below.

Lemma 5.12. *Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function and $p(x) = \sum_{k=0}^{n} p_k x^k$ is a polynomial, then*

$$\int f p dx = pF_1 - p'F_2 + p''F_3 - \cdots + (-1)^n p^{(n)} F_{n+1} + C$$

where

$$F_1 = \int f dx \text{ and } F_{k+1} = \int F_k dx.$$

Proof. This is simply a matter of repeated integration by parts. Explicitly,

$$\int f p dx = \int F_1' p dx = F_1 p - \int F_1 p' dx + C.$$
$$\int F_1 p' dx = F_2 p' - \int F_2 p'' dx + C$$

and hence

$$\begin{aligned}\int f p dx &= \int F_1' p dx = F_1 p - F_2 p' + \int F_2 p'' dx + C \\ &= F_1 p - F_2 p' + F_2 p'' dx - \int F_3 p^{(3)} dx + C,\end{aligned}$$

etc. ■

Using this fact we may now compute the Fourier series of a number of functions.

Example 5.13. Let $(f,g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) g(x) dx$, then

1.

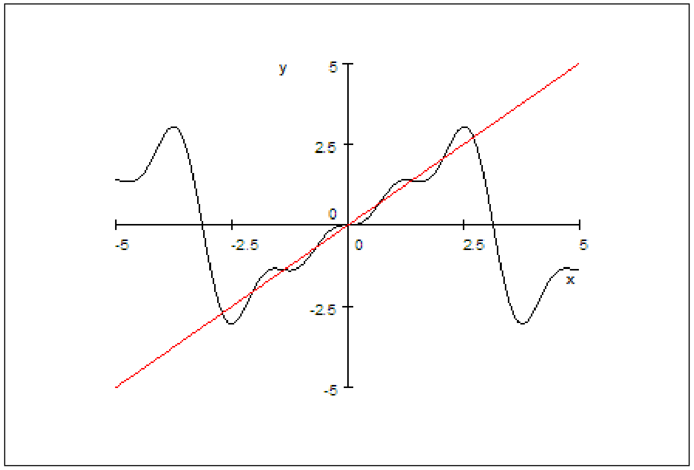
$$\begin{aligned}(x,1) = 0,\ \ &(x,\cos nx) = 0 \text{ by symmetry and} \\ (x,\sin nx) &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx = \left[x\left(-\frac{1}{n}\cos nx\right) - \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} \\ &= \frac{1}{n}\left[-2\cos n\pi\right] = \frac{2}{n}(-1)^{n+1}\end{aligned}$$

wherein we have used

$$\begin{aligned}F_1 &= \int \sin nx dx = -\frac{1}{n}\cos nx \text{ and} \\ F_2 &= -\frac{1}{n}\int \cos nx dx = \frac{1}{n^2}\sin nx.\end{aligned}$$

Thus we expect that

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx = 2\left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\right).$$



Plot of x and

$$\frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x$$

.

2. $f(x) = x^2$ – expansion.

$$\left(x^2, \sin nx\right) = 0 \text{ by symmetry and}$$

$$\begin{aligned}\left(x^2, \cos nx\right) &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx \\ &= \left[x^2\left(\frac{1}{n}\sin nx\right) + 2x\frac{1}{n^2}\cos nx - 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}2\pi\frac{1}{n^2}\left[\cos n\pi + \cos n(-\pi)\right] = 4\frac{1}{n^2}(-1)^n\end{aligned}$$

wherein we have used

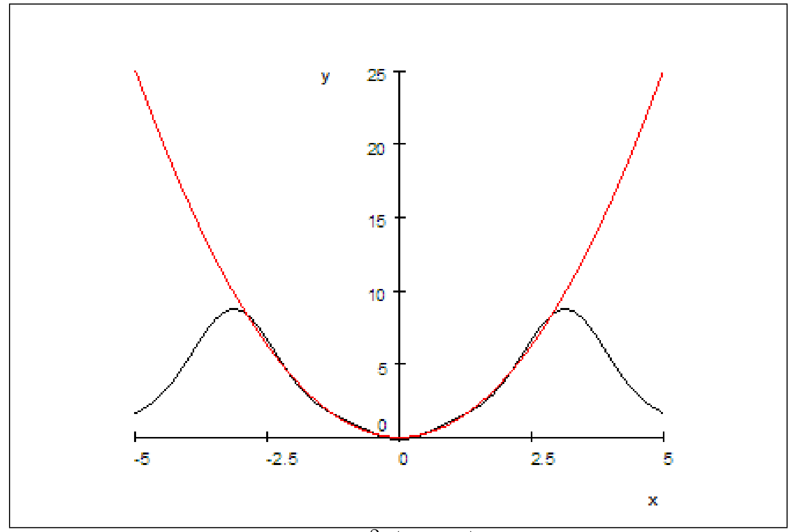
$$\begin{aligned}F_1 &= \int \cos nx dx = \frac{1}{n}\sin nx, \\ F_2 &= \int \frac{1}{n}\sin nx dx = -\frac{1}{n^2}\cos nx \text{ and} \\ F_3 &= \int -\frac{1}{n^2}\cos nx dx = -\frac{1}{n^3}\sin nx.\end{aligned}$$

We also have

$$\left(x^2, 1\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{x^3}{3}|_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

Thus we expect from Eq. (4.5) that

$$\begin{aligned}x^2 &= \frac{1}{2}\left(x^2, 1\right)1 + \sum_{n=1}^{\infty}\left[\left(x^2, \cos n(\cdot)\right)\cos nx + \left(x^2, \sin n(\cdot)\right)\sin nx\right] \\ &= \frac{1}{2}\frac{2\pi^2}{3}1 + \sum_{n=1}^{\infty}4\frac{1}{n^2}(-1)^n\cos nx \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \\ &= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots.\right].\end{aligned}$$



Plot of x^2 (in red) and

$$\frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right]$$

.

3. Integrating the equation,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx,$$

one expects,

$$
\begin{aligned}
\frac{x^3}{3} &= \frac{\pi^2}{3}x + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\sin nx}{n} \\
&= \frac{\pi^2}{3}\left[2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\right] - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx \\
&= \sum_{n=1}^{\infty}\left[\frac{2\pi^2}{3}\frac{(-1)^{n+1}}{n} - 4\frac{(-1)^{n+1}}{n^3}\right]\sin nx \\
&= 2\sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{(\pi n)^2-6}{3n^3}\right]\sin nx
\end{aligned}
$$

and hence that

$$x^3 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\left[\frac{(\pi n)^2-6}{n^3}\right]\sin nx$$

also see top of page 77 of the book

4. The expansions for functions of the form $f(x) = ax + bx^2 + cx^3$ are now easily found. For example, if

$$x^2 - \pi x = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - \pi\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx.$$

The plots of this function and the following approximation,

$$S_2(x) = \frac{\pi^2}{3} + 4\left(\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x\right) - 2\pi\left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x\right),$$

$$
\begin{aligned}
S_3(x) = \frac{\pi^2}{3} &+ 4\left(\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x\right) \\
&- 2\pi\left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x\right)
\end{aligned}
$$

and

$$
\begin{aligned}
S_4(x) = \frac{\pi^2}{3} &+ 4\left(\frac{-1}{1^2}\cos x + \cdots + \frac{1}{4^2}\cos 4x\right) \\
&- 2\pi\left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x\right)
\end{aligned}
$$

are given in Figure 5.6 below.

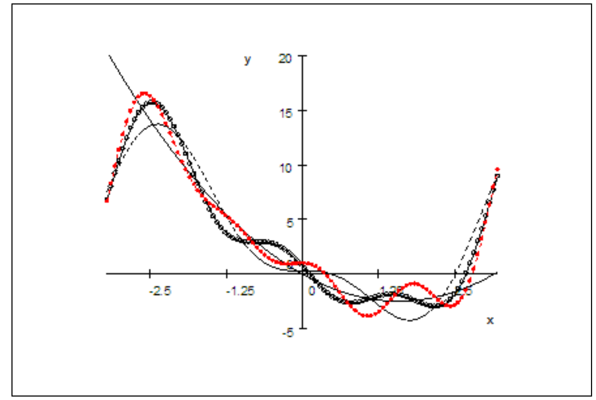


Fig. 5.6. Plot of $x^2 - \pi x$ and the approximats, S_2, S_3, and S_4. Notice that the approximations are not doing to well at the end points. This is because they are convergeing to π^2 at $x = \pi$ and $x = -\pi$.

