13. Complex Measures, Radon-Nikodym Theorem and the Dual of L^p

Definition 13.1. A **signed measure** ν on a measurable space $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ such that

(1) Either $\nu(\mathcal{M}) \subset (-\infty, \infty]$ or $\nu(\mathcal{M}) \subset [-\infty, \infty)$.
(2) ν is countably additive, this is to say if $E = \coprod_{j=1}^{\infty} E_j$ with $E_j \in \mathcal{M}$, then $\nu(E) = \sum\limits_{j=1}^{\infty} \nu(E_j)$.[31]
(3) $\nu(\emptyset) = 0$.

If there exists $X_n \in \mathcal{M}$ such that $|\nu(X_n)| < \infty$ and $X = \cup_{n=1}^{\infty} X_n$, then ν is said to be $\sigma -$ **finite** and if $\nu(\mathcal{M}) \subset \mathbb{R}$ then ν is said to be a **finite signed measure.** Similarly, a countably additive set function $\nu : \mathcal{M} \to \mathbb{C}$ such that $\nu(\emptyset) = 0$ is called a **complex measure.**

A finite signed measure is clearly a complex measure.

Example 13.2. Suppose that μ_+ and μ_- are two positive measures on $\mathcal{M}$ such that either $\mu_+(X) < \infty$ or $\mu_-(X) < \infty$, then $\nu = \mu_+ - \mu_-$ is a signed measure. If both $\mu_+(X)$ and $\mu_-(X)$ are finite then ν is a finite signed measure.

Example 13.3. Suppose that $g : X \to \overline{\mathbb{R}}$ is measurable and either $\int_E g^+ d\mu$ or $\int_E g^- d\mu < \infty$, then

$$\nu(A) = \int_A g d\mu \, \forall A \in \mathcal{M} \tag{13.1}$$

defines a signed measure. This is actually a special case of the last example with $\mu_\pm(A) \equiv \int_A g^\pm d\mu$. Notice that the measure $\mu_\pm$ in this example have the property that they are concentrated on disjoint sets, namely μ_+ "lives" on $\{g > 0\}$ and μ_- "lives" on the set $\{g < 0\}$.

Example 13.4. Suppose that μ is a positive measure on $(X, \mathcal{M})$ and $g \in L^1(\mu)$, then ν given as in Eq. (13.1) is a complex measure on $(X, \mathcal{M})$. Also if $\left\{\mu_\pm^r, \mu_\pm^i\right\}$ is any collection of four positive measures on $(X, \mathcal{M})$, then

$$\nu := \mu_+^r - \mu_-^r + i\left(\mu_+^i - \mu_-^i\right) \tag{13.2}$$

is a complex measure.

If ν is given as in Eq. 13.1, then ν may be written as in Eq. (13.2) with $d\mu_\pm^r = (\operatorname{Re} g)_\pm d\mu$ and $d\mu_\pm^i = (\operatorname{Im} g)_\pm d\mu$.

Definition 13.5. Let ν be a complex or signed measure on $(X, \mathcal{M})$. A set $E \in \mathcal{M}$ is a **null** set or precisely a $\nu -$ null set if $\nu(A) = 0$ for all $A \in \mathcal{M}$ such that $A \subset E$, i.e. $\nu|_{\mathcal{M}_E} = 0$. Recall that $\mathcal{M}_E := \{A \cap E : A \in \mathcal{M}\} = i_E^{-1}(\mathcal{M})$ is the "trace of M on E.

[31] If $\nu(E) \in \mathbb{R}$ then the series $\sum\limits_{j=1}^{\infty} \nu(E_j)$ is absolutely convergent since it is independent of rearrangements.

13.1. **Radon-Nikodym Theorem I.** We will eventually show that every complex and $\sigma-$finite signed measure ν may be described as in Eq. (13.1). The next theorem is the first result in this direction.

Theorem 13.6. *Suppose $(X,\mathcal{M})$ is a measurable space, μ is a positive finite measure on $\mathcal{M}$ and ν is a complex measure on $\mathcal{M}$ such that $|\nu(A)| \le \mu(A)$ for all $A \in \mathcal{M}$. Then $d\nu = \rho d\mu$ where $|\rho| \le 1$. Moreover if ν is a positive measure, then $0 \le \rho \le 1$.*

Proof. For a simple function, $f \in \mathcal{S}(X,\mathcal{M})$, let $\nu(f) := \sum_{a\in\mathbb{C}} a\nu(f=a)$. Then

$$|\nu(f)| \le \sum_{a\in\mathbb{C}} |a|\,|\nu(f=a)| \le \sum_{a\in\mathbb{C}} |a|\,\mu(f=a) = \int_X |f|\,d\mu.$$

So, by the B.L.T. theorem, ν extends to a continuous linear functional on $L^1(\mu)$ satisfying the bounds

$$|\nu(f)| \le \int_X |f|\,d\mu \le \sqrt{\mu(X)}\,\|f\|_{L^2(\mu)} \text{ for all } f \in L^1(\mu).$$

The Riesz representation Theorem (Proposition 10.15) then implies there exists a unique $\rho \in L^2(\mu)$ such that

$$\nu(f) = \int_X f\rho d\mu \text{ for all } f \in L^2(\mu).$$

Taking $f = \overline{\operatorname{sgn}(\rho)}1_A$ in this equation shows

$$\int_A |\rho|\,d\mu = \nu(\overline{\operatorname{sgn}(\rho)}1_A) \le \mu(A) = \int_A 1 d\mu$$

from which it follows that $|\rho| \le 1$, $\mu-$a.e. If ν is a positive measure, then for real f, $0 = \operatorname{Im}[\nu(f)] = \int_X \operatorname{Im}\rho f d\mu$ and taking $f = \operatorname{Im}\rho$ shows $0 = \int_X [\operatorname{Im}\rho]^2\,d\mu$, i.e. $\operatorname{Im}(\rho(x)) = 0$ for $\mu-$a.e. x and we have shown ρ is real a.e. Similarly,

$$0 \le \nu(\operatorname{Re}\rho < 0) = \int_{\{\operatorname{Re}\rho<0\}} \rho d\mu \le 0,$$

shows $\rho \ge 0$ a.e. ∎

Definition 13.7. Let μ and ν be two signed or complex measures on $(X,\mathcal{M})$. Then μ and ν are **mutually singular** (written as $\mu \perp \nu$) if there exists $A \in \mathcal{M}$ such that A is a $\nu-$null set and A^c is a $\mu-$null set. The measure ν **is absolutely continuous relative to** μ (written as $\nu \ll \mu$) provided $\nu(A) = 0$ whenever A is a $\mu-$null set, i.e. all $\mu-$null sets are $\nu-$null sets as well.

Remark 13.8. If μ_1, μ_2 and ν are signed measures on $(X,\mathcal{M})$ such that $\mu_1 \perp \nu$ and $\mu_2 \perp \nu$ and $\mu_1+\mu_2$ is well defined, then $(\mu_1+\mu_2) \perp \nu$. If $\{\mu_i\}_{i=1}^\infty$ is a sequence of positive measures such that $\mu_i \perp \nu$ for all i then $\mu = \sum_{i=1}^\infty \mu_i \perp \nu$ as well.

Proof. In both cases, choose $A_i \in \mathcal{M}$ such that A_i is $\nu-$null and A_i^c is μ_i-null for all i. Then by Lemma 13.17, $A := \cup_i A_i$ is still a $\nu-$null set. Since

$$A^c = \cap_i A_i^c \subset A_m^c \text{ for all } m$$

we see that A^c is a μ_i - null set for all i and is therefore a null set for $\mu = \sum_{i=1}^\infty \mu_i$. This shows that $\mu \perp \nu$. ∎

Throughout the remainder of this section μ will be always be a positive measure.

