5. Measures and Integration

Definition 5.1. A set X equipped with a σ – algebra $\mathcal{M}$ is called a **measurable space**.

Definition 5.2. A measure μ on a measurable space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$ and
2. (Finite Additivity) If $\{A_i\}_{i=1}^n \subset \mathcal{M}$ are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$ when $i \neq j$, then
$$\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$
3. (Continuity) If $A_n \in \mathcal{M}$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.

Remark 5.3. Properties 2) and 3) in Definition 5.2 are equivalent to the following condition. If $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ are pairwise disjoint then

$$\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i). \tag{5.1}$$

To prove this suppose that Properties 2) and 3) in Definition 5.2 and $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ are pairwise disjoint. Let $B_n := \bigcup\limits_{i=1}^n A_i \uparrow B := \bigcup\limits_{i=1}^\infty A_i$, so that

$$\mu(B) \overset{(3)}{=} \lim_{n\to\infty} \mu(B_n) \overset{(2)}{=} \lim_{n\to\infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i).$$

Conversely, if Eq. (5.1) holds we may take $A_j = \emptyset$ for all $j \geq n$ to see that Property 2) of Definition 5.2 holds. Also if $A_n \uparrow A$, let $B_n := A_n \setminus A_{n-1}$. Then $\{B_n\}_{n=1}^\infty$ are pairwise disjoint, $A_n = \cup_{j=1}^n B_j$ and $A = \cup_{j=1}^\infty B_j$. So if Eq. (5.1) holds we have

$$\begin{aligned}\mu(A) &= \mu\left(\cup_{j=1}^\infty B_j\right) = \sum_{j=1}^\infty \mu(B_j)\\ &= \lim_{n\to\infty} \sum_{j=1}^n \mu(B_j) = \lim_{n\to\infty} \mu(\cup_{j=1}^n B_j) = \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

Proposition 5.4 (Basic properties of measures)**.** *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$ and $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$, then :*

1. *If $E \subseteq F$ then $\mu(E) \leq \mu(F)$.*
2. $\mu(\cup E_j) \leq \sum \mu(E_j)$.
3. *If $\mu(E_1) < \infty$ and $E_j \searrow E$, i.e. $E_1 \supset E_2 \supset E_3 \supset \ldots$ and $E = \cap_j E_j$, then $\mu(E_j) \searrow \mu(E)$ as $j \to \infty$.*

Proof. (1) Since $F = E \cup (F \setminus E)$,

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

(2) Let $\widetilde{E}_j = E_j \setminus (E_1 \cup \cdots \cup E_{j-1})$ so that the $\tilde{E}_j$'s are pair-wise disjoint and $E = \cup \widetilde{E}_j$. Since $\tilde{E}_j \subset E_j$ it follows from Remark 5.3 and part (1), that

$$\mu(E) = \sum \mu(\widetilde{E}_j) \leq \sum \mu(E_j).$$

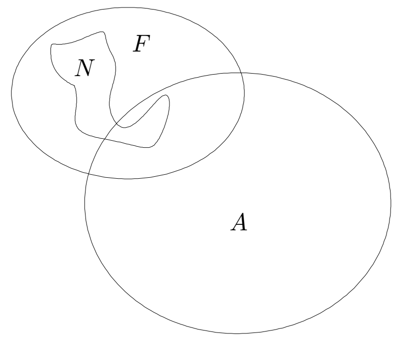

FIGURE 12. Completing a σ − algebra.

(3) Define $D_i \equiv E_1 \setminus E_i$ then $D_i \uparrow E_1 \setminus E$ which implies that

$$\mu(E_1) - \mu(E) = \lim_{i\to\infty} \mu(D_i) = \mu(E_1) - \lim_{i\to\infty} \mu(E_i)$$

which shows that $\lim_{i\to\infty} \mu(E_i) = \mu(E)$. ■

Definition 5.5. A set $E \in \mathcal{M}$ is a **null** set if $\mu(E) = 0$.

Definition 5.6. A measure space $(X, \mathcal{M}, \mu)$ is **complete** if every subset of a null set is in $\mathcal{M}$, i.e. for all $F \subset X$ such that $F \subseteq E \in \mathcal{M}$ with $\mu(E) = 0$ implies that $F \in \mathcal{M}$.

Proposition 5.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Set $\mathcal{N} \equiv \{N \subseteq X$:there exists $F \in \mathcal{M}$ such that $N \subseteq F$ and $\mu(F) = 0\}$.*

$$\bar{\mathcal{M}} = \{A \cup N : A \in \mathcal{M}, N \in \mathcal{M}\},$$

see Fig. 12. Then $\bar{\mathcal{M}}$ is a σ-algebra. Define $\bar{\mu}(A \cup N) = \mu(A)$, then $\bar{\mu}$ is the unique measure on $\bar{\mathcal{M}}$ which extends μ.

Proof. Clearly $X, \emptyset \in \bar{\mathcal{M}}$. Let $A \in \mathcal{M}$ and $N \in \mathcal{N}$ and choose $F \in \mathcal{M}$ such that $N \subseteq F$ and $\mu(F) = 0$. Since $N^c = (F \setminus N) \cup F^c$,

$$(A \cup N)^c = A^c \cap N^c = A^c \cap (F \setminus N \cup F^c) = [A^c \cap (F \setminus N)] \cup [A^c \cap F^c]$$

where $[A^c \cap (F \setminus N)] \in \mathcal{N}$ and $[A^c \cap F^c] \in \mathcal{M}$. Thus $\bar{\mathcal{M}}$ is closed under complements.

If $A_i \in \mathcal{M}$ and $N_i \subseteq F_i \in \mathcal{M}$ such that $\mu(F_i) = 0$ then $\cup(A_i \cup N_i) = (\cup A_i) \cup (\cup N_i) \in \bar{\mathcal{M}}$ since $\cup A_i \in \mathcal{M}$ and $\cup N_i \subseteq \cup F_i$ and $\mu(\cup F_i) \leq \sum \mu(F_i) = 0$. Therefore, $\bar{\mathcal{M}}$ is a σ-algebra.

Suppose $A \cup N_1 = B \cup N_2$ with $A, B \in \mathcal{M}$ and $N_1, N_2, \in \mathcal{N}$. Then $A \subseteq A \cup N_1 \subseteq A \cup N_1 \cup F_2 = B \cup F_2$ which shows that

$$\mu(A) \leq \mu(B) + \mu(F_2) = \mu(B).$$

Similarly, we show that $\mu(B) \leq \mu(A)$ so that $\mu(A) = \mu(B)$ and hence $\bar{\mu}(A \cup N) := \mu(A)$ is well defined. It is left as an exercise to show that $\bar{\mu}$ is a measure, i.e. that it is countable additive. ■

5.1. **Example of Measures.** Most σ – algebras and σ -additive measures are somewhat difficult to describe and define. However, one special case is fairly easy to understand. Namely suppose that $\mathcal{F} \subset \mathcal{P}(X)$ is a countable or finite partition of X and $\mathcal{M} \subset \mathcal{P}(X)$ is the σ – algebra which consists of the collection of set $A \subset X$ such that

$$A = \bigcup_{\alpha \in \mathcal{F} \ni \alpha \subset A} \alpha. \tag{5.2}$$

It is easily seen that $\mathcal{M}$ is a σ – algebra.

Any measure $\mu : \mathcal{M} \to [0, \infty]$ is determined uniquely by its values on $\mathcal{F}$. Conversely, if we are given any function $\lambda : \mathcal{F} \to [0, \infty]$ we may define, for $A \in \mathcal{M}$,

$$\mu(A) = \sum_{\alpha \in \mathcal{F} \ni \alpha \subset A} \lambda(\alpha) = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A}$$

where $1_{\alpha \subset A}$ is one if $\alpha \subset A$ and zero otherwise. We may check that μ is a measure on $\mathcal{M}$. Indeed, if $A = \coprod_{i=1}^{\infty} A_i$ and $\alpha \in \mathcal{F}$, then $\alpha \subset A$ iff $\alpha \subset A_i$ for one and hence exactly one A_i. Therefore,

$$1_{\alpha \subset A} = \sum_{i=1}^{\infty} 1_{\alpha \subset A_i}$$

and hence

$$\begin{aligned} \mu(A) &= \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A} = \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) \sum_{i=1}^{\infty} 1_{\alpha \subset A_i} \\ &= \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{F}} \lambda(\alpha) 1_{\alpha \subset A_i} = \sum_{i=1}^{\infty} \mu(A_i) \end{aligned}$$

as desired. Thus we have shown that there is a one to one correspondence between measures μ on $\mathcal{M}$ and functions $\lambda : \mathcal{F} \to [0, \infty]$.