Example 5.14. Suppose that $f(x)$ is a function defined for $0 \le x \le \pi$. Suppose we extend f to be an odd function by setting

$$F(x) := \begin{cases} f(x) & \text{if } 0 < x < \pi \\ -f(-x) & \text{if } -\pi < x < 0, \end{cases}$$

see Figure 5.7. Then computing the Fourier series of F, we learn

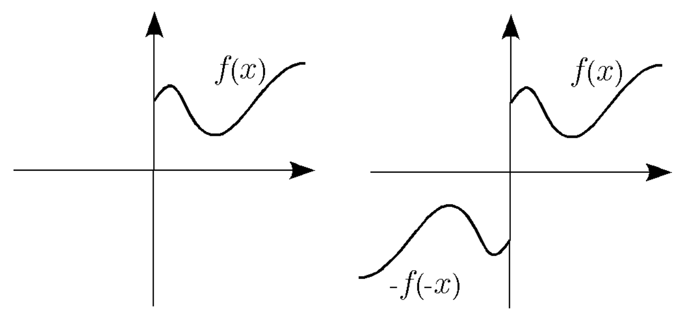


Fig. 5.7. The function f with its extension to an odd function on $[-\pi, \pi]$.

$$a_n := (F, \cos n(\cdot)) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ny\; dy = 0 \text{ for } n = 0, 1, 2, \dots$$

and

$$b_n := (F, \sin n(\cdot)) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin ny\; dy = \frac{2}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\; dy.$$

From this we learn that

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\; dy\right)\sin nx.$$

Notice that $\{\sin ny : n \in \mathbb{N}\}$ form an orthonormal set relative to the inner product,

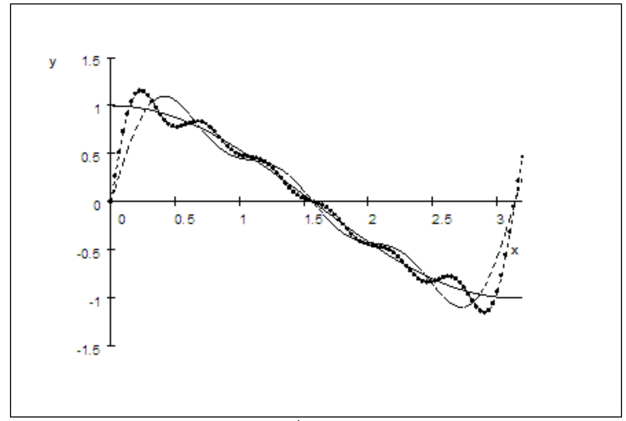
$$(f, g) := \frac{2}{\pi}\int_0^{\pi} f(x)\, g(x)\, dx.$$

As an explicit example, let us consider the sin - series expansion of $\cos x$ for $0 < x < \pi$. For this we have

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}\cos y \sin ny\; dy \\ &= \frac{1}{\pi}\int_0^{\pi}\left(\sin((n+1)y) + \sin((n-1)y)\right) dy \\ &= -\frac{1}{\pi}\left(\frac{\cos((n+1)y)}{n+1} + \frac{\cos((n-1)y)}{n-1}\right)_0^{\pi} \\ &= -\frac{1}{\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right) \\ &= -1_{n\text{-even}}\frac{-2}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) = 1_{n\text{-even}}\frac{2}{\pi}\frac{2n}{n^2-1} \\ &= 1_{n\text{-even}}\frac{4}{\pi}\frac{n}{n^2-1} \end{aligned}$$

and we conclude that

$$\begin{aligned} \cos x &= \frac{4}{\pi}\sum_{n=2,4,6,\dots}\frac{n}{n^2-1}\sin nx \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2-1}\sin 2nx. \end{aligned}$$



Here is a plot of $\cos x$ along with $\frac{4}{\pi}\left(\frac{2}{2^2-1}\sin 2x + \frac{4}{4^2-1}\sin 4x + \frac{6}{6^2-1}\sin 6x\right)$ and $\frac{4}{\pi}\left(\frac{2}{2^2-1}\sin 2x + \cdots + \frac{12}{12^2-1}\sin 12x\right)$

Example 5.15. Suppose that $f(x)$ is a function defined for $0 \le x \le \pi$. Suppose we extend f to be an even function by setting

$$F(x) := \begin{cases} f(x) & \text{if } \;\; 0 < x < \pi \\ f(-x) & \text{if } -\pi < x < 0, \end{cases}$$

see Figure 5.8. Then computing the Fourier series of F, we learn

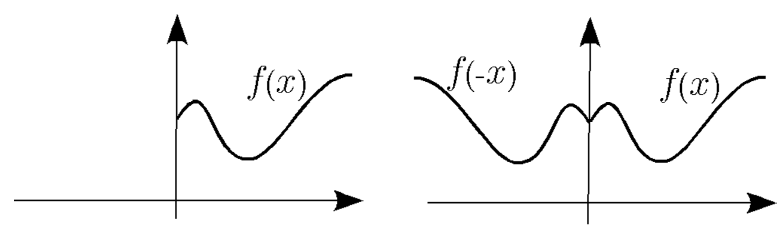


Fig. 5.8. The function f with its extension to an even function on $[-\pi, \pi]$.

$$a_n := (F, \cos n(\cdot)) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ny\; dy = \frac{2}{\pi}\int_0^{\pi} f(y)\cos ny\; dy$$

and

$$b_n := (F, \sin n(\cdot)) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin ny\; dy = 0.$$

From this we learn that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_n := \frac{2}{\pi} \int_0^{\pi} f(y) \cos ny \; dy.$$

Notice that $\{\cos ny : n \in \mathbb{N}\} \cup \left\{\frac{1}{\sqrt{2}}\right\}$ form an orthonormal set relative to the inner product,

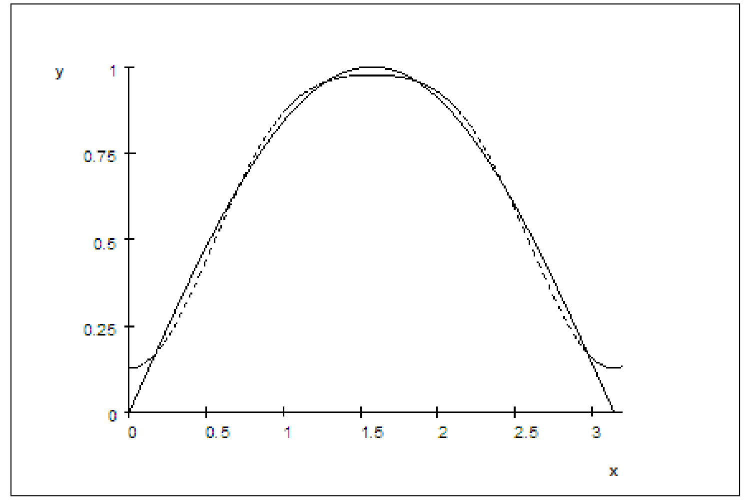
$$(f, g) := \frac{2}{\pi} \int_0^{\pi} f(x) \, g(x) \, dx.$$

As an explicit example, let us consider the cos - series expansion of $\sin x$ for $0 < x < \pi$. For this we have

$$\begin{aligned}
a_n &:= \frac{2}{\pi} \int_0^{\pi} \sin y \cos ny \; dy \\
&= \frac{1}{\pi} \int_0^{\pi} \left(\sin\left((n+1)\, y\right) + \sin\left((1-n)\, y\right)\right) \, dy \\
&= -\frac{1}{\pi} \left(\frac{\cos\left((n+1)\, y\right)}{n+1} + \frac{\cos\left((1-n)\, y\right)}{1-n} \right)_0^{\pi} \\
&= 2 \cdot 1_{n\text{-even}} \frac{1}{\pi} \left(\frac{1}{1+n} + \frac{1}{1-n} \right) \\
&= \frac{4}{\pi} 1_{n\text{-even}} \frac{1}{1-n^2} = -\frac{4}{\pi} 1_{n\text{-even}} \frac{1}{n^2-1}.
\end{aligned}$$

So we conclude that

$$\begin{aligned}
\sin x &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6,\dots} \frac{1}{n^2-1} \cos nx \\
&= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nx \\
&= \frac{2}{\pi} \left(1 - \frac{2}{2^2-1} \cos 2x - \frac{2}{4^2-1} \cos 4x - \frac{2}{6^2-1} \cos 6x - \dots \right).
\end{aligned}$$



Plot of $\sin x$ and the function

$$\frac{2}{\pi} \left(1 - \frac{2}{2^2-1} \cos 2x - \frac{2}{4^2-1} \cos 4x \right)$$

which consists of the first 3 terms in the cosine expansion of $\sin x$.

5.4 Proof of Theorem 5.8

Before giving the proof of Theorem 5.8, we will need the following simple consequence of Bessel's inequality in Corollary 5.7.

Lemma 5.16. *Suppose f is a continuous function on $[-\pi, \pi]$ or more generally any function such that $\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$ and a_n and b_n are given as above, then*

$$\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 \right] \le \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty. \tag{5.13}$$

Moreover we also have

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} f(y) \sin\left(\left(N + \frac{1}{2} \right) y \right) dy = 0. \tag{5.14}$$

Proof. Since $\left\{ \frac{1}{\sqrt{2}}, \cos nx, \sin nx : n \in \mathbb{N} \right\}$ forms an orthonormal set, it follows from Corollary 5.7 that

$$\begin{aligned}
&\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}\left[|a_n|^2+|b_n|^2\right]\\
&\quad=\left(f,\frac{1}{\sqrt{2}}\right)^2+\sum_{n=1}^{\infty}(f,\cos n(\cdot))^2+(f,\sin n(\cdot))^2\\
&\quad\le (f,f)=\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx<\infty.
\end{aligned}$$

In particular this implies that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{-\pi}^{\pi}f(y)\cos ny\;dy=\pi\lim_{n\to\infty}b_n=0 \text{ and}\\
&\lim_{n\to\infty}\int_{-\pi}^{\pi}f(y)\sin ny\;dy=\pi\lim_{n\to\infty}a_n=0.
\end{aligned}$$

Since

$$\sin\left(\left(N+\frac{1}{2}\right)y\right)=\cos\left(\frac{1}{2}y\right)\sin(Ny)+\sin\left(\frac{1}{2}y\right)\cos(Ny),$$

Eq. (5.14) now follows from the previous limit formulas with f replaced by $f(y)\cos\frac{1}{2}y$ and $f(y)\sin\frac{1}{2}y$ respectively. ■

Proof of the Fourier Convergence Theorem 5.8.. To concentrate on the basic ideas of the argument, I am only going to give the proof under the additional assumption that f is continuously differentiable. The full proof may be found in the book.