Definition 13.9 (Lebesgue Decomposition)**.** Suppose that ν is a signed (complex) measure and μ is a positive measure on $(X, \mathcal{M})$. Two signed (complex) measures ν_a and ν_s form a **Lebesgue decomposition** of ν relative to μ if

(1) If $\nu = \nu_a + \nu_s$ where implicit in this statement is the assertion that if ν takes on the value ∞ $(-\infty)$ then ν_a and ν_s do not take on the value $-\infty$ (∞).
(2) $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

Lemma 13.10. *Let ν is a signed (complex) measure and μ is a positive measure on $(X, \mathcal{M})$. If there exists a Lebesgue decomposition of ν relative to μ then it is unique. Moreover,* ***the*** *Lebesgue decomposition satisfies the following properties.*

(1) *If ν is a positive measure then so are ν_s and ν_a.*
(2) *If ν is a σ – finite measure then so are ν_s and ν_a.*

Proof. Since $\nu_s \perp \mu$, there exists $A \in \mathcal{M}$ such that $\mu(A) = 0$ and A^c is ν_s – null and because $\nu_a \ll \mu$, A is also a null set for ν_a. So for $C \in \mathcal{M}$, $\nu_a(C \cap A) = 0$ and $\nu_s\left(C \cap A^c\right) = 0$ from which it follows that

$$\nu(C) = \nu(C \cap A) + \nu(C \cap A^c) = \nu_s(C \cap A) + \nu_a(C \cap A^c)$$

and hence,

$$\begin{aligned} \nu_s(C) &= \nu_s(C \cap A) = \nu(C \cap A) \text{ and} \\ \nu_a(C) &= \nu_a(C \cap A^c) = \nu(C \cap A^c). \end{aligned} \tag{13.3}$$

Item 1. is now obvious from Eq. (13.3). For Item 2., if ν is a σ – finite measure then there exists $X_n \in \mathcal{M}$ such that $X = \cup_{n=1}^\infty X_n$ and $|\nu(X_n)| < \infty$ for all n. Since $\nu(X_n) = \nu_a(X_n) + \nu_s(X_n)$, we must have $\nu_a(X_n) \in \mathbb{R}$ and $\nu_s(X_n) \in \mathbb{R}$ showing ν_a and ν_s are σ – finite as well.

For the uniqueness assertion, if we have another decomposition $\nu = \tilde{\nu}_a + \tilde{\nu}_s$ with $\tilde{\nu}_s \perp \tilde{\mu}$ and $\tilde{\nu}_a \ll \tilde{\mu}$ we may choose $\tilde{A} \in \mathcal{M}$ such that $\mu(\tilde{A}) = 0$ and $\tilde{A}^c$ is $\tilde{\nu}_s$ – null. Letting $B = A \cup \tilde{A}$ we have

$$\mu(B) \le \mu(A) + \mu(\tilde{A}) = 0$$

and $B^c = A^c \cap \tilde{A}^c$ is both a ν_s and a $\tilde{\nu}_s$ null set. Therefore by the same arguments that proves Eqs. (13.3), for all $C \in \mathcal{M}$,

$$\begin{aligned} \nu_s(C) &= \nu(C \cap B) = \tilde{\nu}_s(C) \text{ and} \\ \nu_a(C) &= \nu(C \cap B^c) = \tilde{\nu}_a(C). \end{aligned}$$

■

Lemma 13.11. *Suppose μ is a positive measure on $(X, \mathcal{M})$ and $f, g : X \to \bar{\mathbb{R}}$ are extended integrable functions such that*

$$\int_A f d\mu = \int_A g d\mu \text{ for all } A \in \mathcal{M}, \tag{13.4}$$

$\int_X f_- d\mu < \infty$, $\int_X g_- d\mu < \infty$, and the measures $|f|\, d\mu$ and $|g|\, d\mu$ are σ – finite. Then $f(x) = g(x)$ for μ – a.e. x.

Proof. By assumption there exists $X_n \in \mathcal{M}$ such that $X_n \uparrow X$ and $\int_{X_n} |f|\, d\mu < \infty$ and $\int_{X_n} |g|\, d\mu < \infty$ for all n. Replacing A by $A \cap X_n$ in Eq. (13.4) implies

$$\int_A 1_{X_n} f d\mu = \int_{A \cap X_n} f d\mu = \int_{A \cap X_n} g d\mu = \int_A 1_{X_n} g d\mu$$

for all $A \in \mathcal{M}$. Since $1_{X_n} f$ and $1_{X_n} g$ are in $L^1(\mu)$ for all n, this equation implies $1_{X_n} f = 1_{X_n} g$, μ – a.e. Letting $n \to \infty$ then shows that $f = g$, μ – a.e. ■

Remark 13.12. Suppose that f and g are two positive measurable functions on $(X, \mathcal{M}, \mu)$ such that Eq. (13.4) holds. It is not in general true that $f = g$, μ – a.e. A trivial counter example is to take $\mathcal{M} = \mathcal{P}(X)$, $\mu(A) = \infty$ for all non-empty $A \in \mathcal{M}$, $f = 1_X$ and $g = 2 \cdot 1_X$. Then Eq. (13.4) holds yet $f \neq g$.

Theorem 13.13 (Radon Nikodym Theorem for Positive Measures)**.** *Suppose that μ, ν are σ – finite positive measures on $(X, \mathcal{M})$. Then ν has a unique Lebesgue decomposition $\nu = \nu_a + \nu_s$ relative to μ and there exists a unique (modulo sets of μ – measure* 0) *function $\rho : X \to [0, \infty)$ such that $d\nu_a = \rho d\mu$. Moreover, $\nu_s = 0$ iff $\nu \ll \mu$.*

Proof. The uniqueness assertions follow directly from Lemmas 13.10 and 13.11.

Existence. (Von-Neumann's Proof.) First suppose that μ and ν are finite measures and let $\lambda = \mu + \nu$. By Theorem 13.6, $d\nu = h d\lambda$ with $0 \le h \le 1$ and this implies, for all non-negative measurable functions f, that

$$\nu(f) = \lambda(fh) = \mu(fh) + \nu(fh) \tag{13.5}$$

or equivalently

$$\nu(f(1-h)) = \mu(fh). \tag{13.6}$$

Taking $f = 1_{\{h=1\}}$ and $f = g 1_{\{h<1\}} (1-h)^{-1}$ with $g \ge 0$ in Eq. (13.6)

$$\mu(\{h = 1\}) = 0 \text{ and } \nu(g 1_{\{h<1\}}) = \mu(g 1_{\{h<1\}} (1-h)^{-1} h) = \mu(\rho g)$$

where $\rho := 1_{\{h<1\}} \frac{h}{1-h}$ and $\nu_s(g) := \nu(g 1_{\{h=1\}})$. This gives the desired decomposition[32] since

$$\nu(g) = \nu(g 1_{\{h=1\}}) + \nu(g 1_{\{h<1\}}) = \nu_s(g) + \mu(\rho g)$$

and

$$\nu_s(h \neq 1) = 0 \text{ while } \mu(h = 1) = \mu(\{h \neq 1\}^c) = 0.$$

If $\nu \ll \mu$, then $\mu(h = 1) = 0$ implies $\nu(h = 1) = 0$ and hence that $\nu_s = 0$. If $\nu_s = 0$, then $d\nu = \rho d\mu$ and so if $\mu(A) = 0$, then $\nu(A) = \mu(\rho 1_A) = 0$ as well.

For the σ – finite case, write $X = \coprod_{n=1}^{\infty} X_n$ where $X_n \in \mathcal{M}$ are chosen so that $\mu(X_n) < \infty$ and $\nu(X_n) < \infty$ for all n. Let $d\mu_n = 1_{X_n} d\mu$ and $d\nu_n = 1_{X_n} d\nu$. Then by what we have just proved there exists $\rho_n \in L^1(X, \mu_n)$ and measure ν_n^s such that

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[32] Here is the motivation for this construction. Suppose that $d\nu = d\nu_s + \rho d\mu$ is the Radon-Nikodym decompostion and $X = A \coprod B$ such that $\nu_s(B) = 0$ and $\mu(A) = 0$. Then we find

$$\nu_s(f) + \mu(\rho f) = \nu(f) = \lambda(fg) = \nu(fg) + \mu(fg).$$

Letting $f \to 1_A f$ then implies that

$$\nu_s(1_A f) = \nu(1_A f g)$$

which show that $g = 1$ ν –a.e. on A. Also letting $f \to 1_B f$ implies that

$$\mu(\rho 1_B f(1-g)) = \nu(1_B f(1-g)) = \mu(1_B f g) = \mu(fg)$$

which shows that

$$\rho(1-g) = \rho 1_B (1-g) = g \ \mu - \text{a.e.}.$$

This shows that $\rho = \frac{g}{1-g}$ μ – a.e.

