C

Review

The following topics were covered in this course.

1. **Linear ODE** associated to a symmetric matrix, A, with $\{v_i\}_{i=1}^N$ being an orthogonal basis of eigenvectors of A with $Av_i = \lambda_i v_i$.
   a) If $f \in \mathbb{R}^N$ we have
   $$f = \sum_{i=1}^N \left(f, |v_i|^{-1} v_i\right) |v_i|^{-1} v_i = \sum_{i=1}^N \frac{(f, v_i)}{|v_i|^2} v_i.$$
   b) The first order linear equation
   $$\dot{u}(t) = Au(t) + h(t) \text{ with } u(0) = f$$
   has solution
   $$u(t) = e^{tA} f + \int_0^t e^{(t-\tau)A} h(\tau)\, d\tau.$$
   where
   $$e^{tA} f = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} e^{tA} v_i = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} e^{t\lambda_i} v_i.$$
   c) The second order ordinary differential equation,
   $$\ddot{u}(t) = Au(t) + h(t) \text{ with}$$
   $$u(0) = f \text{ and } \dot{u}(0) = g$$
   has a unique solution given by
   $$u(t) = \left(\cos\sqrt{-A}t\right) f + \frac{\sin\sqrt{-A}t}{\sqrt{-A}} g + \int_0^t \frac{\sin\sqrt{-A}(t-\tau)}{\sqrt{-A}} h(\tau)\, d\tau$$
   where
   $$\cos\sqrt{-A}tf = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} \cos\sqrt{-A}t v_i = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} \left(\cos\sqrt{-\lambda_i}t\right) v_i$$
   and
   $$\frac{\sin\sqrt{-A}t}{\sqrt{-A}} g = \sum_{i=1}^N \frac{(g, v_i)}{\|v_i\|^2} \frac{\sin\sqrt{-A}t}{\sqrt{-A}} v_i = \sum_{i=1}^N \frac{(g, v_i)}{\|v_i\|^2} \left(\frac{\sin\sqrt{-\lambda_i}t}{\sqrt{-\lambda_i}}\right) v_i$$
   See Notation 3.14.
2. **Generalities about inner products** on function spaces, see Chapter 5.1.
3. **PDE Examples**
   a) The Wave Equation
   $$u_{tt}(t,x) = \frac{H}{\delta(x)} u_{xx}(t,x) - g - \frac{k(x)}{\delta(x)} u_t(t,x) \text{ and}$$
   $$u_{tt}(t,x) = a^2 \Delta u(t,x) + h(t,x)$$
   b) d'Alembert's solution to
   $$u_{tt} = a^2 u_{xx} \text{ with}$$
   $$u(0,x) = f(x) \text{ and } u_t(0,x) = g(x)$$
   is given by
   $$u(t,x) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds.$$
   c) Heat Equation
   $$u_t(t,x) = Lu(t,x) + h(t,x)$$
   where
   $$Lf(x) := \frac{1}{p(x)} \frac{d}{dx}\left(\kappa(x) \frac{d}{dx} f(x)\right) + \frac{1}{p(x)} q(x) f(x).$$
   Such equation come from looking at the heat equation
   $$u_t = \Delta u + h \text{ on some domain in } \mathbb{R}^N$$
   in spherical and cylindrical coordinates.
   d) Equilibrium solutions.
4. (**Strurm-Liouville Spectral Theorem.**) Let L be as above, assume $\kappa > 0$ on $[a,b]$ and let $D = D_B$ or $D = D_{per}$ (in which case we assume additionally that $\kappa(b) = \kappa(a)$), then there exists $u_n \in D$ and $\lambda_n \in \mathbb{R}$ such that:
   a) $-Lu_n = \lambda_n u_n$ for all n,
   b) the eigenvalues are increasing, i.e.
   $$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

c) $\lim_{n\to\infty}\lambda_n = \infty$ (in fact $\#\{n : \lambda_n \le a\} \sim a^{1/2}$ or equivalently $\lambda_n \sim n^2$).

d) Every "nice" function f on $[a,b]$ may be expanded as

$$f(x) = \sum_{n=1}^{\infty} (f, u_n)\, u_n(x) = \sum_{n=1}^{\infty} \left[\int_a^b f(x)\, u_n(x)\, p(x)\, dx\right] u_n(x).$$