Let

$$f_N(x)=\frac{1}{2}a_0+\sum_{n=1}^{N}\left[a_n\cos nx+b_n\sin nx\right]. \tag{5.15}$$

We begin by deriving a more tractable form for the function $f_N(x)$. To do this notice that

$$\begin{aligned}
&a_n\cos nx+b_n\sin nx\\
&=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}f(y)\cos ny\;dy\right)\cos nx+\frac{1}{\pi}\left(\int_{-\pi}^{\pi}f(y)\sin ny\;dy\right)\sin nx\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)(\cos ny\cos nx+\sin ny\sin nx)\,dy\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\cos n(x-y)dy
\end{aligned}$$

wherein the last equality we have used Eq. (3.14) with $t=nx$ and $s=-ny$. Using this observation in Eq. (5.15) shows

$$\begin{aligned}
f_N(x)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\,dy+\frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi}f(y)\cos n(x-y)dy\\
&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\,D_N(x-y)dy,
\end{aligned} \tag{5.16}$$

where

$$D_N(\theta)=\frac{1}{2}+\sum_{n=1}^{N}\cos n\theta=\frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}, \tag{5.17}$$

the second equality being Problem 14 of Section 32 of the book. (Also see Remark 5.18 below.)

To see what $D_N(\theta)$ looks like, see Figure 5.9.

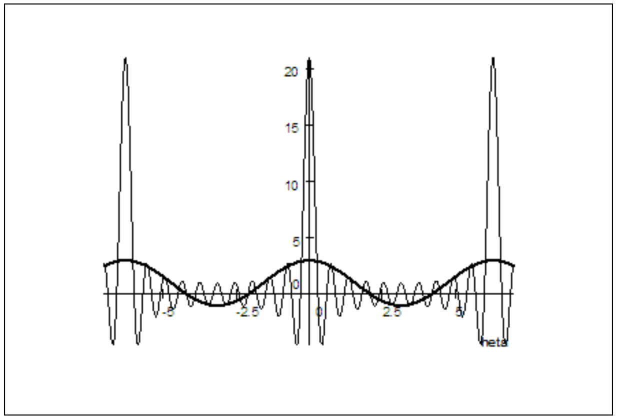


Fig. 5.9. This is a plot D_1 and D_{10}.

Making the change of variables, $z=x-y$ in Eq. (5.16) and using the fact that f and D_N are 2π - periodic, we have

$$\begin{aligned}
f_N(x)&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\,D_N(x-y)dy=-\frac{1}{\pi}\int_{x+\pi}^{x-\pi}f(x-z)\,D_N(z)dz\\
&=\frac{1}{\pi}\int_{\pi-x}^{\pi+x}f(x-z)\,D_N(z)dz=\frac{1}{\pi}\int_{\pi}^{\pi}f(x-z)\,D_N(z)dz.
\end{aligned}$$

Also notice that it follows from Eq. (5.17) that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(y)dy = (1, D_N)$$
$$= \left(1, \frac{1}{2} + \sum_{n=1}^{N} \cos n(\cdot)\right) = \frac{1}{2}(1,1) = 1.$$

Hence we may write

$$\begin{aligned} f_N(x) - f(x) &= \frac{1}{\pi}\int_{\pi}^{\pi} [f(x-z) - f(x)] D_N(z)dz \\ &= \frac{1}{\pi}\int_{\pi}^{\pi} \frac{f(x-z)-f(x)}{2\sin\frac{1}{2}z} \sin\left(\left(N+\frac{1}{2}\right)z\right)dz \\ &= \frac{1}{\pi}\int_{\pi}^{\pi} g(z)\sin\left(\left(N+\frac{1}{2}\right)z\right)dz \end{aligned} \tag{5.18}$$

where

$$g(z) := \begin{cases} \frac{f(x-z)-f(x)}{2\sin\frac{1}{2}z} & \text{if } z \neq 0 \\ -f'(z) & \text{if } z = 0. \end{cases}$$

Notice that, by l'Hopital's rule, g is continuous and hence

$$\frac{1}{\pi}\int_{\pi}^{\pi} |g(z)|^2 dz < \infty$$

and so we may let $N \to \infty$ in Eq. (5.18) with the aid of Lemma 5.16 to find

$$\lim_{N\to\infty} f_N(x) - f(x) = \frac{1}{\pi}\lim_{N\to\infty}\int_{\pi}^{\pi} g(z)\sin\left(\left(N+\frac{1}{2}\right)z\right)dz = 0.$$

■

The following strengthens the convergence of the sum in Eq. (5.7) when $: \mathbb{R} \to \mathbb{R}$ is piecewise C^1.

Theorem 5.17 (Uniform Convergence of Fourier Series). *If $f : \mathbb{R} \to \mathbb{R}$ is piecewise C^1 – the convergence in Eq. (5.7) is uniform. To be explicit, if we let $f_N(x)$ be as in Eq. (5.15), i.e.*

$$f_N(x) = \frac{1}{2\pi}a_0 + \frac{1}{\pi}\sum_{n=1}^{N}[a_n \cos nx + b_n \sin nx],$$

then

$$\max_x |f(x) - f_N(x)| \to 0 \text{ as } N \to \infty. \tag{5.19}$$

Proof. We have

$$\begin{aligned} \max_x |f(x) - f_N(x)| &= \frac{1}{\pi}\max_x \left|\sum_{n=N+1}^{\infty}[a_n \cos nx + b_n \sin nx]\right| \\ &\leq \frac{1}{\pi}\max_x \sum_{n=N+1}^{\infty}|a_n \cos nx + b_n \sin nx| \\ &\leq \frac{1}{\pi}\sum_{n=N+1}^{\infty}|a_n| + \frac{1}{\pi}\sum_{n=N+1}^{\infty}|b_n|. \end{aligned} \tag{5.20}$$

By the Cauchy-Schwarz inequality,

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty} n|a_n| \cdot \frac{1}{n} \leq \sqrt{\sum_{n=1}^{\infty}(n|a_n|)^2 \cdot \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)}. \tag{5.21}$$

Finally, by an integration by parts (where the boundary terms vanish using the 2π - periodicity of the function f) we find

$$\begin{aligned} na_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\, n\cos ny\, dy = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\frac{d}{dy}\cos ny\, dy \\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi} f'(y)\cos ny\, dy = -(f', \cos(n(\cdot))) \end{aligned}$$

Therefore by Bessel's inequality, with f replaced by f', it follows that $\sum_{n=1}^{\infty}(n|a_n|)^2 < \infty$ and so by Eq. (5.21), $\sum_{n=1}^{\infty}|a_n| < \infty$. Similarly we may also show that $\sum_{n=1}^{\infty}|b_n| < \infty$ and hence it Eq. (5.19) follows from Eq. (5.20). ■

Remark 5.18 (The Dirichlet kernel for those who know complex variables). Recall Euler's formula which states, $e^{i\theta} = \cos\theta + i\sin\theta$. With this notation we have

$$\begin{aligned} D_N(\theta) &= \frac{1}{2} + \sum_{n=1}^{N}\cos n\theta \\ &= \frac{1}{2} + \sum_{n=1}^{N}\frac{1}{2}\left(e^{in\theta} - e^{-in\theta}\right) = \frac{1}{2}\sum_{n=-N}^{N} e^{in\theta}. \end{aligned}$$

Letting $\alpha = e^{i\theta/2}$, we have

$$\begin{aligned} 2D_N(\theta) &= \sum_{n=-N}^{N}\alpha^{2n} = \frac{\alpha^{2(N+1)} - \alpha^{-2N}}{\alpha^2 - 1} = \frac{\alpha^{2N+1} - \alpha^{-(2N+1)}}{\alpha - \alpha^{-1}} \\ &= \frac{2i\sin(N+\frac{1}{2})\theta}{2i\sin\frac{1}{2}\theta} = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \end{aligned}$$

and therefore

$$D_N(\theta) := \frac{1}{2}\sum_{n=-N}^{N} e^{in\theta} = \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}, \tag{5.22}$$

with the understanding that the right side of this equation is $N+\frac{1}{2}$ whenever $\theta \in 2\pi\mathbb{Z}$.

5.5 Fourier Series on Other Intervals

Suppose $f(x)$ is defined for $-c < x < c$. By setting $F(y) := f\left(\frac{c}{\pi}y\right)$, we get a function defined for $-\pi < y < \pi$. This function may be expanded into a Fourier series as

$$f\left(\frac{c}{\pi}y\right) = F(y) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos ny + b_n \sin ny\right]$$

where

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}y\right)\cos ny\; dy \text{ and}$$
$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}y\right)\sin ny\; dy.$$

Making the change of variables, $x = \frac{c}{\pi}y$ (or $y = \frac{\pi}{c}x$) in the above equations gives

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\frac{\pi}{c}x + b_n \sin n\frac{\pi}{c}x\right]$$

where

$$a_n := \frac{1}{c}\int_{-c}^{c} f(x)\cos n\frac{\pi}{c}x\; dx$$
$$b_n := \frac{1}{c}\int_{-c}^{c} f(x)\sin n\frac{\pi}{c}x\; dx.$$

The convergence properties of these sum are the same as those for the Fourier series on $(-c, c)$. Similarly if $f(x)$ is defined on $0 < x < c$ then we have the sine and cosine series expansions

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\frac{\pi}{c}x \text{ for } 0 < x < c, \text{ and}$$
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{c}x \text{ for } 0 < x < c,$$

where now

$$a_n := \frac{2}{c}\int_0^c f(x)\cos n\frac{\pi}{c}x\; dx$$
$$b_n := \frac{2}{c}\int_0^c f(x)\sin n\frac{\pi}{c}x\; dx.$$

Example 5.19. We have seen

$$y = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin ny \text{ for } -\pi < y < \pi.$$

By letting $y = \frac{\pi}{c}x$ in the above formula, we conclude that

$$\frac{\pi}{c}x = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin n\frac{\pi}{c}x \text{ for } -c < x < c,$$

i.e.

$$x = \sum_{n=1}^{\infty}\frac{2c}{\pi n}(-1)^{n+1}\sin n\frac{\pi}{c}x \text{ for } -c < x < c.$$