$d\nu_n = \rho_n d\mu_n + d\nu_n^s$ with $\nu_n^s \perp \mu_n$, i.e. there exists $A_n, B_n \in \mathcal{M}_{X_n}$ and $\mu(A_n) = 0$ and $\nu_n^s(B_n) = 0$. Define $\nu_s := \sum_{n=1}^\infty \nu_n^s$ and $\rho := \sum_{n=1}^\infty 1_{X_n}\rho_n$, then

$$\nu = \sum_{n=1}^\infty \nu_n = \sum_{n=1}^\infty (\rho_n\mu_n + \nu_n^s) = \sum_{n=1}^\infty (\rho_n 1_{X_n}\mu + \nu_n^s) = \rho\mu + \nu_s$$

and letting $A := \cup_{n=1}^\infty A_n$ and $B := \cup_{n=1}^\infty B_n$, we have $A = B^c$ and

$$\mu(A) = \sum_{n=1}^\infty \mu(A_n) = 0 \text{ and } \nu(B) = \sum_{n=1}^\infty \nu(B_n) = 0.$$

■

Theorem 13.14. *Let $(X, \mathcal{M}, \mu)$ be a σ – finite measure space and suppose that $p, q \in [1, \infty]$ are conjugate exponents. Then for $p \in [1, \infty)$, the map $g \in L^q \to \phi_g \in (L^p)^*$ is an isometric isomorphism of Banach spaces. (Recall that $\phi_g(f) := \int_X fg d\mu$.) We summarize this by writing $(L^p)^* = L^q$ for all $1 \le p < \infty$.*

Proof. The only point that we have not yet proved is the surjectivity of the map $g \in L^q \to \phi_g \in (L^p)^*$. When $p = 2$ the result follows directly from the Riesz theorem. We will begin the proof under the extra assumption that $\mu(X) < \infty$ in which cased bounded functions are in $L^p(\mu)$ for all p. So let $\phi \in (L^p)^*$. We need to find $g \in L^q(\mu)$ such that $\phi = \phi_g$. When $p \in [1, 2]$, $L^2(\mu) \subset L^p(\mu)$ so that we may restrict ϕ to $L^2(\mu)$ and again the result follows fairly easily from the Riesz Theorem, see Exercise 13.1 below.

To handle general $p \in [1, \infty)$, define $\nu(A) := \phi(1_A)$. If $A = \coprod_{n=1}^\infty A_n$ with $A_n \in \mathcal{M}$, then

$$\|1_A - \sum_{n=1}^N 1_{A_n}\|_{L^p} = \|1_{\cup_{n=N+1}^\infty A_n}\|_{L^p} = \left[\mu(\cup_{n=N+1}^\infty A_n)\right]^{\frac{1}{p}} \to 0 \text{ as } N \to \infty.$$

Therefore

$$\nu(A) = \phi(1_A) = \sum_1^\infty \phi(1_{A_n}) = \sum_1^\infty \nu(A_n)$$

showing ν is a complex measure.[33]

For $A \in \mathcal{M}$, let $|\nu|(A)$ be the "total variation" of A defined by

$$|\nu|(A) := \sup\{|\phi(f1_A)| : |f| \le 1\}$$

and notice that

$$|\nu(A)| \le |\nu|(A) \le \|\phi\|_{(L^p)^*}\mu(A)^{1/p} \text{ for all } A \in \mathcal{M}. \tag{13.7}$$

You are asked to show in Exercise 13.2 that $|\nu|$ is a measure on $(X, \mathcal{M})$. (This can also be deduced from Lemma 13.31 and Proposition 13.35 below.) By Eq. (13.7) $|\nu| \ll \mu$, by Theorem 13.6 $d\nu = hd|\nu|$ for some $|h| \le 1$ and by Theorem 13.13 $d|\nu| = \rho d\mu$ for some $\rho \in L^1(\mu)$. Hence, letting $g = \rho h \in L^1(\mu)$, $d\nu = gd\mu$ or equivalently

$$\phi(1_A) = \int_X g1_A d\mu \;\forall\; A \in \mathcal{M}. \tag{13.8}$$

[33] It is at this point that the proof breaks down when $p = \infty$.

By linearity this equation implies

$$\phi(f)=\int_X gf d\mu \tag{13.9}$$

for all simple functions f on X. Replacing f by $1_{\{|g|\le M\}}f$ in Eq. (13.9) shows

$$\phi(f1_{\{|g|\le M\}})=\int_X 1_{\{|g|\le M\}}gf d\mu$$

holds for all simple functions f and then by continuity for all $f\in L^p(\mu)$. By the converse to Holder's inequality, (Proposition 7.26) we learn that

$$\left\|1_{\{|g|\le M\}}g\right\|_q=\sup_{\|f\|_p=1}\left|\phi(f1_{\{|g|\le M\}})\right|\le\sup_{\|f\|_p=1}\|\phi\|_{(L^p)^*}\left\|f1_{\{|g|\le M\}}\right\|_p\le\|\phi\|_{(L^p)^*}.$$

Using the monotone convergence theorem we may let $M\to\infty$ in the previous equation to learn $\|g\|_q\le\|\phi\|_{(L^p)^*}$ .With this result, Eq. (13.9) extends by continuity to hold for all $f\in L^p(\mu)$ and hence we have shown that $\phi=\phi_g$.

Case 2. Now suppose that μ is $\sigma-$finite and $X_n\in\mathcal{M}$ are sets such that $\mu(X_n)<\infty$ and $X_n\uparrow X$ as $n\to\infty$. We will identify $f\in L^p(X_n,\mu)$ with $f1_{X_n}\in L^p(X,\mu)$ and this way we may consider $L^p(X_n,\mu)$ as a subspace of $L^p(X,\mu)$ for all n and $p\in[1,\infty]$.

By Case 1. there exits $g_n\in L^q(X_n,\mu)$ such that

$$\phi(f)=\int_{X_n}g_n f d\mu \text{ for all } f\in L^p(X_n,\mu)$$

and

$$\|g_n\|_q=\sup\left\{|\phi(f)|:f\in L^p(X_n,\mu)\text{ and }\|f\|_{L^p(X_n,\mu)}=1\right\}\le\|\phi\|_{[L^p(\mu)]^*}.$$

It is easy to see that $g_n=g_m$ a.e. on $X_n\cap X_m$ for all m,n so that $g:=\lim_{n\to\infty}g_n$ exists $\mu-$a.e. By the above inequality and Fatou's lemma, $\|g\|_q\le\|\phi\|_{[L^p(\mu)]^*}<\infty$ and since $\phi(f)=\int_{X_n}gfd\mu$ for all $f\in L^p(X_n,\mu)$ and n and $\cup_{n=1}^\infty L^p(X_n,\mu)$ is dense in $L^p(X,\mu)$ it follows by continuity that $\phi(f)=\int_X gfd\mu$ for all $f\in L^p(X,\mu)$,i.e. $\phi=\phi_g$. ■

Example 13.15. Theorem 13.14 fails in general when $p=\infty$. Consider $X=[0,1]$, $\mathcal{M}=\mathcal{B}$, and $\mu=m$. Then $(L^\infty)^*\ne L^1$.

Proof. Let $M:=C([0,1])$"$\subset$"$L^\infty([0,1],dm)$. It is easily seen for $f\in M$, that $\|f\|_\infty=\sup\{|f(x)|:x\in[0,1]\}$ for all $f\in M$. Therefore M is a closed subspace of L^∞. Define $\ell(f)=f(0)$ for all $f\in M$. Then $\ell\in M^*$ with norm 1. Appealing to the Hahn-Banach Theorem 16.15 below, there exists an extension $L\in(L^\infty)^*$ such that $L=\ell$ on M and $\|L\|=1$. If $L\ne\phi_g$ for some $g\in L^1$, i.e.

$$L(f)=\phi_g(f)=\int_{[0,1]}fgdm \text{ for all } f\in L^\infty,$$

then replacing f by $f_n(x)=(1-nx)\,1_{x\le n^{-1}}$ and letting $n\to\infty$ implies, (using the dominated convergence theorem)

$$1=\lim_{n\to\infty}L(f_n)=\lim_{n\to\infty}\int_{[0,1]}f_n gdm=\int_{\{0\}}gdm=0.$$

From this contradiction, we conclude that $L\ne\phi_g$ for any $g\in L^1$. ■

13.2. **Exercises.**

Exercise 13.1. Prove Theorem 13.14 for $p \in [1,2]$ by directly applying the Riesz theorem to $\phi|_{L^2(\mu)}$.

Exercise 13.2. Show $|\nu|$ be defined as in Eq. (13.7) is a positive measure. Here is an outline.

(1) Show

$$|\nu|(A) + |\nu|(B) \le |\nu|(A \cup B). \tag{13.10}$$

when A, B are disjoint sets in $\mathcal{M}$.