We will leave the issue of constructing measures until Sections 8 and 9. However, let us point out that interesting measures do exist. The following theorem may be found in Theorem 8.22 or Theorem 8.41 in Section 8.

Theorem 5.8. *To every right continuous non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ there exists a unique measure μ_F on $\mathcal{B}_{\mathbb{R}}$ such that*

$$\mu_F((a, b]) = F(b) - F(a) \ \forall \ -\infty < a \le b < \infty \tag{5.3}$$

Moreover, if $A \in \mathcal{B}_{\mathbb{R}}$ then

$$\mu_F(A) = \inf \left\{ \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) : A \subseteq \cup_{i=1}^{\infty} (a_i, b_i] \right\} \tag{5.4}$$

$$= \inf \left\{ \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) : A \subseteq \coprod_{i=1}^{\infty} (a_i, b_i] \right\}. \tag{5.5}$$

In fact the map $F \to \mu_F$ is a one to one correspondence between right continuous functions F with $F(0) = 0$ on one hand and measures μ on $\mathcal{B}_{\mathbb{R}}$ such that $\mu(J) < \infty$ on any bounded set $J \in \mathcal{B}_{\mathbb{R}}$ on the other.

Example 5.9. The most important special case of Theorem 5.8 is when $F(x) = x$, in which case we write m for μ_F. The measure m is called Lebesgue measure.

Theorem 5.10. *Lebesgue measure m is invariant under translations, i.e. for $A \in \mathcal{B}_{\mathbb{R}}$ and $x \in \mathbb{R}$,*

$$m(x+B) = m(B). \tag{5.6}$$

Moreover, m is the unique measure on $\mathcal{B}_{\mathbb{R}}$ such that $m((0,1]) = 1$ and Eq. (5.6) holds for $A \in \mathcal{B}_{\mathbb{R}}$ and $x \in \mathbb{R}$. Moreover, m has the scaling property

$$m(\lambda B) = |\lambda|\, m(B) \tag{5.7}$$

where $\lambda \in \mathbb{R}$, $B \in \mathcal{B}_{\mathbb{R}}$ and $\lambda B := \{\lambda x : x \in B\}$.

Proof. Let $m_x(B) := m(x+B)$, then one easily shows that m_x is a measure on $\mathcal{B}_{\mathbb{R}}$ such that $m_x((a,b]) = b-a$ for all $a<b$. Therefore, $m_x = m$ by the uniqueness assertion in Theorem 5.8.

For the converse, suppose that m is translation invariant and $m((0,1]) = 1$. Given $n \in \mathbb{N}$, we have

$$(0,1] = \cup_{k=1}^{n} (\frac{k-1}{n}, \frac{k}{n}] = \cup_{k=1}^{n} \left(\frac{k-1}{n} + (0, \frac{1}{n}] \right).$$

Therefore,

$$\begin{aligned} 1 = m((0,1]) &= \sum_{k=1}^{n} m\left(\frac{k-1}{n} + (0, \frac{1}{n}] \right) \\ &= \sum_{k=1}^{n} m((0, \frac{1}{n}]) = n \cdot m((0, \frac{1}{n}]). \end{aligned}$$

That is to say

$$m((0, \frac{1}{n}]) = 1/n.$$

Similarly we show that $m((0, \frac{l}{n}]) = l/n$ for all $l, n \in \mathbb{N}$. Using the translation invariance of m, we then learn that

$$m((a,b]) = b-a$$

for all $a, b \in \mathbb{Q}$ such that $a<b$. Finally for $a, b \in \mathbb{R}$ such that $a<b$, choose $a_n, b_n \in \mathbb{Q}$ such that $b_n \downarrow b$ and $a_n \uparrow a$, then $(a_n, b_n] \downarrow (a,b]$ and thus

$$m((a,b]) = \lim_{n\to\infty} m((a_n, b_n]) = \lim_{n\to\infty} (b_n - a_n) = b - a,$$

i.e. m is Lebesgue measure.

To prove Eq. (5.7) we may assume that $\lambda \neq 0$ since this case is trivial to prove. Now let $m_\lambda(B) := |\lambda|^{-1} m(\lambda B)$. It is easily checked that m_λ is again a measure on $\mathcal{B}_{\mathbb{R}}$ which satisfies

$$m_\lambda((a,b]) = \lambda^{-1} m\left((\lambda a, \lambda b]\right) = \lambda^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda > 0$ and

$$m_\lambda((a,b]) = |\lambda|^{-1} m\left([\lambda b, \lambda a)\right) = -|\lambda|^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda < 0$. Hence $m_\lambda = m$. ■

We are now going to develope integration theory relative to a measure. The integral defined in the case the measure is Lebesgue measure m will be an extension of the standard Riemann integral on $\mathbb{R}$.

5.2. **Integrals of Simple functions.** Let $(X, \mathcal{M}, \mu)$ be a fixed measure space in this section.

Definition 5.11. A function $\phi : X \to \mathbb{F}$ is a **simple function** if ϕ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable and $\phi(X)$ is a finite set. Any such simple functions can be written as

$$\phi = \sum_{i=1}^{n} \lambda_i 1_{A_i} \text{ with } A_i \in \mathcal{M} \text{ and } \lambda_i \in \mathbb{F}. \tag{5.8}$$

Indeed, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be an enumeration of the range of ϕ and $A_i = \phi^{-1}(\{\lambda_i\})$. Also note that Eq. (5.8) may be written more intrinsically as

$$\phi = \sum_{y \in \mathbb{F}} y 1_{\phi^{-1}(\{y\})}.$$

The next theorem shows that simple functions are "pointwise dense" in the space of measurable functions.

Theorem 5.12 (Approximation Theorem)**.** *Let $f : X \to [0, \infty]$ be measurable and define*

$$\begin{aligned}\phi_n(x) &\equiv \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{f^{-1}\left((\frac{k}{2^n}, \frac{k+1}{2^n}]\right)}(x) + 2^n 1_{f^{-1}((2^n,\infty])}(x) \\ &= \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{\left\{\frac{k}{2^n} < f \le \frac{k+1}{2^n}\right\}}(x) + 2^n 1_{\{f > 2^n\}}(x)\end{aligned}$$

then $\phi_n \le f$ for all n, $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on the sets $X_M := \{x \in X : f(x) \le M\}$ with $M < \infty$. Moreover, if $f : X \to \mathbb{C}$ is a measurable function, then there exists simple functions ϕ_n such that $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all x and $|\phi_n| \uparrow |f|$ as $n \to \infty$.

Proof. It is clear by construction that $\phi_n(x) \le f(x)$ for all x and that $0 \le f(x) - \phi_n(x) \le 2^{-n}$ if $x \in X_{2^n}$. From this it follows that $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on bounded sets.

Also notice that

$$(\frac{k}{2^n}, \frac{k+1}{2^n}] = (\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}] = (\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}] \cup (\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}]$$

and for $x \in f^{-1}\left((\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}]\right)$, $\phi_n(x) = \phi_{n+1}(x) = \frac{2k}{2^{n+1}}$ and for $x \in f^{-1}\left((\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}]\right)$, $\phi_n(x) = \frac{2k}{2^{n+1}} \le \frac{2k+1}{2^{n+1}} = \phi_{n+1}(x)$. Similarly since

$$(2^n, \infty] = (2^n, 2^{n+1}] \cup (2^{n+1}, \infty],$$

for $x \in f^{-1}((2^{n+1}, \infty])$ $\phi_n(x) = 2^n < 2^{n+1} = \phi_{n+1}(x)$ and for $x \in f^{-1}((2^n, 2^{n+1}])$, $\phi_{n+1}(x) \ge 2^n = \phi_n(x)$. Therefore $\phi_n \le \phi_{n+1}$ for all n and we have completed the proof of the first assertion.