Suppose that we want the cosine expansion of x on the interval, $0 < x < c$. In this case we will have

$$x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{c}x$$

with

$$\begin{aligned} a_n &= \frac{2}{c}\int_0^c x\cos n\frac{\pi}{c}x\; dx = \frac{2}{c}\int_0^c x\left(\frac{c}{\pi n}\right)\frac{d}{dx}\sin n\frac{\pi}{c}x\; dx \\ &= \frac{2}{c}x\left(\frac{c}{\pi n}\right)\sin n\frac{\pi}{c}x|_0^c - \frac{2}{c}\left(\frac{c}{\pi n}\right)\int_0^c \sin n\frac{\pi}{c}x\; dx \\ &= \frac{2}{c}\left(\frac{c}{\pi n}\right)^2\cos n\frac{\pi}{c}x|_0^c = \frac{2}{c}\left(\frac{c}{\pi n}\right)^2\left((-1)^n - 1\right) \end{aligned}$$

for $n \neq 0$ and

$$a_0 = \frac{2}{c}\int_0^c x\; dx = \frac{x^2}{c}|_0^c = c$$

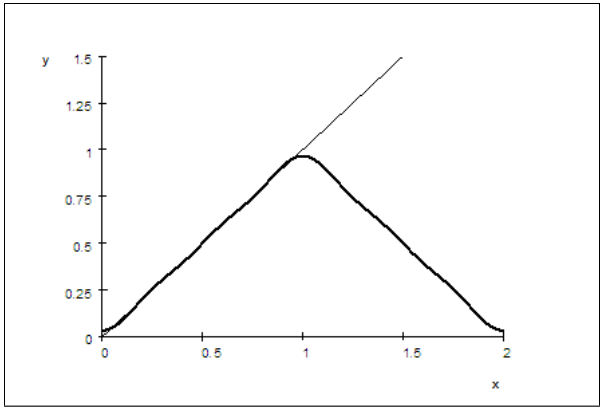
Thus we have

$$x = \frac{c}{2} - \frac{4c}{\pi^2}\sum_{n=1,3,5,\dots}\frac{1}{n^2}\cos n\frac{\pi}{c}x.$$

For example if $c = 1$, we have

$$x = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \dots\right),$$

see Figure 5.19 below.



A plot of x and its Fourier cosine series approximation,
$\frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x\right)$.

This series is convergent for all $0 \le x \le c$. Taking $x = 0$ in this expansion we may conclude that

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

or that

$$\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) = \frac{\pi^2}{8}$$

and taking $x = 1$ in this expansion we find

$$1 = \frac{1}{2} - \frac{4}{\pi^2}\left(-1 - \frac{1}{3^2}\cos 3\pi x - \frac{1}{5^2}\cos 5\pi x + \dots\right)$$

which again gives

$$\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) = \frac{\pi^2}{8}.$$

6

Boundary value generalities

6.1 Linear Algebra of the Strurm-Liouville Eigenvalue Problem

Suppose L is a differential operator on functions on $\Omega = [a, b]$ of the form

$$Lf(x) := \frac{1}{p(x)} \frac{d}{dx}\left(\kappa(x) \frac{d}{dx} f(x)\right) + \frac{1}{p(x)} q(x) f(x). \tag{6.1}$$

Define the **inner product,**

$$(f, g) := \int_{\Omega=[a,b]} f(x) g(x) p(x) dx,$$

for functions $f, g : \Omega \to \mathbb{R}$. With this notation we find

$$\begin{aligned}(Lf, g) &= \int_\Omega \left[\frac{d}{dx}\left(\kappa(x) \frac{d}{dx} f(x)\right) + q(x) f(x)\right] \cdot g(x) dx \\ &= \int_\Omega \left[-\kappa(x) \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x) + q(x) f(x) g(x)\right] dx + \kappa(x) f'(x) g(x)|_a^b \\ &= \int_\Omega f(x) \cdot \left[\frac{d}{dx}\left(\kappa(x) \frac{d}{dx} g(x)\right) + q(x) g(x)\right] dx \\ &+ \left[\kappa(x) \left(f'(x) g(x) - f(x) g'(x)\right)\right]|_a^b. \end{aligned} \tag{6.2}$$

Hence we see that

$$(Lf, g) - (f, Lg) = \left[\kappa(x) \left(f'(x) g(x) - f(x) g'(x)\right)\right]|_a^b. \tag{6.3}$$

Let (a_1, a_2) and (b_1, b_2) be two non-zero vectors in $\mathbb{R}^2$ and define

$$\begin{aligned} Bf(a) &= a_1 f(a) + a_2 f'(a) \text{ and} \\ Bf(b) &= b_1 f(b) + b_2 f'(b). \end{aligned}$$

In the sequel we will be interested on imposing the **boundary conditions** on $Bf(a) = 0 = Bf(b)$. If we assume that f and g satisfy these boundary conditions in Eq. (6.3) then it follows that $(f(a), f'(a))$ and $(g(a), g'(a))$ line on the same line and therefore

$$f'(a) g(a) - f(a) g'(a) = \det \begin{bmatrix} g(a) & g'(a) \\ f(a) & f'(a) \end{bmatrix} = 0.$$

Similar reasoning shows

$$f'(b) g(b) - f(b) g'(b) = \det \begin{bmatrix} g(b) & g'(b) \\ f(b) & f'(b) \end{bmatrix} = 0$$

and therefore it follows from Eq. (6.3) that L satisfies the symmetry condition,

$$(Lf, g) = (f, Lg) \text{ if } Bf = 0 = Bg \text{ on } \partial\Omega = \{a, b\}.$$

It also worth noting that if $\kappa(a) = \kappa(b)$ and f and g satisfy **periodic boundary conditions; i.e.**

$$f(a) = f(b) \text{ and } f'(a) = f'(a) \tag{6.4}$$

then we still have

$$(Lf, g) = (f, Lg).$$

As we will see we are going to be interested in the following eigenvalue problem, namely we will be interested in finding solutions to the eigenfunction equation,

$$Lf = -\lambda f \text{ with } Bf = 0 \text{ on } \partial\Omega.$$

This may be rewritten as

$$\frac{d}{dx}\left[\kappa(x) \frac{d}{dx} f(x)\right] + [q(x) + \lambda p(x)] f(x) = 0 \text{ with } Bf = 0 \text{ on } \partial\Omega.$$

This is the general form of the **Sturm-Liouville eigenvalue problem** as in Chapter 6 of the book. There $\kappa(x) = r(x)$.

Let D_B denote those functions $f : \Omega \to \mathbb{R}$ with are twice continuously differentiable and satisfy the boundary conditions, $Bf = 0$ on $\partial\Omega$ and if $\kappa(a) = \kappa(b)$ let D_{per} denote those functions $f : \Omega \to \mathbb{R}$ with are twice continuously differentiable and satisfy periodic boundary conditions in Eq. (6.4).

The next result is the formal analogue of Corollary 1.4.

Proposition 6.1. *If $f, g \in D_B$ or $\kappa(a) = \kappa(b)$ and $f, g \in D_{per}$ such that $Lf = -\lambda f$ and $Lg = -\mu g$ with $\mu \neq \lambda$, then $(f, g) = 0$.*

Proof. The same proof as that given for Corollary 1.4 work here without change. ∎

The next theorem is an analogue of the spectral theorem for the operator L.

Theorem 6.2 (Strurm-Liouville Spectral Theorem). *Let L be as above, assume $\kappa > 0$ on $[a,b]$ and let $D = D_B$ or $D = D_{per}$ (in which case we assume additionally that $\kappa(b) = \kappa(a)$), then there exists $u_n \in D$ and $\lambda_n \in \mathbb{R}$ such that:*

1. $-Lu_n = \lambda_n u_n$ *for all* n,
2. *the eigenvalues are increasing, i.e.*
$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots,$$
3. $\lim_{n\to\infty} \lambda_n = \infty$ *(in fact* $\#\{n : \lambda_n \le a\} \sim a^{1/2}$ *or equivalently* $\lambda_n \sim n^2$*).*
4. *Every "nice" function f on $[a,b]$ may be expanded as*
$$f(x) = \sum_{n=1}^{\infty} (f, u_n)\, u_n(x) = \sum_{n=1}^{\infty} \left[\int_a^b f(x)\, u_n(x)\, p(x)\, dx\right] u_n(x).$$

6.2 General Elliptic PDE Theory

In this section we will state the generalization of theorem 6.2 to higher dimensional situations. Let Ω be a "nice" open bounded region in $\mathbb{R}^d$ (typically we have in mind $d = 1, 2, 3$ here). Suppose that $\kappa_{i,j}(x)$ and $p(x)$ be smooth

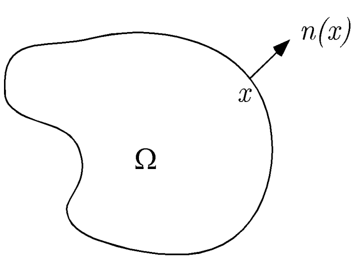


Fig. 6.1. A region represnting some material of some region in space.

functions on Ω such that matrix $\kappa(x)$

$$\kappa(x) := \begin{bmatrix} \kappa_{11}(x) & \kappa_{12}(x) & \dots & \kappa_{1d}(x) \\ \kappa_{21}(x) & \kappa_{22}(x) & \dots & \kappa_{2d}(x) \\ \vdots & & \ddots & \vdots \\ \kappa_{d1}(x) & \kappa_{d2}(x) & \dots & \kappa_{dd}(x) \end{bmatrix}$$

is positive definite, i.e. $\kappa^{\mathrm{tr}} = \kappa$ and $\kappa v \cdot v > 0$ for all $v \in \mathbb{R}^d$. We now form the inner product
$$(u, v) := \int_\Omega u(x)\, v(x)\, p(x)\, dV$$
for function u, v on Ω and let
$$Lu := \sum_{i,j=1}^{d} \frac{1}{p} \partial_i (\kappa_{ij} \partial_j u) + \gamma u. \tag{6.5}$$

Example 6.3. Look at the formula for the Laplacian in polar, cylindrical, and spherical coordinates to find natural operators written in this form. (There are some singularities involved here which are artifices of these coordinates system. We will have to deal with them later.) The general form of the heat equation also produced such operators L as in Eq. (6.5) above.