(2) If $A = \coprod_{n=1}^{\infty} A_n$ with $A_n \in \mathcal{M}$ then

$$|\nu|(A) \le \sum_{n=1}^{\infty} |\nu|(A_n). \tag{13.11}$$

(3) From Eqs. (13.10) and (13.11) it follows that ν is finitely additive, and hence

$$|\nu|(A) = \sum_{n=1}^{N} |\nu|(A_n) + |\nu|(\cup_{n>N} A_n) \ge \sum_{n=1}^{N} |\nu|(A_n).$$

Letting $N \to \infty$ in this inequality shows $|\nu|(A) \ge \sum_{n=1}^{\infty} |\nu|(A_n)$ which combined with Eq. (13.11) shows $|\nu|$ is countable additive.

Exercise 13.3. Suppose μ_i, ν_i are $\sigma -$ finite positive measures on measurable spaces, $(X_i, \mathcal{M}_i)$, for $i = 1,2$. If $\nu_i \ll \mu_i$ for $i = 1,2$ then $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$ and in fact

$$\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}(x_1, x_2) = \rho_1 \otimes \rho_2(x_1, x_2) := \rho_1(x_1)\rho_2(x_2)$$

where $\rho_i := d\nu_i / d\mu_i$ for $i = 1,2$.

Exercise 13.4. Folland 3.13 on p. 92.

13.3. **Signed Measures.**

Definition 13.16. Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$, then

(1) E is **positive** if for all $A \in \mathcal{M}$ such that $A \subset E$, $\nu(A) \ge 0$, i.e. $\nu|_{\mathcal{M}_E} \ge 0$.
(2) E is **negative** if for all $A \in \mathcal{M}$ such that $A \subset E$, $\nu(A) \le 0$, i.e. $\nu|_{\mathcal{M}_E} \le 0$.

Lemma 13.17. *Suppose that ν is a signed measure on $(X, \mathcal{M})$. Then*

(1) *Any subset of a positive set is positive.*
(2) *The countable union of positive (negative or null) sets is still positive (negative or null).*
(3) *Let us now further assume that $\nu(\mathcal{M}) \subset [-\infty, \infty)$ and $E \in \mathcal{M}$ is a set such that $\nu(E) \in (0, \infty)$. Then there exists a positive set $P \subset E$ such that $\nu(P) \ge \nu(E)$.*

Proof. The first assertion is obvious. If $P_j \in \mathcal{M}$ are positive sets, let $P = \bigcup_{n=1}^{\infty} P_n$. By replacing P_n by the positive set $P_n \setminus \left(\bigcup_{j=1}^{n-1} P_j \right)$ we may assume that the $\{P_n\}_{n=1}^{\infty}$ are pairwise disjoint so that $P = \coprod_{n=1}^{\infty} P_n$. Now if $E \subset P$ and $E \in \mathcal{M}$,

$E = \coprod_{n=1}^{\infty} (E \cap P_n)$ so $\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap P_n) \geq 0$.which shows that P is positive. The proof for the negative and the null case is analogous.

The idea for proving the third assertion is to keep removing "big" sets of negative measure from E. The set remaining from this procedure will be P. We now proceed to the formal proof.

For all $A \in \mathcal{M}$ let $n(A) = 1 \wedge \sup\{-\nu(B) : B \subset A\}$. Since $\nu(\emptyset) = 0$, $n(A) \geq 0$ and $n(A) = 0$ iff A is positive. Choose $A_0 \subset E$ such that $-\nu(A_0) \geq \frac{1}{2} n(E)$ and set $E_1 = E \setminus A_0$, then choose $A_1 \subset E_1$ such that $-\nu(A_1) \geq \frac{1}{2} n(E_1)$ and set $E_2 = E \setminus (A_0 \cup A_1)$. Continue this procedure inductively, namely if $A_0, \ldots, A_{k-1}$ have been chosen let $E_k = E \setminus \left(\coprod_{i=0}^{k-1} A_i \right)$ and choose $A_k \subset E_k$ such that $-\nu(A_k) \geq \frac{1}{2} n(E_k)$. We will now show that

$$P := E \setminus \coprod_{k=0}^{\infty} A_k = \bigcap_{k=0}^{\infty} E_k$$

is a positive set such that $\nu(P) \geq \nu(E)$.

Since $E = P \cup \coprod_{k=0}^{\infty} A_k$,

$$\nu(E) - \nu(P) = \nu(E \setminus P) = \sum_{k=0}^{\infty} \nu(A_k) \leq -\frac{1}{2} \sum_{k=0}^{\infty} n(E_k) \tag{13.12}$$

and hence $\nu(E) \leq \nu(P)$. Moreover, $\nu(E) - \nu(P) > -\infty$ since $\nu(E) \geq 0$ and $\nu(P) \neq \infty$ by the assumption $\nu(\mathcal{M}) \subset [-\infty, \infty)$. Therefore we may conclude from Eq. (13.12) that $\sum_{k=0}^{\infty} n(E_k) < \infty$ and in particular $\lim_{k\to\infty} n(E_k) = 0$. Now if $A \subset P$ then $A \subset E_k$ for all k and this implies that $\nu(A) \geq 0$ since by definition of $n(E_k)$,

$$-\nu(A) \leq n(E_k) \to 0 \text{ as } k \to \infty.$$

■

13.3.1. *Hahn Decomposition Theorem.*

Definition 13.18. Suppose that ν is a signed measure on $(X, \mathcal{M})$. A **Hahn decomposition** for ν is a partition $\{P, N\}$ of X such that P is positive and N is negative.

Theorem 13.19 (Hahn Decomposition Theorem)**.** *Every signed measure space $(X, \mathcal{M}, \nu)$ has a Hahn decomposition, $\{P, N\}$. Moreover, if $\{\tilde{P}, \tilde{N}\}$ is another Hahn decomposition, then $P \Delta \tilde{P} = N \Delta \tilde{N}$ is a null set, so the decomposition is unique modulo null sets.*

Proof. With out loss of generality we may assume that $\nu(\mathcal{M}) \subset [-\infty, \infty)$. If not just consider $-\nu$ instead. Let us begin with the uniqueness assertion. Suppose that $A \in \mathcal{M}$, then

$$\nu(A) = \nu(A \cap P) + \nu(A \cap N) \leq \nu(A \cap P) \leq \nu(P)$$

and similarly $\nu(A) \leq \nu(\tilde{P})$ for all $A \in \mathcal{M}$.Therefore

$$\nu(P) \leq \nu(P \cup \tilde{P}) \leq \nu(\tilde{P}) \text{ and } \nu(\tilde{P}) \leq \nu(P \cup \tilde{P}) \leq \nu(P)$$

which shows that

$$s := \nu(\tilde{P}) = \nu(P \cup \tilde{P}) = \nu(P).$$

Since
$$s = \nu(P \cup \tilde{P}) = \nu(P) + \nu(\tilde{P}) - \nu(P \cap \tilde{P}) = 2s - \nu(P \cap \tilde{P})$$
we see that $\nu(P \cap \tilde{P}) = s$ and since
$$s = \nu(P \cup \tilde{P}) = \nu(P \cap \tilde{P}) + \nu(\tilde{P}\Delta P)$$
it follows that $\nu(\tilde{P}\Delta P) = 0$. Thus $N\Delta\tilde{N} = \tilde{P}\Delta P$ is a positive set with zero measure, i.e. $N\Delta\tilde{N} = \tilde{P}\Delta P$ is a null set and this proves the uniqueness assertion.

Let
$$s \equiv \sup\{\nu(A) : A \in \mathcal{M}\}$$
which is non-negative since $\nu(\emptyset) = 0$. If $s = 0$, we are done since $P = \emptyset$ and $N = X$ is the desired decomposition. So assume $s > 0$ and choose $A_n \in \mathcal{M}$ such that $\nu(A_n) > 0$ and $\lim_{n\to\infty} \nu(A_n) = s$. By Lemma 13.17here exists positive sets $P_n \subset A_n$ such that $\nu(P_n) \geq \nu(A_n)$. Then $s \geq \nu(P_n) \geq \nu(A_n) \to s$ as $n \to \infty$ implies that $s = \lim_{n\to\infty} \nu(P_n)$. The set $P \equiv \cup_{n=1}^{\infty} P_n$ is a positive set being the union of positive sets and since $P_n \subset P$ for all n,
$$\nu(P) \geq \nu(P_n) \to s \text{ as } n \to \infty.$$
This shows that $\nu(P) \geq s$ and hence by the definition of s, $s = \nu(P) < \infty$.