For the second assertion, first assume that $f : X \to \mathbb{R}$ is a measurable function and choose $\phi_n^{\pm}$ to be simple functions such that $\phi_n^{\pm} \uparrow f_{\pm}$ as $n \to \infty$ and define $\phi_n = \phi_n^+ - \phi_n^-$. Then

$$|\phi_n| = \phi_n^+ + \phi_n^- \le \phi_{n+1}^+ + \phi_{n+1}^- = |\phi_{n+1}|$$

and clearly $|\phi_n| = \phi_n^+ + \phi_n^- \uparrow f_+ + f_- = |f|$ and $\phi_n = \phi_n^+ - \phi_n^- \to f_+ - f_- = f$ as $n \to \infty$.

Now suppose that $f : X \to \mathbb{C}$ is measurable. We may now choose simple function u_n and v_n such that $|u_n| \uparrow |\operatorname{Re} f|$, $|v_n| \uparrow |\operatorname{Im} f|$, $u_n \to \operatorname{Re} f$ and $v_n \to \operatorname{Im} f$ as $n \to \infty$. Let $\phi_n = u_n + iv_n$, then

$$|\phi_n|^2 = u_n^2 + v_n^2 \uparrow |\operatorname{Re} f|^2 + |\operatorname{Im} f|^2 = |f|^2$$

and $\phi_n = u_n + iv_n \to \operatorname{Re} f + i \operatorname{Im} f = f$ as $n \to \infty$. ■

We are now ready to define the Lebesgue integral. We will start by integrating simple functions and then proceed to general measurable functions.

Definition 5.13. Let $\mathbb{F} = \mathbb{C}$ or $[0, \infty]$ and suppose that $\phi : X \to \mathbb{F}$ is a simple function. If $\mathbb{F} = \mathbb{C}$ assume further that $\mu(\phi^{-1}(\{y\})) < \infty$ for all $y \neq 0$ in $\mathbb{C}$. For such functions ϕ we define $\int \phi = \int_X \phi \, d\mu$ by

$$\int_X \phi \, d\mu = \sum_{y \in \mathbb{F}} y \mu(\phi^{-1}(\{y\})).$$

Proposition 5.14. *The integral has the following properties.*

1. *Suppose that $\lambda \in \mathbb{F}$ then*

$$\int_X \lambda f d\mu = \lambda \int_X f d\mu. \tag{5.9}$$

2. *Suppose that ϕ and ψ are two simple functions, then*

$$\int (\phi + \psi) d\mu = \int \psi d\mu + \int \phi d\mu.$$

3. *If ϕ and ψ are non-negative simple functions such that $\phi \leq \psi$ then*

$$\int \phi d\mu \leq \int \psi d\mu.$$

4. *If ϕ is a non-negative simple function then $A \to \nu(A) := \int_A \phi \, d\mu \equiv \int_X 1_A \phi \, d\mu$ is a measure.*

Proof. Let us write $\{\phi = y\}$ for the set $\phi^{-1}(\{y\}) \subset X$ and $\mu(\phi = y)$ for $\mu(\{\phi = y\}) = \mu(\phi^{-1}(\{y\}))$ so that

$$\int \phi = \sum_{y \in \mathbb{C}} y \mu(\phi = y).$$

We will also write $\{\phi = a, \psi = b\}$ for $\phi^{-1}(\{a\}) \cap \psi^{-1}(\{b\})$. This notation is more intuitive for the purposes of this proof. Suppose that $\lambda \in \mathbb{F}$ then

$$\begin{aligned}
\int_X \lambda \phi \, d\mu &= \sum_{y \in \mathbb{F}} y \ \mu(\lambda \phi = y) \\
&= \sum_{y \in \mathbb{F}} y \ \mu(\phi = y/\lambda) \\
&= \sum_{z \in \mathbb{F}} \lambda z \ \mu(\phi = z) = \lambda \int_X \phi \, d\mu
\end{aligned}$$

provided that $\lambda \neq 0$. The case $\lambda = 0$ is clear, so we have proved 1.

Suppose that ϕ and ψ are two simple functions, then

$$\begin{aligned}
\int (\phi+\psi)\ d\mu &= \sum_{z\in\mathbb{F}} z\ \mu(\phi+\psi=z) \\
&= \sum_{z\in\mathbb{F}} z\ \mu\left(\cup_{\omega\in\mathbb{F}}\left\{\phi=\omega,\ \psi=z-\omega\right\}\right) \\
&= \sum_{z\in\mathbb{F}} z \sum_{\omega\in\mathbb{F}} \mu(\phi=\omega,\ \psi=z-\omega) \\
&= \sum_{z,\omega\in\mathbb{F}} (z+\omega)\mu(\phi=\omega,\ \psi=z) \\
&= \sum_{z\in\mathbb{F}} z\ \mu(\psi=z) + \sum_{\omega\in\mathbb{F}} \omega\ \mu(\phi=\omega) \\
&= \int \psi\ d\mu + \int \phi\ d\mu.
\end{aligned}$$

which proves 2.

For 3. if ϕ and ψ are non-negative simple functions such that $\phi \le \psi$

$$\begin{aligned}
\int \phi &= \sum_{a\ge 0} a\mu(\phi=a) \\
&= \sum_{a,b\ge 0} a\mu(\phi=a,\psi=b) \\
&\le \sum_{a,b\ge 0} b\mu(\phi=a,\psi=b) \\
&= \sum_{b\ge 0} b\mu(\psi=b) = \int \psi,
\end{aligned}$$

where in the third inequality we have used $\{\phi=a,\psi=b\}=\emptyset$ if $a>b$.

Finally for 4., write $\phi=\sum\lambda_i 1_{B_i}$ with $\lambda_i>0$ and $B_i\in\mathcal{M}$, then

$$\nu(A)=\int 1_A\phi\ d\mu=\sum_{i=1}^N \lambda_i\ \mu(A\cap B_i).$$

The latter expression for ν is easily checked to be a measure. ■

5.3. **Integrals of positive functions.**

Definition 5.15. Let $L^+=\{f:X\to[0,\infty]: f \text{ is measurable}\}$. Define

$$\int_X f d\mu = \sup\left\{\int_X \phi\ d\mu : \phi \text{ is simple and } \phi\le f\right\}.$$

Because of item 3. of Proposition 5.14, this definition is consistent with our previous definition of the integral on non-negative simple functions. We say the $f\in L^+$ is **integrable** if

$$\int_X f d\mu < \infty.$$

Remark 5.16. Notice that we still have the monotonicity property: $0\le f\le g$ then

$$\int_X f \le \int_X g$$

and for $c > 0$

$$\int_X cf = c\int_X f.$$

Also notice that if f is integrable, then $\mu(\{f = \infty\}) = 0$.

