Math 21D (Driver) Midterm I (Practice Test and Key)

Compute or explain why the limits in Problems 1. and 2. do not exist.

1. $\lim_{n\to\infty} (-1)^n e^{-n}$

2. $\lim_{n\to\infty} \frac{n}{\ln n}$

Determine if the series in problems 3–6 converge or diverge. Explain your answers.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}}$

4. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ For this problem, find an estimate for $\sum_{n=1}^{\infty} \frac{\ln n}{n^2} - \sum_{n=1}^{N} \frac{\ln n}{n^2}$.

5. $\sum_{n=1}^{\infty} \frac{3^k k!}{(2k)!}$ 6. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$

7. Write down the Taylor polynomial about zero of degree 5 which best approximates the function e^x.

8. Use the remainder theorem to give an estimate of the error between the value of $e^{-0.1}$ and the value given by evaluating the Taylor polynomial above at $x = -0.1$. Please show your work.

9. Find the Maclaurin series for $\frac{x^2}{1-x}$.

10. Explain how to use Taylor's theorem with remainder to show that $f(x) = \sqrt{1+x}$ has a Maclaurin series centered at $a = 0$ which is convergent for $|x| < 1/2$. (In fact the series is valid for $|x| < 1$, but this is harder to show.) (**Note well:** This problems is too **hard.** You should replace it by the following problem.)

10b. Explain how to use Taylor's theorem with remainder to show that $f(x) = \cosh(x)$ has a Taylor series centered at $a = \pi$ which is convergent for all x.

1 Here are some additional series problems.

11. Consider the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(2n)-1}$. Is this series absolutely convergent, divergent, conditionally convergent? Justify your answers.

12. The numbers a_n are given by the inductive relation:

$$a_1 = 1 \quad a_{n+1} = \frac{3}{2}a_n$$

Consider $b_n = (-1)^n a_n$ and $c_n = \frac{b_n}{n!}$. Are the series $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} c_n$ divergent, conditionally, convergent, absolutely convergent? Justify your answers.

13. Give the Maclaurin series of the function

$$f(x) = \cos^2 x - \sin^2 x.$$

Deduce the Maclaurin series of $\cos^2 x$ and $\sin^2 x$.

Hint: Use $\cos 2x = \cos^2 x - \sin^2 x$ and

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}.$$

2 Solution Key

For many of the problems below, I have only given the answer. To get credit on the test you must do more by showing your work and explaining your answer.

1. $\lim\limits_{n\to\infty} (-1)^n e^{-n} = 0$

2. $\lim\limits_{n\to\infty} \frac{n}{\ln n} = \infty$, i.e. diverges to ∞ by l'Hopital's rule.

Determine if the series in problems 3–6 converge or diverge. Explain your answers.

3. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}}$ diverges, since $\lim_{n\to\infty} \frac{(-1)^n}{e^{1/n}} \neq 0$.

4. $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^2}$ For this problem, find an estimate for $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^2} - \sum\limits_{n=1}^{N} \frac{\ln n}{n^2}$.

Set $f(x) = \ln(x)/x^2$, then $f'(x) = \frac{1}{x^3} - \frac{\ln(x)}{x^3} = \frac{1-\ln(x)}{x^3} < 0$ for $x > e$. So f is decreasing for $x > e$ and goes to zero, by problem 2. We may use the integral test to find

$$\left| \sum_{n=1}^{\infty} \frac{\ln n}{n^2} - \sum_{n=1}^{N} \frac{\ln n}{n^2} \right| = \sum_{n=N+1}^{\infty} \frac{\ln n}{n^2} \leq \int_N^{\infty} \frac{\ln(x)}{x^2} dx.$$

Let $y = \ln(x)$, $dy = x^{-1}dx$ and $x = e^y$, we find that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{\ln n}{n^2} - \sum_{n=1}^{N} \frac{\ln n}{n^2} \right| &\leq \int_N^{\infty} \frac{\ln(x)}{x^2} dx \\ &= \int_{\ln(N)}^{\infty} y e^{-y} dy = \frac{\ln N + 1}{N}. \end{aligned}$$

5. $\sum\limits_{n=1}^{\infty} \frac{3^k k!}{(2k)!}$ converges by the ratio test.

6. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ converges by the alternating series test. One should show that $f(x) = \ln(x)/x$ is decreasing for large x.

7. Write down the Taylor polynomial about zero of degree 5 which best approximates the function e^x.

$$T_5(x) = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5!$$

8. Use the remainder theorem to give an estimate of the error between the value of the function at $x = -0.1$ and the value given by evaluating the Taylor polynomial above at $x = -0.1$. Please show your work.

$$R_5(x) = e^x - T_5(x) = e^z \frac{x^6}{6!}$$

for some z between 0 and $x = -0.1$. Therefore,

$$|R_5(-0.1)| \leq e^0 \frac{(-0.1)^6}{6!} = \frac{(0.1)^6}{6!}.$$

9. Find the Maclaurin series for $\frac{x^2}{1-x} = x^2 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+2}$

10. Explain how to use Taylor's theorem with remainder to show that $f(x) = \sqrt{1+x}$ has a Maclaurin series centered at $a = 0$ which is convergent for $|x| < 1/2$. (In fact the series is valid for $|x| < 1$, but this is harder to show.) (**Note well:** This problems is too **hard.** You should replace it by the following problem.)

10b. Explain how to use Taylor's theorem with remainder to show that $f(x) = \cosh(x)$ has a Taylor series centered at $a = \pi$ which is convergent for all x.

Ans for 10b.

$$|R_N(x)| = |\cosh(x) - T_N(x)| = \left|\frac{\cosh^{(N+1)}(z)(x-\pi)^{N+1}}{(N+1)!}\right|.$$

Now $\left|\cosh^{(N+1)}(z)\right|$ is equal to either

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \leq \frac{e^{|z|} + e^{|-z|}}{2} = e^{|z|}$$

or to

$$|\sinh(z)| = \left|\frac{e^z - e^{-z}}{2}\right| \leq \frac{e^{|z|} + e^{|-z|}}{2} = e^{|z|}.$$

Since z is between x and a, $|z| \leq a + |x|$. Putting this altogether shows that

$$|R_N(x)| \leq \left|\frac{e^{a+|x|}(x-\pi)^{N+1}}{(N+1)!}\right| \to 0 \text{ as } N \to \infty.$$

We know this limit is zero since, by the ratio test

$$\sum_{n=0}^{\infty} M^n/n!$$

is convergent for all M, and in particular this implies that $\lim_{n\to\infty} M^n/n! = 0$. This shows that

$$\begin{aligned}\cosh(x) &= \lim_{N\to\infty} T_N(x) = \sum_{n=0}^{\infty} \cosh^{(n)}(\pi)(x-\pi)^n/n! \\ &= \cosh(\pi)\sum_{n=0}^{\infty}\frac{(x-\pi)^{2n}}{(2n)!} + \sinh(\pi)\sum_{n=0}^{\infty}\frac{(x-\pi)^{2n+1}}{(2n+1)!},\end{aligned}$$

which is valid for all x.

3 Here are some additional series problems.

I will leave problems 11. and 12. to you.

13. Give the Maclaurin series of the function

$$f(x) = \cos^2 x - \sin^2 x.$$

Deduce the Maclaurin series of $\cos^2 x$ and $\sin^2 x$.

Hint: Use $\cos 2x = \cos^2 x - \sin^2 x$ and

$$\cos t = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}.$$

$$\begin{aligned}f(x) &= \cos^2 x - \sin^2 x = \cos(2x) = \sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{(-2)^n}{(2n)!}x^{2n}.\end{aligned}$$

Now

$$f(x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1.$$

Thus

$$\begin{aligned} \cos^2(x) &= \frac{f(x)+1}{2} = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!} x^{2n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{2}\frac{(-2)^n}{(2n)!} x^{2n}. \end{aligned}$$

The formula for

$$\sin^2(x) = 1 - \cos^2(x) = -\sum_{n=1}^{\infty} \frac{1}{2}\frac{(-2)^n}{(2n)!} x^{2n}.$$