We now claim that $N = P^c$ is a negative set and therefore, $\{P, N\}$ is the desired Hahn decomposition. If N were not negative, we could find $E \subset N = P^c$ such that $\nu(E) > 0$. We then would have
$$\nu(P \cup E) = \nu(P) + \nu(E) = s + \nu(E) > s$$
which contradicts the definition of s. ■

13.3.2. *Jordan Decomposition.*

Definition 13.20. Let $X = P \cup N$ be a Hahn decomposition of ν and define
$$\nu_+(E) = \nu(P \cap E) \text{ and } \nu_-(E) = -\nu(N \cap E) \ \forall\ E \in \mathcal{M}.$$

Suppose $X = \widetilde{P} \cup \widetilde{N}$ is another Hahn Decomposition and $\widetilde{\nu}_\pm$ are define as above with P and N replaced by $\widetilde{P}$ and $\widetilde{N}$ respectively. Then
$$\widetilde{\nu}_+(E) = \nu(E \cap \widetilde{P}) = \nu(E \cap \widetilde{P} \cap P) + \nu((E \cap \widetilde{P} \cap N) = \nu(E \cap \widetilde{P} \cap P)$$
since $N \cap \tilde{P}$ is both positive and negative and hence null. Similarly $\nu_+(E) = \nu(E \cap \widetilde{P} \cap P)$ showing that $\nu_+ = \widetilde{\nu}_+$ and therefore also that $\nu_- = \widetilde{\nu}_-$.

Theorem 13.21 (Jordan Decomposition)**.** *There exists unique positive measure $\nu_\pm$ such that $\nu_+ \perp \nu_-$ and $\nu = \nu_+ - \nu_-$.*

Proof. Existence has been proved. For uniqueness suppose $\nu = \nu_+ - \nu_-$ is a Jordan Decomposition. Since $\nu_+ \perp \nu_-$ there exists $P, N = P^c \in \mathcal{M}$ such that $\nu_+(N) = 0$ and $\nu_-(P) = 0$. Then clearly P is positive for ν and N is negative for ν. Now $\nu(E \cap P) = \nu_+(E)$ and $\nu(E \cap N) = \nu_-(E)$. The uniqueness now follows from the remarks after Definition 13.20. ■

Definition 13.22. $|\nu|(E) = \nu_+(E) + \nu_-(E)$ is called the total variation of ν. A signed measure is called $\sigma -$ **finite** provided that $|\nu| := \nu_+ + \nu_-$ is a σ finite measure.

Lemma 13.23. *Let ν be a signed measure on $(X,\mathcal{M})$ and $A\in\mathcal{M}$. If $\nu(A)\in\mathbb{R}$ then $\nu(B)\in\mathbb{R}$ for all $B\subset A$. Moreover, $\nu(A)\in\mathbb{R}$ iff $|\nu|(A)<\infty$. In particular, ν is σ finite iff $|\nu|$ is σ – finite. Furthermore if $P,N\in\mathcal{M}$ is a Hahn decomposition for ν and $g=1_P-1_N$, then $d\nu=gd|\nu|$, i.e.*

$$\nu(A)=\int_A gd|\nu| \text{ for all } A\in\mathcal{M}.$$

Proof. Suppose that $B\subset A$ and $|\nu(B)|=\infty$ then since $\nu(A)=\nu(B)+\nu(A\setminus B)$ we must have $|\nu(A)|=\infty$. Let $P,N\in\mathcal{M}$ be a Hahn decomposition for ν, then

$$\nu(A)=\nu(A\cap P)+\nu(A\cap N)=|\nu(A\cap P)|-|\nu(A\cap N)| \text{ and}$$
$$|\nu|(A)=\nu(A\cap P)-\nu(A\cap N)=|\nu(A\cap P)|+|\nu(A\cap N)|. \tag{13.13}$$

Therefore $\nu(A)\in\mathbb{R}$ iff $\nu(A\cap P)\in\mathbb{R}$ and $\nu(A\cap N)\in\mathbb{R}$ iff $|\nu|(A)<\infty$. Finally,

$$\begin{aligned}\nu(A)&=\nu(A\cap P)+\nu(A\cap N)\\&=|\nu|(A\cap P)-|\nu|(A\cap N)\\&=\int_A(1_P-1_N)d|\nu|\end{aligned}$$

which shows that $d\nu=gd|\nu|$. ■

Definition 13.24. Let ν be a signed measure on $(X,\mathcal{M})$, let

$$L^1(\nu):=L^1(\nu^+)\cap L^1(\nu^-)=L^1(|\nu|)$$

and for $f\in L^1(\nu)$ we define

$$\int_X fd\nu=\int_X fd\nu_+-\int_X fd\nu_-.$$

Lemma 13.25. *Let μ be a positive measure on $(X,\mathcal{M})$, g be an extended integrable function on $(X,\mathcal{M},\mu)$ and $d\nu=gd\mu$. Then $L^1(\nu)=L^1(|g|d\mu)$ and for $f\in L^1(\nu)$,*

$$\int_X fd\nu=\int_X fgd\mu.$$

Proof. We have already seen that $d\nu_+=g_+d\mu$, $d\nu_-=g_-d\mu$, and $d|\nu|=|g|d\mu$ so that $L^1(\nu)=L^1(|\nu|)=L^1(|g|d\mu)$ and for $f\in L^1(\nu)$,

$$\begin{aligned}\int_X fd\nu&=\int_X fd\nu_+-\int_X fd\nu_-=\int_X fg_+d\mu-\int_X fg_-d\mu\\&=\int_X f(g_+-g_-)d\mu=\int_X fgd\mu.\end{aligned}$$

■

Lemma 13.26. *Suppose that μ is a positive measure on $(X,\mathcal{M})$ and $g:X\to\mathbb{R}$ is an extended integrable function. If ν is the signed measure $d\nu=gd\mu$, then $d\nu_\pm=g_\pm d\mu$ and $d|\nu|=|g|d\mu$. We also have*

$$|\nu|(A)=\sup\{\int_A f\ d\nu:|f|\le 1\} \text{ for all } A\in\mathcal{M}. \tag{13.14}$$

Proof. The pair, $P=\{g>0\}$ and $N=\{g\le 0\}=P^c$ is a Hahn decomposition for ν. Therefore

$$\nu_+(A)=\nu(A\cap P)=\int_{A\cap P}gd\mu=\int_A 1_{\{g>0\}}gd\mu=\int_A g_+d\mu,$$

$$\nu_-(A) = -\nu(A \cap N) = -\int_{A\cap N} g d\mu = -\int_A 1_{\{g\le 0\}} g d\mu = -\int_A g_- d\mu.$$

and

$$|\nu|(A) = \nu_+(A) + \nu_-(A) = \int_A g_+ d\mu - \int_A g_- d\mu$$
$$= \int_A (g_+ - g_-) d\mu = \int_A |g| d\mu.$$

If $A \in \mathcal{M}$ and $|f| \le 1$, then

$$\left|\int_A f\ d\nu\right| = \left|\int_A f\ d\nu_+ - \int_A f\ d\nu^-\right| \le \left|\int_A f\ d\nu_+\right| + \left|\int_A f\ d\nu^-\right|$$
$$\le \int_A |f| d\nu_+ + \int_A |f| d\nu_- = \int_A |f|\ d|\nu| \le |\nu|(A).$$

For the reverse inequality, let $f \equiv 1_P - 1_N$ then

$$\int_A f\ d\nu = \nu(A\cap P) - \nu(A \cap N) = \nu^+(A) + \nu^-(A) = |\nu|(A).$$

■

Lemma 13.27. *Suppose ν is a signed measure, μ is a positive measure and $\nu = \nu_a + \nu_s$ is a Lebesgue decomposition of ν relative to μ, then $|\nu| = |\nu_a| + |\nu_s|$.*