5. **Fourier Series**

a) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π - periodic function which is piecewise continuous on $(-\pi,\pi)$. Then at points $x \in \mathbb{R}$ where $f'(x\pm)$ exist we have

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] = \frac{f(x+) + f(x-)}{2}$$

where

$$a_n := (f, \cos n(\cdot)) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \cos ny\ dy \text{ for } n = 0, 1, 2, \dots$$

and

$$b_n := (f, \sin n(\cdot)) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \sin ny\ dy \text{ for } n = 1, 2, \dots.$$

Moreover we have

$$\begin{aligned}(f,f) &= \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \left(\frac{1}{2}a_0\right)^2 2\pi + \sum_{n=1}^{\infty}\left[a_n^2 \int_{-\pi}^{\pi} \cos^2 nx dx + b_n^2 \int_{-\pi}^{\pi} \sin^2 nx dx\right] \\ &= \frac{\pi}{2}a_0^2 + \pi \sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right].\end{aligned}$$

This comes from the Strurm-Liouville problems with $L = \frac{d^2}{dx^2}$ with periodic boundary conditions.

b) (**Cosine Expansion**) Let $f : [0,\pi] \to \mathbb{R}$ be a piecewise continuous function, then at points $x \in [0,\pi]$ where $f'(x\pm)$ exist we have

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n := \frac{2}{\pi}\int_0^{\pi} f(y) \cos ny\ dy$$

and

$$\begin{aligned}(f,f) = \int_0^{\pi} |f(x)|^2 dx &= \left(\frac{1}{2}a_0\right)^2 \pi + \sum_{n=1}^{\infty}\left[a_n^2 \int_0^{\pi} \cos^2 nx dx\right] \\ &= \frac{\pi}{4}a_0^2 + \frac{\pi}{2}\sum_{n=1}^{\infty} a_n^2.\end{aligned}$$

This comes from the Strurm-Liouville problems with $L = \frac{d^2}{dx^2}$ with Neumann boundary conditions.

c) (**Sine Expansion**) Let $f : [0,\pi] \to \mathbb{R}$ be a piecewise continuous function, then at points $x \in [0,\pi]$ where $f'(x\pm)$ exist we have

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin nx$$

(except at $x = 0$ and $x = \pi$ where the right side is always zero), where

$$b_n := \frac{2}{\pi}\int_0^{\pi} f(y) \sin ny\ dy.$$

Moreover,

$$(f,f) = \int_0^{\pi} |f(x)|^2 dx = \sum_{n=1}^{\infty}\left[b_n^2 \int_0^{\pi} \sin^2 nx dx\right] = \frac{\pi}{2}\sum_{n=1}^{\infty} a_n^2.$$

This comes from the Strurm-Liouville problems with $L = \frac{d^2}{dx^2}$ with Dirichlet boundary conditions.

6. **Separation of variables techniques** for solving PDE. For example if we want to find the eigenfunctions and eigenvalues to $\Delta u = \lambda u$ with $u = u(x,y)$ for $(x,y) \in [0,\pi]^2$ with Dirichlet Boundary conditions. Then separation of variables would give

$$u_{m,n}(x,y) = \sin mx \cdot \sin ny \tag{C.1}$$

is an orthogonal basis with $\Delta u_{m,n} = -\left(m^2 + n^2\right) u_{m,n}$ and

$$\int_0^{\pi} dx \int_0^{\pi} dy u_{m,n}(x,y)\, u_{m',n'}(x,y) = \delta_{m,m'}\delta_{n,n'}\left(\frac{\pi}{2}\right)^2.$$

Therefore we have the expansion

$$f(x,y)=\sum_{m,n=1}^{\infty}B_{m,n}\sin mx\cdot\sin ny$$

where

$$B_{m,n}=\frac{\int_0^\pi dx\int_0^\pi dy\ u_{m,n}(x,y)\,f(x,y)}{\left(\frac{\pi}{2}\right)^2}=\left(\frac{2}{\pi}\right)^2\int_0^\pi dx\int_0^\pi dy\,u_{m,n}(x,y)\,f(x,y)\,.$$