By the divergence Theorem 1.9,
$$\begin{aligned}
(Lu, v) &= \int_\Omega \left[\sum_{i,j=1}^{d} \frac{1}{p} \partial_i (\kappa_{ij} \partial_j u) + \gamma u\right] v p dV \\
&= \int_\Omega \left[\sum_{i,j=1}^{d} \partial_i (\kappa_{ij} \partial_j u) \cdot v + \gamma u v p\right] dV \\
&= \int_\Omega \left[-\sum_{i,j=1}^{d} \kappa_{ij} \partial_j u \partial_i v + \gamma u v p\right] dV + \int_{\partial\Omega} \sum_{i,j=1}^{d} (\kappa_{ij} \partial_j u)\, n_i v d\sigma \\
&= \int_\Omega [-\kappa \nabla u \cdot \nabla v + \gamma u v]\, p \cdot dV + \int_{\partial\Omega} (\kappa \nabla u \cdot n)\, v \; d\sigma
\end{aligned}$$

where $n(x)$ is the outward pointing normal and $d\sigma$ is the surface measure on $\partial\Omega$. By interchanging the roles of u and v in the above formula, it follows that
$$\begin{aligned}
(u, Lv) &= -\int_\Omega [\nabla u \cdot \kappa \nabla v + \gamma u v]\, dV + \int_{\partial\Omega} (\kappa \nabla v \cdot n)\, u \; d\sigma \\
&= -\int_\Omega [\kappa \nabla u \cdot \nabla v + \gamma u v]\, dV + \int_{\partial\Omega} (\kappa \nabla v \cdot n)\, u \; d\sigma
\end{aligned}$$

and therefore,

$$(Lu, v) - (u, Lv) = \int_{\partial\Omega} [(\kappa\nabla u \cdot n)\, v - (\kappa\nabla v \cdot n)\, u] \; d\sigma. \tag{6.6}$$

Let us now further suppose λ is a given function on $\partial\Omega$ and u and v satisfy the boundary conditions,

$$Bu\,(x) := \kappa\,(x)\, \nabla u\,(x) \cdot n\,(x) + \alpha\,(x)\, u\,(x) \overset{\text{set}}{=} 0$$

where we allow for $\alpha\,(x) = \infty$ by which we mean $u\,(x) = 0$ at such points. Then using these boundary condition in Eq. (6.6) shows that

$$[(\kappa\nabla u \cdot n)\, v - (\kappa\nabla v \cdot n)\, u] = \alpha vu - \alpha uv = 0 \text{ on } \partial\Omega$$

and hence we have

$$(Lu, v) = (u, Lv) \;\text{ whenever } Bu = Bv = 0.$$

The following is an analogue of the spectral theorem for matrices in this context.

Theorem 6.4. *Keeping the above set up, there exists an orthonormal set $\{u_n\}_{n=1}^{\infty}$ of eigenvectors for $(-L, B)$, i.e. $Bu_n = 0$ on $\partial\Omega$ and $-Lu_n = \lambda_n u_n$. Moreover these may be chosen so that:*

1. *the eigenvalues are increasing, i.e.*

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots,$$

2. $\lim_{n\to\infty} \lambda_n = \infty$ *(in fact* $\#\{n : \lambda_n \le a\} \sim a^{d/2}$ *or equivalently* $\lambda_n \sim n^{2/d}$*).*
3. *Every "nice" function f on Ω may be expanded as*

$$\begin{aligned} f\,(x) &= \sum_{n=1}^{\infty} (f, u_n)\, u_n\,(x) \\ &= \sum_{n=1}^{\infty} \left[\int_{\Omega} f\,(x)\, u_n\,(x)\, p\,(x)\, dV\right] u_n\,(x)\,. \end{aligned}$$

7

PDE Applications and Duhamel's Principle

7.1 Interpretation of d'Alembert's solution to the 1-d wave equation

Example 7.1. We may use d'Alembert's solution to the wave equation to formally work out the meaning of $\cos\left(t\sqrt{-a^2\partial_x^2}\right)$ and $\frac{\sin\left(t\sqrt{-a^2\partial_x^2}\right)}{\sqrt{-a^2\partial_x^2}}$. To see what we should get, let $A^2 = -a^2\partial_x^2$ and $A = \sqrt{-a^2\partial_x^2}$ then

$$y(t,x) = \cos\left(t\sqrt{-a^2\partial_x^2}\right) f(x) + \frac{\sin\left(t\sqrt{-a^2\partial_x^2}\right)}{\sqrt{-a^2\partial_x^2}} g(x) \text{ for } -\infty < x < \infty$$

should solve Eqs. (2.5) and (2.6). By comparing this with Eq. (2.7), d'Alembert's solution which I recall here,

$$y(t,x) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds,$$

we conclude that

$$\cos\left(t\sqrt{-a^2\partial_x^2}\right) f(x) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] \text{ and} \tag{7.1}$$

$$\frac{\sin\left(t\sqrt{-a^2\partial_x^2}\right)}{\sqrt{-a^2\partial_x^2}} g(x) = \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds. \tag{7.2}$$

We will use these results and the results of the next section to allow of to solve the forced wave equation.

7.2 Solving 1^{st} - order equations using 2^{nd} - order solutions

Lemma 7.2 (A Key Fourier Transform Formula). *For all $\lambda \in \mathbb{R}$ and $t > 0$,*

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cos(\lambda s)\, ds = e^{-t\lambda^2}. \tag{7.3}$$

Proof. Fix $t > 0$ and let

$$g(\lambda) := \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cos(\lambda s)\, ds.$$

Then

$$\begin{aligned} g'(\lambda) &= -\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} s \sin(\lambda s)\, ds \\ &= 2t\int_{-\infty}^{\infty} \frac{d}{ds}\frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cdot \sin(\lambda s)\, ds \\ &= -2t\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cdot \frac{d}{ds}\sin(\lambda s)\, ds \\ &= -2t\lambda\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cdot \cos(\lambda s)\, ds = -2t\lambda g(\lambda). \end{aligned}$$

Solving this ODE for g gives,

$$g(\lambda) = e^{-t\lambda^2} g(0).$$

This completes the proof since $g(0) = 1$ as we now show. Letting $s = \sqrt{t}x$,

$$\begin{aligned} g^2(0) &= \left(\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} ds\right)^2 = \left(\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4}x^2}}{\sqrt{4\pi}} dx\right)^2 \\ &= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4}x^2}}{\sqrt{4\pi}} dx \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4}y^2}}{\sqrt{4\pi}} dy = \iint_{\mathbb{R}^2} \frac{e^{-\frac{1}{4}\left(x^2+y^2\right)}}{4\pi} dxdy \\ &= \int_0^{\infty}\int_0^{2\pi} \frac{e^{-\frac{1}{4}r^2}}{4\pi} r dr d\theta = \frac{1}{2}\int_0^{\infty} e^{-\frac{1}{4}r^2} r dr = -e^{-\frac{1}{4}r^2}|_0^{\infty} = 1. \end{aligned}$$

wherein the fifth equality we have gone to polar coordinates. ■

Theorem 7.3 (Solving for e^{tA} via $\cos\left(\sqrt{-A}t\right)$). *Suppose A is a $N \times N$ symmetric matrix with all non-positive eigenvalues. Then*

$$e^{tA} = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cos\left(\sqrt{-A}s\right) ds \text{ for all } t > 0. \tag{7.4}$$

Proof. Formally we are taking $\lambda = \sqrt{-A}$ in Eq. (7.3). To rigorously prove Eq. (7.4), let $\{v_i\}_{i=1}^N$ be an orthonormal basis of eigenvectors for A. By assumption we may write eigenvalue for v_i as $-\lambda_i^2$, i.e. $Av_i = -\lambda_i^2 v_i$.

Then

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cos\left(\sqrt{-A}s\right) v_i ds = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cos\left(\lambda_i s\right) v_i ds$$
$$= e^{-t\lambda_i^2} v_i = e^{tA} v_i$$

and the result follows since both sides of Eq. (7.4) are linear. ∎

7.2.1 The Solution to the Heat Equation on $\mathbb{R}$

Example 7.4 (Heat Equation). Let us try to formally use Theorem 7.3 to solve the heat equation,

$$u_t(t,x) = \partial_x^2 u(t,x) \text{ with } u(0,x) = f(x).$$

According to theorem 7.3 and Example 7.1 with $a = 1$, the solution should be given by

$$\begin{aligned} u(t,x) = \left(e^{t\partial_x^2} f\right)(x) &= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \cos\left(\sqrt{-\partial_x^2}s\right) f(x)\, ds \\ &= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} \frac{1}{2}\left[f(x+s) + f(x-s)\right] ds \\ &= \int_{-\infty}^{\infty} f(x-s) \frac{e^{-\frac{1}{4t}s^2}}{\sqrt{4\pi t}} ds = \int_{-\infty}^{\infty} f(y) \frac{e^{-\frac{1}{4t}(x-y)^2}}{\sqrt{4\pi t}} dy. \end{aligned}$$

Exercise 7.1. Suppose f is a bounded continuous function, show

$$u(t,x) := \int_{-\infty}^{\infty} f(y) \frac{e^{-\frac{1}{4t}(x-y)^2}}{\sqrt{4\pi t}} dy = \int_{-\infty}^{\infty} f(y) p(t, x-y)\, dy \tag{7.5}$$

solves the heat equation, $u_t(t,x) = \partial_x^2 u(t,x)$ for $t > 0$ where

$$p(t,x) := \frac{e^{-\frac{1}{4t}x^2}}{\sqrt{4\pi t}}. \tag{7.6}$$

Hint: first show $p(t,x)$ solves the heat equation for $t > 0$. Then check u solves the heat equation by differentiating past the integral, which you should **assume** to be valid here.