Lemma 5.17. *Let X be a set and $\rho : X \to [0,\infty]$ be a function, let $\mu = \sum_{x\in X}\rho(x)\delta_x$ on $\mathcal{M} = \mathcal{P}(X)$, i.e.*

$$\mu(A) = \sum_{x\in A}\rho(x).$$

If $f : X \to [0,\infty]$ is a function (which is necessarily measurable), then

$$\int_X f d\mu = \sum_X \rho f.$$

Proof. Suppose that $\phi : X \to [0,\infty]$ is a simple function, then $\phi = \sum_{z\in[0,\infty]} z 1_{\phi^{-1}(\{z\})}$ and

$$\begin{aligned}\sum_X \rho\phi &= \sum_{x\in X}\rho(x)\sum_{z\in[0,\infty]} z 1_{\phi^{-1}(\{z\})}(x)\\ &= \sum_{z\in[0,\infty]} z\sum_{x\in X}\rho(x)1_{\phi^{-1}(\{z\})}(x)\\ &= \sum_{z\in[0,\infty]} z\mu(\phi^{-1}(\{z\})) = \int_X \phi d\mu.\end{aligned}$$

So on simple function $\phi : X \to [0,\infty]$,

$$\sum_X \rho\phi = \int_X \phi d\mu.$$

Suppose that $\phi : X \to [0,\infty)$ is a simple function such that $\phi \le f$, then

$$\int_X \phi d\mu = \sum_X \rho\phi \le \sum_X \rho f.$$

Taking the sup over ϕ in this last equation then shows that

$$\int_X f d\mu \le \sum_X \rho f.$$

For the reverse inequality, let $\Lambda \subset\subset X$ be a finite set and $N \in (0,\infty)$. Set $f^N(x) = \min\{N, f(x)\}$ and let $\phi_{N,\Lambda}$ be the simple function given by $\phi_{N,\Lambda}(x) := 1_\Lambda(x) f^N(x)$. Because $\phi_{N,\Lambda}(x) \le f(x)$,

$$\sum_\Lambda \rho f^N = \sum_X \rho\phi_{N,\Lambda} = \int_X \phi_{N,\Lambda} d\mu \le \int_X f d\mu.$$

Since $f^N \uparrow f$ as $N \to \infty$, we may let $N \to \infty$ in this last equation to concluded that

$$\sum_\Lambda \rho f \le \int_X f d\mu$$

and since Λ is arbitrary we learn that

$$\sum_X \rho f \leq \int_X f d\mu.$$

■

Theorem 5.18 (Monotone Convergence Theorem)**.** *Suppose $f_n \in L^+$ is a sequence of functions such that $f_n \uparrow f$ (necessarily in L^+) then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

Proof. Since $f_n \leq f_m \leq f$, for all $n \leq m < \infty$,

$$\int f_n \leq \int f_m \leq \int f$$

from which if follows $\int f_n$ is increasing in n and

$$\lim_{n\to\infty} \int f_n \leq \int f.$$

For the opposite inequality, let ϕ be a simple function such that $0 \leq \phi \leq f$ and let $\alpha \in (0,1)$. Notice that

$$E_n \equiv \{f_n \geq \alpha\phi\} \uparrow X \text{ as } n \to \infty$$

and that, by Proposition 5.14,

$$\int f_n \geq \int 1_{E_n} f_n \geq \int_{E_n} \alpha\phi = \alpha \int_{E_n} \phi. \tag{5.10}$$

Because $E \to \alpha \int_E \phi$ is a measure and $E_n \uparrow X$,

$$\lim_{n\to\infty} \int_{E_n} \phi = \int_X \phi d\mu.$$

Hence we may pass to the limit in Eq. (5.10) to get

$$\lim_{n\to\infty} \int f_n \geq \alpha \int \phi.$$

Because this equation is valid for all simple functions $0 \leq \phi \leq f$, by the definition of $\int f$ we have

$$\lim_{n\to\infty} \int f_n \geq \alpha \int f.$$

Since $\alpha \in (0,1)$ is arbitrary we conclude that

$$\lim_{n\to\infty} \int f_n \geq \int f.$$

■

Corollary 5.19. *If $f_n \in L^+$ is a sequence of functions then*

$$\int \sum_n f_n = \sum_n \int f_n.$$

Proof. First off we show that

$$\int (f_1 + f_2) = \int f_1 + \int f_2$$

by choosing non-negative simple function ϕ_n and ψ_n such that $\phi_n \uparrow f_1$ and $\psi_n \uparrow f_2$. Then $(\phi_n + \psi_n)$ is simple as well and $(\phi_n + \psi_n) \uparrow (f_1 + f_2)$ so that by the monotone convergence theorem,

$$\begin{aligned}\int (f_1 + f_2) &= \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \left(\int \phi_n + \int \psi_n \right) \\ &= \lim_{n\to\infty} \int \phi_n + \lim_{n\to\infty} \int \psi_n = \int f_1 + \int f_2.\end{aligned}$$

Now to the general case. Let $g_N \equiv \sum\limits_{n=1}^{N} f_n$ and $g = \sum\limits_{1}^{\infty} f_n$, then $g_N \uparrow g$ and so by monotone convergence theorem and the additivity just proved,

$$\begin{aligned}\sum_{n=1}^{\infty} \int f_n &:= \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \\ &= \lim_{N\to\infty} \int g_N = \int g = \sum_{n=1}^{\infty} \int f_n.\end{aligned}$$

■

The following Lemma is a simple application of this Corollary.

Lemma 5.20 (First Borell-Carnteli- Lemma.)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $A_n \in \mathcal{M}$, and set*

$$\begin{aligned}\{A_n \text{ i.o.}\} &= \{x \in X : x \in A_n \text{ for infinitely many } n\text{'s}\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} A_n.\end{aligned}$$

If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu(\{A_n \text{ i.o.}\}) = 0$.

Proof. (First Proof.) Let us first observe that

$$\{A_n \text{ i.o.}\} = \left\{ x \in X : \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right\}.$$

Hence if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then

$$\infty > \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \int_X 1_{A_n} \, d\mu = \int_X \sum_{n=1}^{\infty} 1_{A_n} \, d\mu$$

implies that $\sum\limits_{n=1}^{\infty} 1_{A_n}(x) < \infty$ for μ - a.e. x. That is to say $\mu(\{A_n \text{ i.o.}\}) = 0$.

(Second Proof.) Of course we may give a strictly measure theoretic proof of this fact:

$$\begin{aligned}\mu(A_n \text{ i.o.}) &= \lim_{N\to\infty} \mu \left(\bigcup_{n\geq N} A_n \right) \\ &\leq \lim_{N\to\infty} \sum_{n\geq N} \mu(A_n)\end{aligned}$$

and the last limit is zero since $\sum_{n=1}^{\infty}\mu(A_n)<\infty$. ■

Example 5.21. Suppose that $f\in C([0,1])$ and $f\geq 0$. Let $\pi_k=\{0=a_0<a_1<\cdots<a_{n_k}=1\}$ be a sequence of refining partitions such that $\text{mesh}(\pi_k)\to 0$ as $k\to\infty$. Let

$$f_k(x)=f(0)1_{\{0\}}+\sum_{\pi_k}\min\{f(x):a_k\leq x\leq a_{k+1}\}\,1_{(a_k,a_{k+1}]}(x)$$

then $f_k\uparrow f$ as $k\to\infty$ so that by the monotone convergence theorem,

$$\begin{aligned}\int_0^1 f dm &= \lim_{k\to\infty}\int_0^1 f_k\ dm\\ &= \lim_{k\to\infty}\sum_{\pi_k}\min\{f(x):a_k\leq x\leq a_{k+1}\}\,m\left((a_{k+1}a_k]\right)\\ &= \int_0^1 f(x)dx\end{aligned}$$

where the latter integral is the Riemann integral.

Example 5.22. Let m be Lebesgue measure on $\mathbb{R}$, then

$$\begin{aligned}\int_{(0,1]}\frac{1}{x^p}\ dm(x) &= \lim_{n\to\infty}\int_0^1 1_{(\frac{1}{n},1]}(x)\frac{1}{x^p}dm(x)\\ &= \lim_{n\to\infty}\int_{\frac{1}{n}}^1\frac{1}{x^p}dx=\lim_{n\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_{1/n}^1\\ &= \begin{cases}\frac{1}{1-p} & \text{if } \ p<1\\ \infty & \text{if } \ p>1\end{cases}\end{aligned}$$

If $p=1$ we find

$$\int_{(0,1]}\frac{1}{x^p}\ dm(x)=\lim_{n\to\infty}\int_{\frac{1}{n}}^1\frac{1}{x}dx=\lim_{n\to\infty}\ln(x)|_{1/n}^1=\infty.$$

Example 5.23. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the points in $\mathbb{Q}\cap[0,1]$ and define

$$\frac{1}{\sqrt{|x-r_n|}}=5 \text{ if } x=r_n$$

and

$$f(x)=\sum_{n=1}^{\infty}2^{-n}\frac{1}{\sqrt{|x-r_n|}}.$$

Then

$$\int_0^1\frac{1}{\sqrt{|x-r_n|}}\ dx\leq 4\int_0^1 f(x)dx\leq 4$$

and hence

$$\int_{[0,1]}f(x)dm(x)\leq 4<\infty$$

which shows that $m(f = \infty) = 0$, i.e. that $f < \infty$ for almost every $x \in [0,1]$ and this implies that

$$\sum_{n=1}^{\infty} 2^{-n} \frac{1}{\sqrt{|x - r_n|}} < \infty \text{ for a.e. } x.$$

The following simple lemma will often be useful.