7. **Solving Heat and Wave Equations.** All of the formula for linear ODE hold for heat and wave partial differential equations. For example if $u_{m,n}(x,y)$ are as in Eq. (C.1) then

$$u(t,x,y)=e^{t\Delta}f(x,y)=\sum_{m,n=1}^{\infty}B_{m,n}e^{t\Delta}(\sin mx\cdot\sin ny)$$
$$=\sum_{m,n=1}^{\infty}B_{m,n}e^{-t\left(m^2+n^2\right)}(\sin mx\cdot\sin ny)\,,$$

solves the heat equation

$$u_t=\Delta u \text{ with } u(0,x,y)=f(x,y)$$
$$\text{and } u=0 \text{ on the boundary.}$$

More generally if

$$h(t,x,y)=\sum_{m,n=1}^{\infty}H_{m,n}(t)(\sin mx\cdot\sin ny)$$

then

$$u(t,x,y)=e^{t\Delta}f(x,y)+\int_0^t e^{(t-\tau)\Delta}h(\tau,x,y)\,d\tau$$
$$=\sum_{m,n=1}^{\infty}B_{m,n}e^{-t\left(m^2+n^2\right)}(\sin mx\cdot\sin ny)$$
$$+\sum_{m,n=1}^{\infty}\int_0^t H_{m,n}(\tau)\,e^{(t-\tau)\Delta}(\sin mx\cdot\sin ny)\,d\tau$$
$$=\sum_{m,n=1}^{\infty}\left[B_{m,n}e^{-t\left(m^2+n^2\right)}+\int_0^t e^{-(t-\tau)\left(m^2+n^2\right)}H_{m,n}(\tau)\,d\tau\right](\sin mx\cdot\sin ny)$$

solves the heat equation

$$u_t(t,x,y)=\Delta u(t,x,y)+h(t,x,y) \text{ with } u(0,x,y)=f(x,y)$$
$$\text{and } u=0 \text{ on the boundary.}$$

Similar statements hold for the wave equation,

$$u_{tt}(t,x,y)=\Delta u(t,x,y)+h(t,x,y) \text{ with } u=0 \text{ on the boundary}$$
$$\text{and } u(0,x,y)=f(x,y) \text{ and } u_t(0,x,y)=g(x,y)\,.$$

Namely

$$u(t,x,y)=\cos\left(t\sqrt{-\Delta}\right)f(x,y)+\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}g(x,y)$$
$$+\int_0^t\frac{\sin\left((t-\tau)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}h(\tau,x,y)\,d\tau$$

where

$$\cos\left(t\sqrt{-\Delta}\right)f(x,y)=\sum_{m,n=1}^{\infty}B_{m,n}\cos\left(t\sqrt{-\left(-\left(m^2+n^2\right)\right)}\right)(\sin mx\cdot\sin ny)$$
$$=\sum_{m,n=1}^{\infty}B_{m,n}\cos\left(t\sqrt{m^2+n^2}\right)(\sin mx\cdot\sin ny)$$

and

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f(x,y)=\sum_{m,n=1}^{\infty}B_{m,n}\frac{\sin\left(t\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}(\sin mx\cdot\sin ny)\,.$$

8. **Laplacian in other coordinate systems.**
   a) Laplacian in Cylindrical Coordinates is
   $$\Delta f=\frac{1}{\rho}\partial_\rho(\rho\partial_\rho f)+\frac{1}{\rho^2}\partial_\theta^2 f+\partial_z^2 f.$$
   This form of the Laplacian gives rise to Bessel's equation and Bessel functions.
   b) Laplacian in Spherical coordinates
   $$\Delta f=\frac{1}{r^2}\partial_r(r^2\partial_r f)+\frac{1}{r^2\sin\varphi}\partial_\varphi(\sin\varphi\partial_\varphi f)+\frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f$$
   $$=\frac{1}{r}\partial_r^2(rf)+\frac{1}{r^2\sin\varphi}\partial_\varphi(\sin\varphi\partial_\varphi f)+\frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f.$$
   This form of the Laplacian gives rise to "Legendre polynomials" and more generally "spherical harmonics."
   c) You are responsible for problems in these coordinates which do **not** involve the above mentioned special functions (i.e. Bessel functions and Legendre polynomials.)