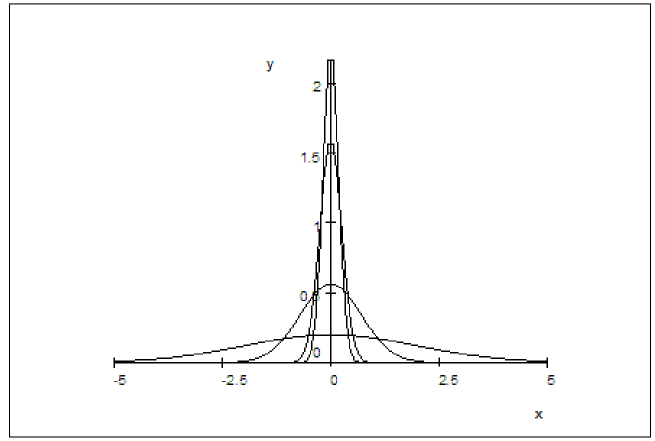


Fig. 7.1. Plots of $x \to p(t,x)$ for $t = 2$, $t = \frac{1}{4}$, $t = \frac{1}{32}$ and $t = \frac{1}{64}$. Notice that $p(t,x)$ is being more and more concentrated near $x = 0$ as $t \downarrow 0$ while always keeping the total area under $x \to p(t,x)$ equal to one.

It is a fact that we will discuss later that

$$\lim_{t\downarrow 0} u(t,x) = f(x). \tag{7.7}$$

This is based on the idea that $p(t,x)$ is approximating a "δ – function," see Figure 7.1

We will abbreviate all this by the suggestive formula,

$$\left(e^{t\partial_x^2} f\right)(x) = \int_{-\infty}^{\infty} f(y) p(t, x-y)\, dy. \tag{7.8}$$

7.3 Duhamel's Principle

Theorem 7.5 (Duhamel's Principle I). *Suppose A is an $N \times N$, $f \in \mathbb{R}^N$ and $h(t) \in \mathbb{R}^N$ be given. Then the ordinary differential equation,*

$$\dot{u}(t) = Au(t) + h(t) \text{ with} \tag{7.9}$$
$$u(0) = f \tag{7.10}$$

has a unique solution given by

$$u(t) = e^{tA} f + \int_0^t e^{(t-\tau)A} h(\tau)\, d\tau. \tag{7.11}$$

In words, $u(t)$ is constructed by "adding" together the solutions to a bunch of initial value problems where $h \equiv 0$. Namely, $e^{tA}f$ is the solution to

$$\dot{u}(t) = Au(t) \ \text{ with } u(0) = f$$

while $e^{(t-\tau)A}h(\tau)$ is the solution to

$$\dot{u}(t) = Au(t) \ \text{ with } u(\tau) = h(\tau).$$

Hence we have

$$u(t) = \begin{pmatrix} \text{solution at time } t \text{ to} \\ \dot{u}(t) = Au(t) \ \& \ u(0) = f \end{pmatrix} + \int_0^t \begin{pmatrix} \text{solution at time } t \text{ to} \\ \dot{u}(t) = Au(t) \ \& \ u(\tau) = h(\tau) \end{pmatrix} d\tau.$$

Proof. Suppose u solves Eq. (7.9), then by the product rule

$$\begin{aligned} \frac{d}{dt}\left[e^{-tA}u(t)\right] &= -Ae^{-tA}u(t) + e^{-tA}\dot{u}(t) \\ &= -e^{-tA}Au(t) + e^{-tA}\left(Au(t) + h(t)\right) \\ &= e^{-tA}h(t). \end{aligned}$$

Integrating this equation then shows

$$e^{-tA}u(t) = u(0) + \int_0^t e^{-\tau A}h(\tau)\,d\tau = f + \int_0^t e^{-\tau A}h(\tau)\,d\tau.$$

Multiplying this equation by e^{tA} on the left shows that if u exists it must be given by Eq. (7.11).

To prove existence, let u now be defined by Eq. (7.11) and notice that we may write it as

$$u(t) = e^{tA}\left[f + \int_0^t e^{-\tau A}h(\tau)\,d\tau\right].$$

Thus $u(0) = f$ and, by the product rule and the fundamental theorem of calculus,

$$\begin{aligned} \dot{u}(t) &= Ae^{tA}\left[f + \int_0^t e^{-\tau A}h(\tau)\,d\tau\right] + e^{tA}\left[e^{-tA}h(t)\right] \\ &= Au(t) + h(t). \end{aligned}$$

■

Example 7.6. Continuing the notation and using the results of Example 1.8,

$$A := \begin{bmatrix} 1 & 7 & -2 \\ 7 & 1 & -2 \\ -2 & -2 & 10 \end{bmatrix}$$

with eigenvectors/eigenvalues given by

$$v_1 := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow -6, \ v_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow 6, \ v_3 := \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \leftrightarrow 12.$$

We wish to solve,

$$\dot{u}(t) = Au(t) + tf \text{ with } u(0) = 0$$

where

$$f = (1,2,3)^{\text{tr}} = \frac{1}{2}v_1 + 2v_2 + \frac{1}{2}v_3$$

The solution is

$$\begin{aligned} u(t) &= \int_0^t \tau e^{(t-\tau)A} f\,d\tau \\ &= \int_0^t \tau\left(\frac{1}{2}e^{-6(t-\tau)}v_1 + 2e^{6(t-\tau)}v_2 + \frac{1}{2}e^{12(t-\tau)}v_3\right)d\tau. \end{aligned}$$

To work this out we notice that

$$\begin{aligned} \int_0^t \tau e^{-\lambda\tau}d\tau &= -\frac{d}{d\lambda}\int_0^t e^{-\lambda\tau}d\tau = -\frac{d}{d\lambda}\left[\frac{e^{-\lambda\tau}}{-\lambda}\Big|_{\tau=0}^{\tau=t}\right] \\ &= \frac{d}{d\lambda}\frac{e^{-\lambda t}-1}{\lambda} \\ &= \frac{1}{\lambda^2}\left(1-(t\lambda+1)e^{-t\lambda}\right) \end{aligned}$$

and therefore,

$$\begin{aligned} \int_0^t \tau e^{\lambda(t-\tau)}d\tau &= e^{\lambda t}\frac{1}{\lambda^2}\left(1-(t\lambda+1)e^{-t\lambda}\right) \\ &= \frac{1}{\lambda^2}\left(e^{\lambda t}-(1+t\lambda)\right). \end{aligned}$$

Hence the answer is given by

$$u(t) = \begin{pmatrix} \frac{1}{2\cdot 6^2}\left(e^{-6t}-1+6t\right)v_1 + \frac{2}{6^2}\left(e^{6t}-1-6t\right)v_2 \\ +\frac{1}{2\cdot 12^2}\left(e^{12t}-1-12t\right)v_3 \end{pmatrix}.$$

Theorem 7.7 (Duhamel' s Principle II). *Suppose A is an $N \times N$, $f, g \in \mathbb{R}^N$ and $h(t) \in \mathbb{R}^N$ be given. Then the ordinary differential equation,*

$$\ddot{u}(t) = Au(t) + h(t) \text{ with} \tag{7.12}$$

$$u(0) = f \text{ and } \dot{u}(0) = g \tag{7.13}$$

has a unique solution given by

$$u(t) = \left(\cos\sqrt{-A}t\right) f + \frac{\sin\sqrt{-A}t}{\sqrt{-A}} g + \int_0^t \frac{\sin\sqrt{-A}(t-\tau)}{\sqrt{-A}} h(\tau)\, d\tau. \tag{7.14}$$

Again, in words, $u(t)$ is constructed by "adding" the solutions to a bunch of initial value problems where $h \equiv 0$. Namely,

$$\left(\cos\sqrt{-A}t\right) f + \frac{\sin\sqrt{-A}t}{\sqrt{-A}} g$$

is the solution to

$$\ddot{u}(t) = Au(t) \text{ with } u(0) = f \text{ and } \dot{u}(0) = g$$

while $\frac{\sin\sqrt{-A}(t-\tau)}{\sqrt{-A}} h(\tau)$ is the solution to

$$\ddot{u}(t) = Au(t) \text{ with } u(\tau) = 0 \text{ and } \dot{u}(\tau) = h(\tau)$$

so

$$u(t) = \begin{pmatrix} \text{solution at time } t \text{ to} \\ \ddot{u}(t) = Au(t) \text{ with} \\ u(0) = f \ \&\ \dot{u}(0) = g \end{pmatrix} + \int_0^t \begin{pmatrix} \text{solution at time } t \text{ to} \\ \ddot{u}(t) = Au(t) \text{ with} \\ u(\tau) = 0 \ \&\ \dot{u}(\tau) = h(\tau) \end{pmatrix} d\tau.$$

Proof. The best way to understand this theorem is to reduce it to the first version of Duhamel's principle in Theorem 7.5. To this end, let $v(t) = \dot{u}(t)$, then the pair $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in \mathbb{R}^N \times \mathbb{R}^N$ solves the equation,

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \ddot{u}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ Au(t) + h(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ h(t) \end{pmatrix}.$$