Lemma 5.24 (Chevbyshev's Inequality)**.** *Suppose that $f \geq 0$ is a measurable function, then for any $\epsilon > 0$,*

$$\mu(\{f \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X f d\mu. \tag{5.11}$$

Proof. Since $1_{\{f \geq \epsilon\}} \leq 1_{\{f \geq \epsilon\}} \frac{1}{\epsilon} f \leq \frac{1}{\epsilon} f$,

$$\mu(\{f \geq \epsilon\}) = \int_X 1_{\{f \geq \epsilon\}} d\mu \leq \int_X 1_{\{f \geq \epsilon\}} \frac{1}{\epsilon} f d\mu \leq \frac{1}{\epsilon} \int_X f d\mu.$$

■

Proposition 5.25. *Suppose that $f \geq 0$ is a measurable function. Then $\int_X f d\mu = 0$ iff $f = 0$ a.e. Also if $f, g \geq 0$ are measurable functions such that $f \leq g$ a.e. then $\int f d\mu \leq \int g d\mu$. In particular if $f = g$ a.e. then $\int f d\mu = \int g d\mu$.*

Proof. If $f = 0$ a.e. and $\phi \leq f$ is a simple function then $\phi = 0$ a.e. This implies that $\mu(\phi^{-1}(\{y\})) = 0$ for all $y > 0$ and hence $\int_X \phi d\mu = 0$ and therefore $\int_X f d\mu = 0$.

Conversely, if $\int f d\mu = 0$, let $E_n = \{f \geq \frac{1}{n}\}$. Then

$$0 = \int_{E_n} f \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n} \mu(E_n)$$

which shows that $\mu(E_n) = 0$ for all n. Since $\{f > 0\} = \cup E_n$, we have

$$\mu(\{f > 0\}) \leq \sum_n \mu(E_n) = 0,$$

i.e. $f = 0$ a.e.

For the second assertion let $E \in \mathcal{M}$ be a set such that $\mu(E^c) = 0$ and $1_E f \leq 1_E g$ everywhere. Because $g = 1_E g + 1_{E^c} g$ and $1_{E^c} g = 0$ a.e.,

$$\int g d\mu = \int 1_E g d\mu + \int 1_{E^c} g d\mu = \int 1_E g d\mu$$

and similarly $\int f d\mu = \int 1_E f d\mu$. Since $1_E f \leq 1_E g$ everywhere,

$$\int f d\mu = \int 1_E f d\mu \leq \int 1_E g d\mu = \int g d\mu.$$

■

Corollary 5.26. *Suppose that $\{f_n\}$ is a sequence of non-negative functions and f is a measurable function such that off a set of measure zero, $f_n \uparrow f$, then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

Proof. Let $E \subseteq X$ such that $\mu(X \setminus E) = 0$ and $f_n 1_E \uparrow f 1_E$. Then by the monotone convergence theorem,

$$\int f_n = \int f_n 1_E \uparrow \int f 1_E = \int f \text{ as } n \to \infty.$$

■

Lemma 5.27 (Fatou's Lemma)**.** *If $f_n : X \to [0,\infty]$ is a sequence of measurable functions then*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

Proof. Define $g_k \equiv \inf_{n\ge k} f_n$ so that $g_k \uparrow \liminf_{n\to\infty} f_n$ as $k \to \infty$. Since $g_k \le f_n$ for all $k \le n$ we have

$$\int g_k \le \int f_n \text{ for all } n \ge k$$

and therefore

$$\int g_k \le \lim\inf_{n\to\infty} \int f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k \to \infty$ to find

$$\int \lim\inf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \int g_k \le \lim\inf_{n\to\infty} \int f_n.$$

■

5.3.1. *Integrals of Complex Valued Functions.*

Definition 5.28. A measurable function $f : \mathbb{R} \to [-\infty,\infty]$ is **integrable** if $f_+ \equiv f1_{\{f\ge 0\}}$ and $f_- = -f\ 1_{\{f\le 0\}}$ are **integrable**. We write L^1 for the space of integrable functions. For $f \in L^1$, let

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

Remark 5.29*.* Notice that if f is integrable, then

$$f_\pm \le |f| \le f_+ + f_-$$

so that f is integrable iff

$$\int |f|\ d\mu < \infty.$$

Proposition 5.30. *The map*

$$f \in L^1 \to \int_X f d\mu \in \mathbb{R}$$

is linear. Also if $f,g \in L^1$ are real valued functions such that $f \le g$, the $\int f d\mu \le \int g d\mu$.

Proof. If $f,g \in L^1$ and $a,b \in \mathbb{R}$, then

$$|af + bg| \le |a||f| + |b|\,|g| \in L^1.$$

For $a \in \mathbb{R}$, say $a < 0$,

$$(af)_+ = -af_- \text{ and } (af)_- = -af_+$$

so that

$$\int af = -a\int f_- + a\int f_+ = a(\int f_+ - \int f_-) = a\int f.$$

A similar calculation works for $a > 0$ and the case $a = 0$ is trivial so we have shown that

$$\int af = a \int f.$$

Now set $h = f + g$. Since $h = h_+ - h_-$,

$$h_+ - h_- = f_+ - f_- + g_+ - g_-$$

or

$$h_+ + f_- + g_- = h_- + f_+ + g_+.$$

Therefore,

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+$$

and hence

$$\int h = \int h_+ - \int h_- = \int f_+ + \int g_+ - \int f_- - \int g_- = \int f + \int g.$$

Finally if $f_+ - f_- = f \leq g = g_+ - g_-$ then $f_+ + g_- \leq g_+ + f_-$ which implies that

$$\int f_+ + \int g_- \leq \int g_+ + \int f_-$$

or equivalently that

$$\int f = \int f_+ - \int f_- \leq \int g_+ - \int g_- = \int g.$$

■

Definition 5.31. A measurable function $f : X \to \mathbb{C}$ is integrable if $\int_X |f| \, d\mu < \infty$, again we write $f \in L^1$. One shows that $\int |f| \, d\mu < \infty$ iff

$$\int |\mathrm{Re}\, f| \, d\mu + \int |\mathrm{Im}\, f| \, d\mu < \infty.$$

For $f \in L^1$ define

$$\int f \ d\mu = \int \mathrm{Re}\, f \ d\mu + i \int \mathrm{Im}\, f \ d\mu.$$

It is routine to show that the integral is still linear on the complex L^1 (prove!).