Proof. Let $A \in \mathcal{M}$ be chosen so that A is a null set for ν_a and A^c is a null set for ν_s. Let $A = P' \coprod N'$ be a Hahn decomposition of $\nu_s|_{\mathcal{M}_A}$ and $A^c = \tilde{P} \coprod \tilde{N}$ be a Hahn decomposition of $\nu_a|_{\mathcal{M}_{A^c}}$. Let $P = P' \cup \tilde{P}$ and $N = N' \cup \tilde{N}$. Since for $C \in \mathcal{M}$,

$$\nu(C\cap P) = \nu(C \cap P') + \nu(C\cap \tilde{P})$$
$$= \nu_s(C\cap P') + \nu_a(C \cap \tilde{P}) \ge 0$$

and

$$\nu(C\cap N) = \nu(C \cap N') + \nu(C\cap \tilde{N})$$
$$= \nu_s(C\cap N') + \nu_a(C \cap \tilde{N}) \le 0$$

we see that $\{P, N\}$ is a Hahn decomposition for ν. It also easy to see that $\{P, N\}$ is a Hahn decomposition for both ν_s and ν_a as well. Therefore,

$$|\nu|(C) = \nu(C\cap P) - \nu(C\cap N)$$
$$= \nu_s(C\cap P) - \nu_s(C\cap N) + \nu_a(C\cap P) - \nu_a(C \cap N)$$
$$= |\nu_s|(C) + |\nu_a|(C).$$

■

Lemma 13.28. *1) Let ν be a signed measure and μ be a positive measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu \equiv 0$. 2) Suppose that $\nu = \sum_{i=1}^\infty \nu_i$ where ν_i are positive measures on $(X, \mathcal{M})$ such that $\nu_i \ll \mu$, then $\nu \ll \mu$. Also if ν_1 and ν_2 are two signed measure such that $\nu_i \ll \mu$ for $i = 1, 2$ and $\nu = \nu_1 + \nu_2$ is well defined, then $\nu \ll \mu$.*

Proof. (1) Because $\nu \perp \mu$, there exists $A \in \mathcal{M}$ such that A is a ν – null set and $B = A^c$ is a μ - null set. Since B is μ – null and $\nu \ll \mu$, B is also ν – null. This shows by Lemma 13.17 that $X = A \cup B$ is also ν – null, i.e. ν is the zero measure. The proof of (2) is easy and is left to the reader. ■

Theorem 13.29 (Radon Nikodym Theorem for Signed Measures)**.** *Let ν be a σ – finite signed measure and μ be a σ – finite positive measure on $(X, \mathcal{M})$. Then ν has a unique Lebesgue decomposition $\nu = \nu_a + \nu_s$ relative to μ and there exists a unique (modulo sets of μ – measure 0) extended integrable function $\rho : X \to \mathbb{R}$ such that $d\nu_a = \rho d\mu$. Moreover, $\nu_s = 0$ iff $\nu \ll \mu$, i.e. $d\nu = \rho d\mu$ iff $\nu \ll \mu$.*

Proof. Uniqueness. Is a direct consequence of Lemmas 13.10 and 13.11.

Existence. Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition of ν. Assume, without loss of generality, that $\nu_+(X) < \infty$, i.e. $\nu(A) < \infty$ for all $A \in \mathcal{M}$. By the Radon Nikodym Theorem 13.13 for positive measures there exist functions $f_\pm : X \to [0, \infty)$ and measures $\lambda_\pm$ such that $\nu_\pm = \mu_{f_\pm} + \lambda_\pm$ with $\lambda_\pm \perp \mu$. Since

$$\infty > \nu_+(X) = \mu_{f_+}(X) + \lambda_+(X),$$

$f_+ \in L^1(\mu)$ and $\lambda_+(X) < \infty$ so that $f = f_+ - f_-$ is an extended integrable function, $d\nu_a := f d\mu$ and $\nu_s = \lambda_+ - \lambda_-$ are signed measures. This finishes the existence proof since

$$\nu = \nu_+ - \nu_- = \mu_{f_+} + \lambda_+ - \left(\mu_{f_-} + \lambda_-\right) = \nu_a + \nu_s$$

and $\nu_s = (\lambda_+ - \lambda_-) \perp \mu$ by Remark 13.8.

For the final statement, if $\nu_s = 0$, then $d\nu = \rho d\mu$ and hence $\nu \ll \mu$. Conversely if $\nu \ll \mu$, then $d\nu_s = d\nu - \rho d\mu \ll \mu$, so by Lemma 13.17, $\nu_s = 0$. Alternatively just use the uniqueness of the Lebesgue decomposition to conclude $\nu_a = \nu$ and $\nu_s = 0$. Or more directly, choose $B \in \mathcal{M}$ such that $\mu(B^c) = 0$ and B is a ν_s – null set. Since $\nu \ll \mu$, B^c is also a ν – null set so that, for $A \in \mathcal{M}$,

$$\nu(A) = \nu(A \cap B) = \nu_a(A \cap B) + \nu_s(A \cap B) = \nu_a(A \cap B).$$

■

Notation 13.30. The function f is called the Radon-Nikodym derivative of ν relative to μ and we will denote this function by $\frac{d\nu}{d\mu}$.

13.4. **Complex Measures II.** Suppose that ν is a complex measure on $(X, \mathcal{M})$, let $\nu_r := \operatorname{Re}\nu$, $\nu_i := \operatorname{Im}\nu$ and $\mu := |\nu_r| + |\nu_i|$. Then μ is a finite positive measure on $\mathcal{M}$ such that $\nu_r \ll \mu$ and $\nu_i \ll \mu$. By the Radon-Nikodym Theorem 13.29, there exists real functions $h, k \in L^1(\mu)$ such that $d\nu_r = h\ d\mu$ and $d\nu_i = k\ d\mu$. So letting $g := h + ik \in L^1(\mu)$,

$$d\nu = (h + ik)d\mu = gd\mu$$

showing every complex measure may be written as in Eq. (13.1).

Lemma 13.31. *Suppose that ν is a complex measure on $(X, \mathcal{M})$, and for $i = 1, 2$ let μ_i be a finite positive measure on $(X, \mathcal{M})$ such that $d\nu = g_i d\mu_i$ with $g_i \in L^1(\mu_i)$. Then*

$$\int_A |g_1|\, d\mu_1 = \int_A |g_2|\, d\mu_2 \text{ for all } A \in \mathcal{M}.$$

In particular, we may define a positive measure $|\nu|$ on $(X, \mathcal{M})$ by

$$|\nu|(A) = \int_A |g_1|\, d\mu_1 \text{ for all } A \in \mathcal{M}.$$

The finite positive measure $|\nu|$ is called the ***total variation measure*** *of ν.*

Proof. Let $\lambda = \mu_1 + \mu_2$ so that $\mu_i \ll \lambda$. Let $\rho_i = d\mu_i/d\lambda \geq 0$ and $h_i = \rho_i g_i$. Since

$$\nu(A) = \int_A g_i d\mu_i = \int_A g_i \rho_i d\lambda = \int_A h_i d\lambda \text{ for all } A \in \mathcal{M},$$

$h_1 = h_2$, λ –a.e. Therefore

$$\int_A |g_1|\, d\mu_1 = \int_A |g_1|\, \rho_1 d\lambda = \int_A |h_1|\, d\lambda = \int_A |h_2|\, d\lambda = \int_A |g_2|\, \rho_2 d\lambda = \int_A |g_2|\, d\mu_2.$$

■

Definition 13.32. Given a complex measure ν, let $\nu_r = \operatorname{Re}\nu$ and $\nu_i = \operatorname{Im}\nu$ so that ν_r and ν_i are finite signed measures such that

$$\nu(A) = \nu_r(A) + i\nu_i(A) \text{ for all } A \in \mathcal{M}.$$

Let $L^1(\nu) := L^1(\nu_r) \cap L^1(\nu_i)$ and for $f \in L^1(\nu)$ define

$$\int_X f d\nu := \int_X f d\nu_r + i \int_X f d\nu_i.$$

Example 13.33. Suppose that μ is a positive measure on $(X, \mathcal{M})$, $g \in L^1(\mu)$ and $\nu(A) = \int_A g d\mu$ as in Example 13.4, then $L^1(\nu) = L^1(|g|\, d\mu)$ and for $f \in L^1(\nu)$

$$\int_X f d\nu = \int_X f g d\mu. \tag{13.15}$$

To check Eq. (13.15), notice that $d\nu_r = \operatorname{Re} g\ d\mu$ and $d\nu_i = \operatorname{Im} g\ d\mu$ so that (using Lemma 13.25)

$$L^1(\nu) = L^1(\operatorname{Re} g d\mu) \cap L^1(\operatorname{Im} g d\mu) = L^1(|\operatorname{Re} g|\, d\mu) \cap L^1(|\operatorname{Im} g|\, d\mu) = L^1(|g|\, d\mu).$$