Let

$$B := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

– a $2N \times 2N$ matrix. Then by Theorem 7.5, we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{tB}\begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{(t-\tau)B}\begin{pmatrix} 0 \\ h(\tau) \end{pmatrix} d\tau. \tag{7.15}$$

When $h \equiv 0$, we know that

$$u(t) = \left(\cos\sqrt{-A}t\right) f + \frac{\sin\sqrt{-A}t}{\sqrt{-A}} g$$

from which we deduce

$$e^{tB}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \left(\cos\sqrt{-A}t\right) f + \frac{\sin\sqrt{-A}t}{\sqrt{-A}} g \\ * \end{pmatrix}. \tag{7.16}$$

The second component of $e^{tB}\begin{pmatrix} f \\ g \end{pmatrix}$ is easily found as well (just differentiate the first component) but we will not need it. Because of Eq. (7.16),

$$e^{(t-\tau)B}\begin{pmatrix} 0 \\ h(\tau) \end{pmatrix} = \begin{pmatrix} \frac{\sin\sqrt{-A}(t-\tau)}{\sqrt{-A}} h(\tau) \\ * \end{pmatrix}. \tag{7.17}$$

Hence taking the first component of Eq. (7.15), using Eqs. (7.16) and (7.17), gives Eq. (7.14). ■

Example 7.8. Continuing the notation and using the results of Example 1.8,

$$A := \begin{bmatrix} 1 & 7 & -2 \\ 7 & 1 & -2 \\ -2 & -2 & 10 \end{bmatrix}$$

with eigenvectors/eigenvalues given by

$$v_1 := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \leftrightarrow -6, \ v_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow 6, \ v_3 := \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \leftrightarrow 12.$$

We wish to solve,

$$\ddot{u}(t) = Au(t) + h \text{ with } u(0) = 0 = \dot{u}(0)$$

where

$$h = (1,2,3)^{\text{tr}} = \frac{1}{2}v_1 + 2v_2 + \frac{1}{2}v_3.$$

The solution is given by

$$\begin{aligned} u(t) &= \int_0^t \frac{\sin\sqrt{-A}\,(t-\tau)}{\sqrt{-A}} h d\tau \\ &= \int_0^t \frac{\sin\sqrt{-A}\,(t-\tau)}{\sqrt{-A}} \left(\frac{1}{2}v_1 + 2v_2 + \frac{1}{2}v_3\right) d\tau \\ &= \int_0^t \begin{pmatrix} \frac{\sin\sqrt{6}(t-\tau)}{\sqrt{6}}\frac{1}{2}v_1 + \frac{\sinh\sqrt{6}(t-\tau)}{\sqrt{6}}2v_2 \\ +\frac{\sinh\sqrt{12}(t-\tau)}{\sqrt{12}}\frac{1}{2}v_3 \end{pmatrix} d\tau. \end{aligned}$$

Now

$$\int_0^t \frac{\sin a(t-\tau)}{a} d\tau = \frac{\cos a(t-\tau)}{a^2}|_{\tau=0}^{\tau=t} = \frac{1-\cos at}{a^2}$$

and similarly,

$$\int_0^t \frac{\sinh a(t-\tau)}{a} d\tau = -\frac{\cosh a(t-\tau)}{a^2}|_{\tau=0}^{\tau=t} = -\frac{1-\cosh at}{a^2}$$

so that

$$u(t) = \frac{1-\cos\sqrt{6}t}{6}\frac{1}{2}v_1 + \frac{\cosh\sqrt{6}t - 1}{6}2v_2 + \frac{\cosh\sqrt{12}t - 1}{12}\frac{1}{2}v_3.$$

Let us do a quick check that this solution is correct. For example let us check that

$$u(t) = \frac{1-\cos\sqrt{6}t}{6}v_1$$

solves $\ddot{u}(t) = Au(t) + v_1$. This is the case since,

$$\begin{aligned} \ddot{u}(t) - Au(t) &= \cos\sqrt{6}t \cdot v_1 - Au(t) \\ &= \cos\sqrt{6}t \cdot v_1 - \frac{1-\cos\sqrt{6}t}{6}Av_1 \\ &= \cos\sqrt{6}t \cdot v_1 + \left(1-\cos\sqrt{6}t\right)v_1 = v_1 \end{aligned}$$

as desired.

Exercise 7.2. Let

$$A := \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and $h = (-1,1,0)^{\text{tr}}$. Solve the following equations for u,

$$\begin{aligned} &\dot{u}(t) = Au(t) + h \text{ with } u(0) = \mathbf{0} = (0,0,0)^{\text{tr}} \text{ and} \\ &\ddot{u}(t) = Au(t) + h \text{ with } u(0) = \mathbf{0} \text{ and } \dot{u}(0) = \mathbf{0}. \end{aligned}$$

Write your solutions in the form

$$u(t) = \sum_{i=1}^3 a_i(t)\,v_i$$

where the functions a_i are to be determined.

7.4 Application of Duhamel's principle to 1 - *d* wave and heat equations

Using the formulas for $\cos\left(t\sqrt{-\partial_x^2}\right)$ and $\frac{\sin\left(t\sqrt{-\partial_x^2}\right)}{\sqrt{-\partial_x^2}}$ in Eqs. (7.1) and (7.2) respectively we are now in a position to formally apply Duhamel's principle in order to solve the forced wave equation;

$$u_{tt} = u_{xx} + h \text{ with } u(0,\cdot) = f \text{ and } u_t(0,\cdot) = g. \tag{7.18}$$

Theorem 7.9. *If $f \in C^2(\mathbb{R},\mathbb{R})$ and $g \in C^1(\mathbb{R},\mathbb{R})$, and $h \in C(\mathbb{R}^2,\mathbb{R})$ such that h_x exists and $h_x \in C(\mathbb{R}^2,\mathbb{R})$, then Eq. (7.18) has a unique solution $u(t,x)$ given by Eq. (7.19).*

Proof. By a formal application of Theorem 7.5 with $A = \partial_x^2$ suggest that

$$u(t,\cdot) = \cos(t\sqrt{-\partial_x^2})f + \frac{\sin(t\sqrt{-\partial_x^2})}{\sqrt{-\partial_x^2}}g + \int_0^t \frac{\sin((t-\tau)\sqrt{-\partial_x^2})}{\sqrt{-\partial_x^2}}h(\tau,\cdot)d\tau.$$

Moreover using the formulas in Eqs. (7.1) and (7.2) then implies

$$u(t,x) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{-t}^t g(x+s)ds + \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} dy\ h(\tau,y). \tag{7.19}$$

To verify that u defined in Eq. (7.19) satisfies Eq. (7.18) it suffices (by what we have already done) to assume $f = g = 0$ so that

$$u(t,x) = \frac{1}{2}\int_0^t d\tau \int_{x-t+\tau}^{x+t-\tau} dy\ h(\tau,y).$$

Now

$$
\begin{aligned}
u_t &= \frac{1}{2}\int_0^t \left[h(\tau, x+t-\tau) + h(\tau, x-t+\tau)\right] d\tau, \\
u_{tt} &= \frac{1}{2}\int_0^t \left[h_x(\tau, x+t-\tau) - h_x(\tau, x-t+\tau)\right] d\tau + h(t,x) \\
u_x(t,x) &= \frac{1}{2}\int_0^t d\tau \left[h(\tau, x+t-\tau) - h(\tau, x-t+\tau)\right] \text{ and} \\
u_{xx}(t,x) &= \frac{1}{2}\int_0^t d\tau \left[h_x(\tau, x+t-\tau) - h_x(\tau, x-t+\tau)\right]
\end{aligned}
$$

so that $u_{tt} - u_{xx} = h$ and $u(0,x) = u_t(0,x) = 0$.

The only thing left to prove is the uniqueness assertion. For this suppose that v is another solution, then $(u-v)$ solves the wave equation (7.18) with $f = g = 0$ and hence by the uniqueness assertion in Theorem 2.4 (with $a = 1$), $u - v \equiv 0$. ■

Similarly we may solve the forced heat equation as well.

Theorem 7.10 (The Forced Heat Equation). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are bounded continuous functions, then the function*

$$
u(t,x) = \int_{-\infty}^{\infty} p(t, x-y) f(y)\, dy + \int_0^t d\tau \int_{-\infty}^{\infty} p(t-\tau, x-y)\, h(\tau, y)\, dy \tag{7.20}
$$

solves the heat equation

$$
u_t = u_{xx} + h \text{ with } \lim_{t \downarrow 0} u(t,x) = f(x). \tag{7.21}
$$

Proof. Formally applying Theorem 7.5 with $A = \partial_x^2$ suggests that

$$
u(t,x) = e^{t\partial_x^2} f(x) + \int_0^t \left[e^{(t-\tau)\partial_x^2} h(\tau, \cdot)\right](x)\, d\tau.
$$

In light of Eq. (7.8), this equation then gives rise to Eq. (7.20). It is of course possible to directly check that Eq. (7.20) solves Eq. (7.21), however I will not stop to do it here. ■

A

Some Complex Variables Facts

Here we suppose $w(t) = c(t) + id(t)$ where $c(t)$ and $d(t)$ are two real valued functions of t. An important example of such a complex valued function is found in the next definition.

Definition A.1 (Euler's Formula). *For $t \in \mathbb{R}$ let*

$$e^{it} := \cos t + i \sin t \tag{A.1}$$

and for $z = x + iy$ let

$$e^{z} := e^{x} e^{iy} = e^{x} (\cos y + i \sin y). \tag{A.2}$$

Notice that any complex number, $z = x + iy$, may be written as $z = re^{i\theta}$ where (r, θ) are the polar coordinates of the point $(x, y) \in \mathbb{R}^2$.