Proposition 5.32. *Suppose that $f \in L^1$, then*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Proof. Start by writing $\int_X f \ d\mu = Re^{i\theta}$. Then

$$\left| \int_X f d\mu \right| = R = e^{-i\theta} \int_X f d\mu = \int_X e^{-i\theta} f d\mu$$
$$= \int_X \mathrm{Re}\left(e^{-i\theta} f\right) d\mu.$$

Let $g := \operatorname{Re}\left(e^{-i\theta} f\right) = g_+ - g_-$ then combining the previous equation with the following estimate proves the theorem.

$$\begin{aligned}\int_X g &= \int_X g_+ - \int_X g \le \int_X g_+ + \int_X g_- \\ &= \int_X g_+ + g_- = \int_X |g| d\mu \\ &= \int_X |\operatorname{Re}(e^{-i\theta} f)| d\mu \le \int_X |f| d\mu.\end{aligned}$$

■

Proposition 5.33. *$f, g \in L^1$, then*

1. *The set $\{f \neq 0\}$ is σ-finite, i.e. there exists $E_n \in \mathcal{M}$ such that $\mu(E_n) < \infty$ and $E_n \uparrow \{f \neq 0\}$.*
2. *The following are equivalent*
   (a) *$\int_E f = \int_E g$ for all $E \in \mathcal{M}$*
   (b) *$\int_X |f - g| = 0$*
   (c) *$f = g$ a.e.*

Proof. 1. The sets $E_n := \{|f| \ge \frac{1}{n}\}$ satisfy the conditions in item 1. since clearly $E_n \uparrow \{f \neq 0\}$ and by Chebyshev's inequality (5.11),

$$\mu(E_n) \le \frac{1}{\epsilon} \int_X |f| d\mu < \infty.$$

2. (a) $\Longrightarrow$ (c)Notice that

$$\int_E f = \int_E g \Leftrightarrow \int_E (f - g) = 0$$

for all $E \in \mathcal{M}$. Taking $E = \{\operatorname{Re}(f - g) > 0\}$ and using $1_E \operatorname{Re}(f - g) \ge 0$, we learn that

$$0 = \operatorname{Re} \int_E (f - g) d\mu = \int 1_E \operatorname{Re}(f - g) \Longrightarrow 1_E \operatorname{Re}(f - g) = 0 \text{ a.e.}$$

This implies that $1_E = 0$ a.e. which happens iff

$$\mu\left(\{\operatorname{Re}(f - g) > 0\}\right) = \mu(E) = 0.$$

Similar $\mu(\operatorname{Re}(f - g) < 0) = 0$ so that $\operatorname{Re}(f - g) = 0$ a.e. Similarly, $\operatorname{Im}(f - g) = 0$ a.e and hence $f - g = 0$ a.e., i.e. $f = g$ a.e.

(c) $\Longrightarrow$ (b) is clear and so is (b) $\Longrightarrow$ (a) since

$$\left| \int_E f - \int_E g \right| \le \int |f - g| = 0.$$

■

Corollary 5.34. *Suppose that $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$ is a collection of sets such that $\mu(A_i \cap A_j) = 0$ for all $i \neq j$, then*

$$\mu\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n).$$

Proof. Since

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \int_X 1_{\cup_{n=1}^{\infty} A_n} d\mu \text{ and}$$
$$\sum_{n=1}^{\infty} \mu(A_n) = \int_X \sum_{n=1}^{\infty} 1_{A_n} d\mu$$

it suffices to show that

$$\sum_{n=1}^{\infty} 1_{A_n} = 1_{\cup_{n=1}^{\infty} A_n} \ \mu - \text{a.e.} \tag{5.12}$$

Now $\sum_{n=1}^{\infty} 1_{A_n} \geq 1_{\cup_{n=1}^{\infty} A_n}$ and $\sum_{n=1}^{\infty} 1_{A_n}(x) \neq 1_{\cup_{n=1}^{\infty} A_n}(x)$ iff $x \in A_i \cap A_j$ for some $i \neq j$, that is

$$\left\{x : \sum_{n=1}^{\infty} 1_{A_n}(x) \neq 1_{\cup_{n=1}^{\infty} A_n}(x)\right\} = \cup_{i<j} A_i \cap A_j$$

and the later set has measure 0 being the countable union of sets of measure zero. This proves Eq. (5.12) and hence the corollary. ■

Definition 5.35. Let $(X, \mathcal{M}, \mu)$ be a measure space and $L^1(\mu) = L^1(X, \mathcal{M}, \mu)$ denote the set of L^1 functions modulo the equivalence relation $f \sim g$ iff $f = g$ a.e. We make this into a normed space using the norm

$$\|f - g\|_{L^1} = \int |f - g| \, d\mu$$

and into a metric space using $\rho_1(f, g) = \|f - g\|_{L^1}$.

Remark 5.36. More generally we may define $L^p(\mu) = L^p(X, \mathcal{M}, \mu)$ for $p \in [1, \infty)$ as the set of measurable functions f such that

$$\int_X |f|^p \, d\mu < \infty$$

modulo the equivalence relation $f \sim g$ iff $f = g$ a.e.

We will see in Section 7 that

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu\right)^{1/p} \text{ for } f \in L^p(\mu)$$

is a norm and $(L^p(\mu), \|\cdot\|_{L^p})$ is a Banach space in this norm.

Theorem 5.37 (Dominated Convergence Theorem)**.** *Suppose $f_n \to f$ a.e. $|f_n| \leq g \in L^1$. Then $f \in L^1$ and*

$$\int_X f d\mu = \lim_{h \to \infty} \int_X f_n d\mu.$$

Proof. Notice that $|f| = \lim |f_n| \leq g$ a.e. so that $f \in L^1$. By considering the real and imaginary parts of f separately, it suffices to prove the theorem in the case where f is real. By Fatou's Lemma,

$$\begin{aligned}\int_X (g \pm f) d\mu &= \int_X \liminf \left(g \pm f_n\right) d\mu \leq \liminf \int_X \left(g \pm f_n\right) d\mu \\ &= \int_X g d\mu + \liminf \left(\pm \int_X f_n d\mu\right).\end{aligned}$$

Since $\liminf(-a_n) = -\limsup a_n$, we have shown,

$$\int_X g d\mu \pm \int_X f d\mu \leq \int_X g d\mu + \begin{cases} \liminf \int_X f_n d\mu \\ -\limsup \int_X f_n d\mu \end{cases}$$

and therefore

$$\limsup \int_X f_n d\mu \leq \int_X f d\mu \leq \liminf \int_X f_n d\mu.$$

This shows that $\lim_{n\to\infty} \int_X f_n d\mu$ exists and is equal to $\int_X f d\mu$. ■

Corollary 5.38 (Differentiation Under the Integral)**.** *Suppose that $J \subset \mathbb{R}$ is an open interval and $f : J \times X \to \mathbb{C}$ is a function such that*

1. $f(t_0, \cdot) \in L^1$ *for some* $t_0 \in J$,
2. $\frac{\partial f}{\partial t}(t, x)$ *exists for all* (t, x)
3. *There is a function* $g \in L^1$ *such that* $\left|\frac{\partial f}{\partial t}(t, x)\right| \leq g(x) \in L^1$.

Then $f(t, \cdot) \in L^1$ for some $t \in J$ and

$$\frac{d}{dt}\int_X f(t, x) d\mu(x) = \int_X \frac{\partial f}{\partial t}(t, x) d\mu(x).$$

Proof. (The proof is the same as for sums.) By considering the real and imaginary parts of f separately, we may assume that f is real. By the mean value theorem,

$$|f(t, x) - f(t_0, x)| \leq g(x) |t - t_0| \text{ for all } t \in J. \tag{5.13}$$

In particular,

$$|f(t, x)| \leq |f(t, x) - f(t_0, x)| + |f(t_0, x)| \leq g(x) |t - t_0| + |f(t_0, x)|$$

which shows $f(t, \cdot) \in L^1(\mu)$ for all $t \in J$. Let $G(t) := \int_X f(t, x) d\mu(x)$, then

$$\frac{G(t) - G(t_0)}{t - t_0} = \int_X \frac{f(t, x) - f(t_0, x)}{t - t_0} d\mu(x).$$

By assumption,

$$\lim_{t\to t_0} \frac{f(t, x) - f(t_0, x)}{t - t_0} = \frac{\partial f}{\partial t}(t, x) \text{ for all } x \in X$$

and by Eq. (5.13),

$$\left|\frac{f(t, x) - f(t_0, x)}{t - t_0}\right| \leq g(x) \text{ for all } t \in J \text{ and } x \in X.$$

Therefore, we my apply the dominated convergence theorem to conclude

$$\begin{aligned} \lim_{n\to\infty} \frac{G(t_n) - G(t_0)}{t_n - t_0} &= \lim_{n\to\infty} \int_X \frac{f(t_n, x) - f(t_0, x)}{t_n - t_0} d\mu(x) = \int_X \lim_{n\to\infty} \frac{f(t_n, x) - f(t_0, x)}{t_n - t_0} d\mu(x) \\ &= \int_X \frac{\partial f}{\partial t}(t_0, x) d\mu(x) \end{aligned}$$

for **all** sequences $t_n \in J \setminus \{t_0\}$ such that $t_n \to t_0$. Therefore, $\dot{G}(t_0) = \lim_{t\to t_0} \frac{G(t)-G(t_0)}{t-t_0}$ exists and

$$\dot{G}(t_0) = \int_X \frac{\partial f}{\partial t}(t_0, x) d\mu(x).$$

■

5.4. **Measurability on Complete Measure Spaces.** In this subsection we will discuss a couple of measurability results concerning completions of measure spaces.