If $f \in L^1(\nu)$, then

$$\int_X f d\nu := \int_X f \operatorname{Re} g d\mu + i \int_X f \operatorname{Im} g d\mu = \int_X f g d\mu.$$

Remark 13.34. Suppose that ν is a complex measure on $(X, \mathcal{M})$ such that $d\nu = g d\mu$ and as above $d\,|\nu| = |g|\, d\mu$. Letting

$$\rho = \operatorname{sgn}(\rho) := \begin{cases} \frac{g}{|g|} & \text{if} \quad |g| \neq 0 \\ 1 & \text{if} \quad |g| = 0 \end{cases}$$

we see that

$$d\nu = g d\mu = \rho\, |g|\, d\mu = \rho d\, |\nu|$$

and $|\rho| = 1$ and ρ is uniquely defined modulo $|\nu|$ – null sets. We will denote ρ by $d\nu/d\,|\nu|$. With this notation, it follows from Example 13.33 that $L^1(\nu) := L^1(|\nu|)$ and for $f \in L^1(\nu)$,

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\,|\nu|} d\,|\nu|\,.$$

Proposition 13.35 (Total Variation)**.** *Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, $\mathcal{M} = \sigma(\mathcal{A})$, ν is a complex (or a signed measure which is σ – finite on $\mathcal{A}$) on $(X, \mathcal{M})$*

and for $E \in \mathcal{M}$ let

$$\mu_0(E) = \sup\left\{\sum_1^n |\nu(E_j)| : E_j \in \mathcal{A}_E \ \ni E_i \cap E_j = \delta_{ij} E_i,\ n = 1, 2, \dots\right\}$$

$$\mu_1(E) = \sup\left\{\sum_1^n |\nu(E_j)| : E_j \in \mathcal{M}_E \ \ni E_i \cap E_j = \delta_{ij} E_i,\ n = 1, 2, \dots\right\}$$

$$\mu_2(E) = \sup\left\{\sum_1^\infty |\nu(E_j)| : E_j \in \mathcal{M}_E \ \ni E_i \cap E_j = \delta_{ij} E_i\right\}$$

$$\mu_3(E) = \sup\left\{\left|\int_E f d\nu\right| : f \text{ is measurable with } |f| \le 1\right\}$$

$$\mu_4(E) = \sup\left\{\left|\int_E f d\nu\right| : f \in \mathbb{S}_f(\mathcal{A}, |\nu|) \text{ with } |f| \le 1\right\}.$$

then $\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = |\nu|$.

Proof. Let $\rho = d\nu/d|\nu|$ and recall that $|\rho| = 1$, $|\nu|$ – a.e. We will start by showing $|\nu| = \mu_3 = \mu_4$. If f is measurable with $|f| \le 1$ then

$$\left|\int_E f\ d\nu\right| = \left|\int_E f\ \rho d|\nu|\right| \le \int_E |f|\ d|\nu| \le \int_E 1 d|\nu| = |\nu|(E)$$

from which we conclude that $\mu_4 \le \mu_3 \le |\nu|$. Taking $f = \bar{\rho}$ above shows

$$\left|\int_E f\ d\nu\right| = \int_E \bar{\rho}\ \rho\ d|\nu| = \int_E 1\ d|\nu| = |\nu|(E)$$

which shows that $|\nu| \le \mu_3$ and hence $|\nu| = \mu_3$. To show $|\nu| = \mu_4$ as well let $X_m \in \mathcal{A}$ be chosen so that $|\nu|(X_m) < \infty$ and $X_m \uparrow X$ as $m \to \infty$. By Theorem 9.3 of Corollary 11.27, there exists $\rho_n \in \mathbb{S}_f(\mathcal{A}, \mu)$ such that $\rho_n \to \rho 1_{X_m}$ in $L^1(|\nu|)$ and each ρ_n may be written in the form

$$\rho_n = \sum_{k=1}^N z_k 1_{A_k} \tag{13.16}$$

where $z_k \in \mathbb{C}$ and $A_k \in \mathcal{A}$ and $A_k \cap A_j = \emptyset$ if $k \ne j$. I claim that we may assume that $|z_k| \le 1$ in Eq. (13.16) for if $|z_k| > 1$ and $x \in A_k$,

$$|\rho(x) - z_k| \ge \left|\rho(x) - |z_k|^{-1} z_k\right|.$$

This is evident from Figure 26 and formally follows from the fact that

$$\frac{d}{dt}\left|\rho(x) - t\,|z_k|^{-1} z_k\right|^2 = 2\left[t - \operatorname{Re}(|z_k|^{-1} z_k \overline{\rho(x)})\right] \ge 0$$

when $t \ge 1$.

Therefore if we define

$$w_k := \begin{cases} |z_k|^{-1} z_k & \text{if} \quad |z_k| > 1 \\ z_k & \text{if} \quad |z_k| \le 1 \end{cases}$$

and $\tilde{\rho}_n = \sum\limits_{k=1}^N w_k 1_{A_k}$ then

$$|\rho(x) - \rho_n(x)| \ge |\rho(x) - \tilde{\rho}_n(x)|$$

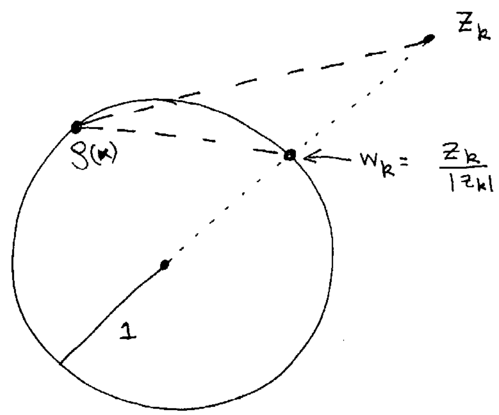

FIGURE 26. Sliding points to the unit circle.

and therefore $\tilde{\rho}_n \to \rho 1_{X_m}$ in $L^1(|\nu|)$. So we now assume that ρ_n is as in Eq. (13.16) with $|z_k| \le 1$.

Now

$$\left|\int_E \bar{\rho}_n d\nu - \int_E \bar{\rho} 1_{X_m} d\nu\right| \le \left|\int_E \left(\bar{\rho}_n d\nu - \bar{\rho} 1_{X_m}\right) \rho d|\nu|\right| \le \int_E |\bar{\rho}_n - \bar{\rho} 1_{X_m}| \, d|\nu| \to 0 \text{ as } n \to \infty$$

and hence

$$\mu_4(E) \ge \left|\int_E \bar{\rho} 1_{X_m} d\nu\right| = |\nu|\,(E \cap X_m) \text{ for all } m.$$

Letting $m \uparrow \infty$ in this equation shows $\mu_4 \ge |\nu|$.

We will now show $\mu_0 = \mu_1 = \mu_2 = |\nu|$. Clearly $\mu_0 \le \mu_1 \le \mu_2$. Suppose $E_j \in \mathcal{M}_E$ such that $E_i \cap E_j = \delta_{ij} E_i$, then

$$\sum |\nu(E_j)| = \sum |\int_{E_j} \rho d|\nu| \le \sum |\nu|(E_j) = |\nu|(\cup E_j) \le |\nu|\,(E)$$

which shows that $\mu_2 \le |\nu| = \mu_4$. So it suffices to show $\mu_4 \le \mu_0$. But if $f \in \mathbb{S}_f(\mathcal{A}, |\nu|)$ with $|f| \le 1$, then f may be expressed as $f = \sum_{k=1}^N z_k 1_{A_k}$ with $|z_k| \le 1$ and $A_k \cap A_j = \delta_{ij} A_k$. Therefore,

$$\left|\int_E f d\nu\right| = \left|\sum_{k=1}^N z_k \nu(A_k \cap E)\right| \le \sum_{k=1}^N |z_k|\,|\nu(A_k \cap E)| \le \sum_{k=1}^N |\nu(A_k \cap E)| \le \mu_0(A).$$

Since this equation holds for all $f \in \mathbb{S}_f(\mathcal{A}, |\nu|)$ with $|f| \le 1$, $\mu_4 \le \mu_0$ as claimed. ■

13.5. **Absolute Continuity on an Algebra.** The following results will be useful in Section 14.4 below.

Lemma 13.36. *Let ν be a complex or a signed measure and μ be a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff $|\nu| \ll \mu$.*

Proof. In all cases we have $|\nu(A)| \le |\nu|\,(A)$ for all $A \in \mathcal{M}$ which clearly shows $\nu \ll \mu$ if $|\nu| \ll \mu$.