Definition A.2. *If $c(t)$ and $d(t)$ are differentiable, then we define*

$$\dot{w}(t) := \dot{c}(t) + i\dot{d}(t)$$

and

$$\int_{\alpha}^{\beta} w(t)\, dt := \int_{\alpha}^{\beta} c(t)\, dt + i \int_{\alpha}^{\beta} d(t)\, dt$$

Example A.3. If $w(t) = e^{t} + i \sin t$, then

$$\dot{w}(t) = e^{t} - i \cos t \text{ and}$$

$$\int_{0}^{\pi/2} w(t)\, dt = \int_{0}^{\pi/2} \left(e^{t} + i \sin t\right) dt = e^{\frac{1}{2}\pi} - 1 + i.$$

Example A.4. Suppose $w(t) = e^{it}$, then

$$\begin{aligned} \frac{d}{dt} e^{it} &= \frac{d}{dt} (\cos t + i \sin t) = -\sin t + i \cos t \\ &= i (\cos t + i \sin t) = ie^{it} \end{aligned}$$

and

$$\begin{aligned} \int_{a}^{b} e^{it} dt = \int_{a}^{b} (\cos t + i \sin t)\, dt &= \int_{a}^{b} \cos t dt + i \int_{a}^{b} \sin t dt \\ = (\sin t - i \cos t)|_{a}^{b} &= \frac{e^{it}}{i}|_{a}^{b}. \end{aligned}$$

Example A.6 below, generalizes this result.

Theorem A.5 (These definitions work just as in real variables). *If $z(t) = a(t) + ib(t)$ and $w(t) = c(t) + id(t)$ and $\lambda = u + iv \in \mathbb{C}$ then*

1. $\frac{d}{dt}(w(t) + z(t)) = \dot{w}(t) + \dot{z}(t)$
2. $\frac{d}{dt}[w(t)\, z(t)] = w\dot{z} + \dot{w}z$
3. $\int_{\alpha}^{\beta} [w(t) + \lambda z(t)]\, dt = \int_{\alpha}^{\beta} w(t)\, dt + \lambda \int_{\alpha}^{\beta} z(t)\, dt$
4. $\int_{\alpha}^{\beta} \dot{w}(t) dt = w(\beta) - w(\alpha)$ *In particular if $\dot{w} = 0$ then w is constant.*
5. $$\int_{\alpha}^{\beta} \dot{w}(t) z(t) dt = -\int_{\alpha}^{\beta} w(t) \dot{z}(t) dt + w(t)\, z(t)\, |_{\alpha}^{\beta}.$$
6. $$\left|\int_{\alpha}^{\beta} w(t)\, dt\right| \le \int_{\alpha}^{\beta} |w(t)|\, dt.$$

Proof. 1. and 4. are easy.
2.

$$\begin{aligned} \frac{d}{dt}[wz] &= \frac{d}{dt}(ac - bd) + i \frac{d}{dt}(bc + ad) \\ &= (\dot{a}c - \dot{b}d) + i(\dot{b}c + \dot{a}d) \\ &+ (a\dot{c} - b\dot{d}) + i(b\dot{c} + a\dot{d}) \\ &= \dot{w}z + w\dot{z}. \end{aligned}$$

3. The only interesting thing to check is that

$$\int_{\alpha}^{\beta} \lambda z(t)\, dt = \lambda \int_{\alpha}^{\beta} z(t)\, dt.$$

Again we simply write out the real and imaginary parts:

$$\begin{aligned} \int_{\alpha}^{\beta} \lambda z(t)\, dt &= \int_{\alpha}^{\beta} (u + iv)(a(t) + ib(t))\, dt \\ &= \int_{\alpha}^{\beta} (ua(t) - vb(t) + i[ub(t) + va(t)])\, dt \\ &= \int_{\alpha}^{\beta} (ua(t) - vb(t))\, dt + i \int_{\alpha}^{\beta} [ub(t) + va(t)]\, dt \end{aligned}$$

while

$$\begin{aligned}\int_\alpha^\beta \lambda z(t)\,dt &= (u+iv)\int_\alpha^\beta [a(t)+ib(t)]\,dt\\ &= (u+iv)\left(\int_\alpha^\beta a(t)\,dt + i\int_\alpha^\beta b(t)\,dt\right)\\ &= \int_\alpha^\beta (ua(t)-vb(t))\,dt + i\int_\alpha^\beta [ub(t)+va(t)]\,dt.\end{aligned}$$

Shorter Alternative: Just check it for $\lambda = i$, this is the only new thing over the real variable theory.

5.

$$w(t)\,z(t)\,|_\alpha^\beta = \int_\alpha^\beta \frac{d}{dt}[w(t)\,z(t)]\,dt = \int_\alpha^\beta \dot{w}(t)z(t)dt + \int_\alpha^\beta w(t)\dot{z}(t)dt.$$

6. (Skip this one!) Let $\rho \geq 0$ and $\theta \in \mathbb{R}$ be chosen so that

$$\int_\alpha^\beta w(t)\,dt = \rho e^{i\theta},$$

then

$$\begin{aligned}\left|\int_\alpha^\beta w(t)\,dt\right| &= \rho = e^{-i\theta}\int_\alpha^\beta w(t)\,dt = \int_\alpha^\beta e^{-i\theta}w(t)\,dt\\ &= \int_\alpha^\beta \operatorname{Re}\left[e^{-i\theta}w(t)\right]dt \leq \int_\alpha^\beta \left|\operatorname{Re}\left[e^{-i\theta}w(t)\right]\right|dt\\ &\leq \int_\alpha^\beta \left|e^{-i\theta}w(t)\right|dt = \int_\alpha^\beta |w(t)|\,dt.\end{aligned}$$

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Example A.6. Suppose $z = x+iy$, then

$$e^{zt} := e^{xt}e^{iyt} := e^{xt}\cos yt + ie^{xt}\sin yt$$

and so, again by definition,

$$\begin{aligned}\frac{d}{dt}e^{zt} &= \frac{d}{dt}\left(e^{xt}\cos yt + ie^{xt}\sin yt\right)\\ &= e^{xt}(x\cos yt - y\sin yt) + ie^{xt}(x\sin yt + y\cos yt)\\ &= (x+iy)\left(e^{xt}\cos yt + ie^{xt}\sin yt\right) = ze^{zt}.\end{aligned}$$

A better proof. By the product rule and Example A.4,

$$\frac{d}{dt}e^{tz} = \frac{d}{dt}\left[e^{tx}e^{ity}\right] = xe^{tx}e^{ity} + e^{tx}iye^{ity} = ze^{tx}e^{ity} = ze^{tz}.$$

Using this fact and item 4. of Theorem A.5 we may conclude,

$$\int_a^b e^{zt}dt = \frac{e^{zt}}{z}|_a^b.$$

If we write out what this means by comparing the real and imaginary parts of both sides we find

$$\int_a^b e^{zt}dt = \int_a^b e^{xt}\cos(yt)\,dt + i\int_a^b e^{xt}\sin(yt)\,dt$$

while

$$\begin{aligned}\frac{e^{zt}}{z} &= \frac{e^{xt}\cos yt + ie^{xt}\sin yt}{x+iy}\frac{x-iy}{x-iy}\\ &= \frac{e^{xt}}{x^2+y^2}\left[x\cos yt + y\sin yt + i(x\sin yt - y\cos yt)\right]\end{aligned}$$

from which we may conclude that

$$\begin{aligned}\int_a^b e^{xt}\cos(yt)\,dt &= \frac{e^{xt}}{x^2+y^2}\left[x\cos yt + y\sin yt\right]\Big|_a^b \text{ and}\\ \int_a^b e^{xt}\sin(yt)\,dt &= \frac{e^{xt}}{x^2+y^2}\left[x\sin yt - y\cos yt\right]\Big|_a^b.\end{aligned}$$

Theorem A.7 (Addition formula for e^z). *The function e^z defined by Eq. (A.2) satisfies*

Proposition A.8. *1. $e^{-z} = \frac{1}{e^z}$ and*
2. $e^{w+z} = e^w e^z$.

Proof. By the previous example we know

$$\frac{d}{dt}e^{tz} = ze^{tz} \text{ with } e^{0z} = e^{0+i0} = 1.$$

Similarly, using the chain rule or by direct computation, one shows

$$\frac{d}{dt}e^{-tz} = -ze^{tz} \text{ with } e^{0z} = e^{0+i0} = 1.$$

1. By the product rule,

$$\frac{d}{dt}\left[e^{-tz}e^{tz}\right] = -ze^{-tz}e^{tz} + e^{-tz}ze^{tz} = 0$$

and therefore $e^{-tz}e^{tz}$ is independent of t and hence $e^{-tz}e^{tz} = e^{-0z}e^{0z} = 1$. Taking $t = 1$ proves 1.

2. Again by the product rule shows

$$\begin{aligned}\frac{d}{dt}\left[e^{-t(w+z)}e^{tw}e^{tz}\right] &= \begin{bmatrix} -(w+z)\,e^{-t(w+z)}e^{tw}e^{tz} \\ +e^{-t(w+z)}we^{tw}e^{tz} + e^{-t(w+z)}e^{tw}ze^{tz}\end{bmatrix} \\ &= 0\end{aligned}$$

and so $e^{-t(w+z)}e^{tw}e^{tz} = e^{-t(w+z)}e^{tw}e^{tz}|_{t=0} = 1$. Taking $t = 1$ then shows $e^{-(w+z)}e^{w}e^{z} = 1$ and then using Item 1. we get item 2. ■

Corollary A.9 (Addition formulas cos and sin). *For $\alpha, \beta \in \mathbb{R}$ we have*

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \cos\beta\sin\alpha.\end{aligned}$$

Proof. These follow by comparing the real and imaginary parts of the identity

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

while

$$\begin{aligned}e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i\left(\cos\alpha\sin\beta + \cos\beta\sin\alpha\right).\end{aligned}$$

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