Proposition 5.39. *Suppose that $(X,\mathcal{M},\mu)$ is a complete measure space*[8] *and $f : X \to \mathbb{R}$ is measurable.*

1. *If $g : X \to \mathbb{R}$ is a function such that $f(x) = g(x)$ for μ – a.e. x, then g is measurable.*
2. *If $f_n : X \to \mathbb{R}$ are measurable and $f : X \to \mathbb{R}$ is a function such that $\lim_{n\to\infty} f_n = f$, μ - a.e., then f is measurable as well.*

Proof. 1. Let $E = \{x : f(x) \neq g(x)\}$ which is assumed to be in $\mathcal{M}$ and $\mu(E) = 0$. Then $g = 1_{E^c} f + 1_E g$ since $f = g$ on E^c. Now $1_{E^c} f$ is measurable so g will be measurable if we show $1_E g$ is measurable. For this consider,

$$(1_E g)^{-1}(A) = \begin{cases} E^c \cup (1_E g)^{-1}(A \setminus \{0\}) & \text{if } 0 \in A \\ (1_E g)^{-1}(A) & \text{if } 0 \notin A \end{cases} \tag{5.14}$$

Since $(1_E g)^{-1}(B) \subseteq E$ if $0 \notin B$ and $\mu(E) = 0$, it follow by completeness of $\mathcal{M}$ that $(1_E g)^{-1}(B) \in \mathcal{M}$ if $0 \notin B$ Therefore Eq. (5.14) shows that $1_E g$ is measurable.

2. Let $E = \{x : \lim\limits_{n\to\infty} f_n(x) \neq f(x)\}$ by assumption $E \in \mathcal{M}$ and $\mu(E) = 0$. Since $g \equiv 1_E f = \lim_{n\to\infty} 1_{E^c} f_n$, g is measurable. Because $f = g$ on E^c and $\mu(E) = 0$, $f = g$ a.e. so by part 1. f is also measurable. ■

The above results are in general false if $(X,\mathcal{M},\mu)$ is not complete. For example, let $X = \{0, 1, 2\}$ $\mathcal{M} = \{\{0\}, \{1, 2\}, X, \phi\}$ and $\mu = \delta_0$ Take $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, then $g = 0$ a.e. yet g is not measurable.

Lemma 5.40. *Suppose that $(X,\mathcal{M},\mu)$ is a measure space and $\bar{\mathcal{M}}$ is the completion of $\mathcal{M}$ relative to μ and $\bar{\mu}$ is the extension of μ to $\bar{\mathcal{M}}$. Then a function $f : X \to \mathbb{R}$ is $(\bar{\mathcal{M}}, \mathcal{B} = \mathcal{B}_{\mathbb{R}})$ – measurable iff there exists a function $g : X \to \mathbb{R}$ that is $(\mathcal{M},\mathcal{B})$ – measurable such $E = \{x : f(x) \neq g(x)\} \in \bar{\mathcal{M}}$ and $\bar{\mu}(E) = 0$, i.e. $f(x) = g(x)$ for $\bar{\mu}$ – a.e. x.*

Proof. Suppose first that such a function g exists so that $\bar{\mu}(E) = 0$. Since g is also $(\bar{\mathcal{M}},\mathcal{B})$ – measurable, we see from Proposition 5.39 that f is $(\bar{\mathcal{M}},\mathcal{B})$ – measurable.

Conversely if f is $(\bar{\mathcal{M}},\mathcal{B})$ – measurable, by considering $f_\pm$ we may assume that $f \geq 0$. Choose $(\bar{\mathcal{M}},\mathcal{B})$ – measurable simple function $\phi_n \geq 0$ such that $\phi_n \uparrow f$ as $n \to \infty$. Writing

$$\phi_n = \sum a_k 1_{A_k}$$

with $A_k \in \bar{\mathcal{M}}$, we may choose $B_k \in \mathcal{M}$ such that $B_k \subset A_k$ and $\bar{\mu}(A_k \setminus B_k) = 0$. Letting

$$\tilde{\phi}_n := \sum a_k 1_{B_k}$$

we have produced a $(\mathcal{M},\mathcal{B})$ – measurable simple function $\tilde{\phi}_n \geq 0$ such that $E_n := \{\phi_n \neq \tilde{\phi}_n\}$ has zero $\bar{\mu}$ – measure. Since $\bar{\mu}(\cup_n E_n) \leq \sum_n \bar{\mu}(E_n)$, there exists $F \in \mathcal{M}$ such that $\cup_n E_n \subset F$ and $\mu(F) = 0$. It now follows that

$$1_F \tilde{\phi}_n = 1_F \phi_n \uparrow g := 1_F f \text{ as } n \to \infty.$$

[8]Recall this means that if $N \subset X$ is a set such that $N \subset A \in \mathcal{M}$ and $\mu(A) = 0$, then $N \in \mathcal{M}$ as well.

This shows that $g = 1_F f$ is $(\mathcal{M}, \mathcal{B})$ – measurable and that $\{f \neq g\} \subset F$ has $\bar{\mu}$ – measure zero. ■

5.5. **Comparison of the Lebesgue and the Riemann Integral.** In this section, suppose $-\infty < a < b < \infty$ and $f : [a,b] \to \mathbb{R}$ be a bounded function. To each partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\} \tag{5.15}$$

of $[a,b]$ let

$$S_P f = \sum M_j (t_j - t_{j-1})$$
$$s_P f = \sum m_j (t_j - t_{j-1})$$

where

$$M_j = \sup\{f(x) : t_j < x \le t_{j-1}\}$$
$$m_j = \inf\{f(x) : t_j < x \le t_{j-1}\}$$

and define the upper and lower Riemann integrals by

$$\overline{\int_a^b} f(x)dx = \inf_P S_P f \text{ and}$$
$$\underline{\int_b^a} f(x)dx = \sup_P \; s_P f$$

respectively.

Fact 5.41. Recall the following fact from the theory of Riemann integrals. There exists a refining sequence of partitions P_k (i.e. the P_k 's are increasing) such that

$$S_{P_k} f \searrow \overline{\int_a^b} f \text{ as } k \to \infty \text{ and}$$
$$s_{P_k} f \uparrow \underline{\int_a^b} f \text{ as } k \to \infty.$$

Definition 5.42. The function f is **Riemann integrable** iff $\overline{\int_a^b} f = \underline{\int_a^b} f$ and which case the Riemann integral $\int_a^b f$ is defined to be the common value:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

For a partition P as in Eq. (5.15) let

$$G_P = \sum_1^n M_j 1_{(t_{j-1}, t_j]} \text{ and } g_P = \sum_1^n m_j 1_{(t_{j-1}, t_j]}.$$

If P_k is a sequence of refining partitions as in Fact 5.41, then G_{P_k} is a decreasing sequence, g_{P_k} is an increasing sequence and $g_{P_k} \le f \le G_{P_k}$ for all k. Define

$$G \equiv \lim_{k\to\infty} G_{P_k} \text{ and } g \equiv \lim_{k\to\infty} g_{P_k}. \tag{5.16}$$

and notice that $g \le f \le G$. By the dominated convergence theorem,

$$\int_{[a,b]} g dm = \lim_{k\to\infty} \int_{[a,b]} g_{P_k} = \lim_{k\to\infty} s_{P_k} f = \underline{\int_a^b} f(x)dx$$

and

$$\int_{[a,b]} G dm = \lim_{k\to\infty} \int_{[a,b]} G_{P_k} = \lim_{k\to\infty} S_{P_k} f = \overline{\int_a^b} f(x)dx.$$

Therefore f is Riemann integrable iff $\int_{[a,b]} G = \int_{[a,b]} g$ i.e. iff $\int_{[a,b]} G - g = 0$. Since $G \ge f \ge g$ this happens iff $G = g$ a.e. Hence we have proved the following theorem.

Theorem 5.43. *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff the Borel measurable functions $g, G : [a,b] \to \mathbb{R}$ defined in Eq. (5.16) satisfy $g(x) = G(x)$ for m – a.e. $x \in [a,b]$. Moreover if f is Riemann integrable, then*

$$\int_a^b f(x)dx = \int_{[a,b]} g dm = \int_{[a,b]} G dm.$$

The function f need not be Borel measurable but it is necessarily Lebesgue measurable, i.e. f is $\mathcal{L}/\mathcal{B}$ – measurable where $\mathcal{L}$ is the Lebesgue σ – algebra and $\mathcal{B}$ is the Borel σ – algebra on $[a,b]$. If we let $\bar{m}$ denote the completion of m, then we may also write

$$\int_a^b f(x)dx = \int_{[a,b]} f d\bar{m}.$$

5.6. **Exercises.**

Exercise 5.1. Let μ be a measure on an algebra $\mathcal{A} \subset \mathcal{P}(X)$, then $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$ for all $A, B \in \mathcal{A}$.

Exercise 5.2. Problem 12 on p. 27. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and for $A, B \in \mathcal{M}$ let $\rho(A,B) = \mu(A\Delta B)$ where $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Define $A \sim B$ iff $\mu(A\Delta B) = 0$. Show "$\sim$ " is an equivalence relation, ρ is a metric on $\mathcal{M}/\sim$ and $\mu(A) = \mu(B)$ if $A \sim B$. Also show that $\mu : (\mathcal{M}/\sim) \to [0,\infty)$ is a continuous function relative to the metric ρ.

Exercise 5.3. Suppose that $\mu_n : \mathcal{M} \to [0,\infty]$ are measures on $\mathcal{M}$ for $n \in \mathbb{N}$. Also suppose that $\mu_n(A)$ is increasing in n for all $A \in \mathcal{M}$. Prove that $\mu : \mathcal{M} \to [0,\infty]$ defined by $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ is also a measure.

Exercise 5.4. Now suppose that Λ is some index set and for each $\lambda \in \Lambda$, $\mu_\lambda : \mathcal{M} \to [0,\infty]$ is a measure on $\mathcal{M}$. Define $\mu : \mathcal{M} \to [0,\infty]$ by $\mu(A) = \sum_{\lambda\in\Lambda} \mu_\lambda(A)$ for each $A \in \mathcal{M}$. Show that μ is also a measure.

Exercise 5.5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\rho : X \to [0,\infty]$ be a measurable function. For $A \in \mathcal{M}$, set $\nu(A) := \int_A \rho d\mu$.

1. Show $\nu : \mathcal{M} \to [0,\infty]$ is a measure.
2. Let $f : X \to [0,\infty]$ be a measurable function, show

$$\int_X f d\nu = \int_X f\rho d\mu. \tag{5.17}$$

Hint: first prove the relationship for characteristic functions, then for simple functions, and then for general positive measurable functions.

3. Show that $f \in L^1(\nu)$ iff $f\rho \in L^1(\mu)$ and if $f \in L^1(\nu)$ then Eq. (5.17) still holds.

Notation 5.44. It is customary to informally describe ν defined in Exercise 5.5 by writing $d\nu = \rho d\mu$.

Exercise 5.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, $(Y, \mathcal{F})$ be a measurable space and $f : X \to Y$ be a measurable map. Define a function $\nu : \mathcal{F} \to [0, \infty]$ by $\nu(A) := \mu(f^{-1}(A))$ for all $A \in \mathcal{F}$.

1. Show ν is a measure. (We will write $\nu = f_*\mu$ or $\nu = \mu \circ f^{-1}$.)
2. Show

$$\int_Y g d\nu = \int_X (g \circ f)\, d\mu \tag{5.18}$$

for all measurable functions $g : Y \to [0, \infty]$. **Hint:** see the hint from Exercise 5.5.

3. Show $g \in L^1(\nu)$ iff $g \circ f \in L^1(\mu)$ and that Eq. (5.18) holds for all $g \in L^1(\nu)$.

Exercise 5.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that $f'(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x\to\pm\infty} f(x) = \pm\infty$. Let m be Lebesgue measure and $\lambda = f_*m = m \circ f^{-1}$. Show $d\lambda = f' dm$.

Exercise 5.8. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$, show

$$\mu(\{A_n \text{ a.a.}\}) \le \liminf_{n\to\infty} \mu(A_n)$$

and if $\mu(\cup_{m\ge n} A_m) < \infty$ for some n, then

$$\mu(\{A_n \text{ i.o.}\}) \ge \limsup_{n\to\infty} \mu(A_n).$$

Exercise 5.9. Show

$$\lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n dm(x) = 1.$$

Exercise 5.10 (Peano's Existence Theorem)**.** Suppose $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function. Then for each $T < \infty$[9] there exists a solution to the differential equation

$$\dot{x}(t) = f(t, x(t)) \text{ for } 0 \le t \le T \text{ with } x(0) = x_0. \tag{5.19}$$

Do this by filling in the following outline for the proof.

1. Given $\epsilon > 0$, show there exists a unique function $x_\epsilon \in C([-\epsilon, \infty) \to \mathbb{R}^d)$ such that $x_\epsilon(t) \equiv x_0$ for $-\epsilon \le t \le 0$ and

$$x_\epsilon(t) = x_0 + \int_0^t f(\tau, x_\epsilon(\tau - \epsilon))d\tau \text{ for all } t \ge 0. \tag{5.20}$$

Here

$$\int_0^t f(\tau, x_\epsilon(\tau - \epsilon))d\tau = \left(\int_0^t f_1(\tau, x_\epsilon(\tau - \epsilon))d\tau, \dots, \int_0^t f_d(\tau, x_\epsilon(\tau - \epsilon))d\tau\right)$$

where $f = (f_1, \dots, f_d)$ and the integrals are either the Lebesgue or the Riemann integral since they are equal on continuous functions.

[9] Using Corollary 26.19 below, we may in fact allow $T = \infty$.

2. Then use Exercise 3.38 to show there exists $\{\epsilon_k\}_{k=1}^{\infty} \subset (0,\infty)$ such that $\lim_{k\to\infty}\epsilon_k = 0$ and x_{ϵ_k} converges to some $x \in C([0,T])$ (relative to the sup-norm: $\|x\|_\infty = \sup_{t\in[0,T]}|x(t)|$) as $k\to\infty$.
3. Pass to the limit in Eq. (5.20) with ϵ replaced by ϵ_k to show x satisfies

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))d\tau \;\forall\, t \in [0,T].$$

4. Conclude from this that $\dot{x}(t)$ exists for $t \in (0,T)$ and that x solves Eq. (5.19).

Exercise 5.11. Folland 2.10 on p.49.

Exercise 5.12. Folland 2.12 on p. 52.

Exercise 5.13. Folland 2.13 on p. 52.

Exercise 5.14. Folland 2.14 on p. 52.

Exercise 5.15. Give examples of measurable functions $\{f_n\}$ on $\mathbb{R}$ such that f_n decreases to 0 uniformly yet $\int f_n dm = \infty$ for all n. Also give an example of a sequence of measurable functions $\{g_n\}$ on $[0,1]$ such that $g_n \to 0$ while $\int g_n dm = 1$ for all n.

Exercise 5.16. Folland 2.19 on p. 59.

Exercise 5.17. Folland 2.20 on p. 59.

Exercise 5.18. Folland 2.23 on p. 59.

Exercise 5.19. Folland 2.26 on p. 59.

Exercise 5.20. Folland 2.28 on p. 59.

Exercise 5.21. Folland 2.31b on p. 60.