Now suppose that ν is a signed measure such that $\nu \ll \mu$. Let $P \in \mathcal{M}$ be chosen so that $\{P, N = P^c\}$ is a Hahn decomposition for ν. If $A \in \mathcal{M}$ and $\mu(A) = 0$ then $\nu(A \cap P) = 0$ and $\nu(A \cap N) = 0$ since $\mu(A \cap P) = 0$ and $\mu(A \cap N) = 0$. Therefore

$$|\nu|(A) = \nu(A \cap P) - \nu(A \cap N) = 0$$

and this shows $|\nu| \ll \mu$.

Now suppose that ν is a complex measure such that $\nu \ll \mu$, then $\nu_r := \operatorname{Re} \nu \ll \mu$ and $\nu_i := \operatorname{Im} \nu \ll \mu$ which implies that $|\nu_r| \ll \mu$ and $|\nu_i| \ll \mu$. Since $|\nu| \leq |\nu_r| + |\nu_i|$, this shows that $|\nu| \ll \mu$.

Here are some alternative proofs in the complex case.

1) Let $\rho = \frac{d\nu}{d|\nu|}$. If $A \in \mathcal{M}$ and $\mu(A) = 0$ then by assumption

$$0 = \nu(B) = \int_B \rho d|\nu|$$

for all $B \in \mathcal{M}_A$. This shows that $\rho 1_A = 0$ for $|\nu|$ – a.e. and hence

$$|\nu|(A) = \int_A |\rho| \, d|\nu| = \int_X 1_A |\rho| \, d|\nu| = 0,$$

i.e. $\mu(A) = 0$ implies $|\nu|(A) = 0$.

2) If $\nu \ll \mu$ and $\mu(A) = 0$, then by Proposition 13.35

$$|\nu|(A) = \sup \left\{ \sum_1^\infty |\nu(E_j)| : E_j \in \mathcal{M}_A \ni E_i \cap E_j = \delta_{ij} E_i \right\} = 0$$

since $E_j \subset A$ implies $\mu(E_j) = 0$ and hence $\nu(E_j) = 0$. ■

Theorem 13.37 ($\epsilon - \delta$ Definition of Absolute Continuity)**.** *Let ν be a complex measure and μ be a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(A)| < \epsilon$ whenever $A \in \mathcal{M}$ and $\mu(A) < \delta$.*

Proof. ($\Longrightarrow$) If $\mu(A) = 0$ then $|\nu(A)| < \epsilon$ for all $\epsilon > 0$ which shows that $\nu(A) = 0$, i.e. $\nu \ll \mu$.

($\Longleftarrow$) Since $|\nu(A)| \leq |\nu|(A)$ it suffices to assume $\nu \geq 0$ with $\nu(X) < \infty$. Suppose for the sake of contradiction there exists $\epsilon > 0$ and $A_n \in \mathcal{M}$ such that $\nu(A_n) \geq \epsilon > 0$ while $\mu(A_n) \leq \frac{1}{2^n}$. Let

$$A = \{A_n \text{ i.o.}\} = \bigcap_{N=1}^\infty \bigcup_{n \geq N} A_n$$

so that

$$\mu(A) = \lim_{N \to \infty} \mu\left(\cup_{n \geq N} A_n\right) \leq \lim_{N \to \infty} \sum_{n=N}^\infty \mu(A_n) \leq \lim_{N \to \infty} 2^{-(N-1)} = 0.$$

On the other hand,

$$\nu(A) = \lim_{N \to \infty} \nu\left(\cup_{n \geq N} A_n\right) \geq \lim_{n \to \infty} \inf \nu(A_n) \geq \epsilon > 0$$

showing that ν is not absolutely continuous relative to μ. ■

Corollary 13.38. *Let μ be a positive measure on $(X, \mathcal{M})$ and $f \in L^1(d\mu)$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\left| \int_A f \, d\mu \right| < \epsilon$ for all $A \in \mathcal{M}$ such that $\mu(A) < \delta$.*

Proof. Apply theorem 13.37 to the signed measure $\nu(A) = \int_A f\, d\mu$ for all $A \in \mathcal{M}$. ■

Theorem 13.39 (Absolute Continuity on an Algebra)**.** *Let ν be a complex measure and μ be a positive measure on $(X, \mathcal{M})$. Suppose that $\mathcal{A} \subset \mathcal{M}$ is an algebra such that $\sigma(\mathcal{A}) = \mathcal{M}$ and that μ is σ – finite on $\mathcal{A}$. Then $\nu \ll \mu$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(A)| < \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$.*

Proof. ($\Longrightarrow$) This implication is a consequence of Theorem 13.37.

($\Longleftarrow$) Let us begin by showing the hypothesis $|\nu(A)| < \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$ implies $|\nu|(A) \leq 4\epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$. To prove this decompose ν into its real and imaginary parts; $\nu = \nu_r + i\nu_i$.and suppose that $A = \coprod_{j=1}^n A_j$ with $A_j \in \mathcal{A}$. Then

$$\begin{aligned}\sum_{j=1}^n |\nu_r(A_j)| &= \sum_{j:\nu_r(A_j)\geq 0} \nu_r(A_j) - \sum_{j:\nu_r(A_j)\leq 0} \nu_r(A_j) \\ &= \nu_r(\cup_{j:\nu_r(A_j)\geq 0} A_j) - \nu_r(\cup_{j:\nu_r(A_j)\leq 0} A_j) \\ &\leq \left|\nu(\cup_{j:\nu_r(A_j)\geq 0} A_j)\right| + \left|\nu(\cup_{j:\nu_r(A_j)\leq 0} A_j)\right| \\ &< 2\epsilon\end{aligned}$$

using the hypothesis and the fact $\mu\left(\cup_{j:\nu_r(A_j)\geq 0} A_j\right) \leq \mu(A) < \delta$ and $\mu\left(\cup_{j:\nu_r(A_j)\leq 0} A_j\right) \leq \mu(A) < \delta$. Similarly, $\sum_{j=1}^n |\nu_i(A_j)| < 2\epsilon$ and therefore

$$\sum_{j=1}^n |\nu(A_j)| \leq \sum_{j=1}^n |\nu_r(A_j)| + \sum_{j=1}^n |\nu_i(A_j)| < 4\epsilon.$$

Using Proposition 13.35, it follows that

$$|\nu|(A) = \sup\left\{\sum_{j=1}^n |\nu(A_j)| : A = \coprod_{j=1}^n A_j \text{ with } A_j \in \mathcal{A} \text{ and } n \in \mathbb{N}\right\} \leq 4\epsilon.$$

Because of this argument, we may now replace ν by $|\nu|$ and hence we may assume that ν is a positive finite measure.

Let $\epsilon > 0$ and $\delta > 0$ be such that $\nu(A) < \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$. Suppose that $B \in \mathcal{M}$ with $\mu(B) < \delta$. Use the regularity Theorem 6.40 or Corollary 11.27 to find $A \in \mathcal{A}_\sigma$ such that $B \subset A$ and $\mu(B) \leq \mu(A) < \delta$. Write $A = \cup_n A_n$ with $A_n \in \mathcal{A}$. By replacing A_n by $\cup_{j=1}^n A_j$ if necessary we may assume that A_n is increasing in n. Then $\mu(A_n) \leq \mu(A) < \delta$ for each n and hence by assumption $\nu(A_n) < \epsilon$. Since $B \subset A = \cup_n A_n$ it follows that $\nu(B) \leq \nu(A) = \lim_{n\to\infty} \nu(A_n) \leq \epsilon$. Thus we have shown that $\nu(B) \leq \epsilon$ for all $B \in \mathcal{M}$ such that $\mu(B) < \delta$. ■

13.6. **Dual Spaces and the Complex Riesz Theorem.**

Proposition 13.40. *Let $\mathbb{S}$ be a vector lattice of bounded real functions on a set X. We equip $\mathbb{S}$ with the sup-norm topology and suppose $I \in \mathbb{S}^*$. Then there exists $I_\pm \in \mathbb{S}^*$ which are positive such that then $I = I_+ - I_-$.*

Proof. For $f \in \mathbb{S}^+$, let

$$I_+(f) := \sup\left\{I(g) : g \in \mathbb{S}^+ \text{ and } g \leq f\right\}.$$

One easily sees that $|I_+(f)| \le \|I\| \, \|f\|$ for all $f \in \mathbb{S}^+$ and $I_+(cf) = cI_+(f)$ for all $f \in \mathbb{S}^+$ and $c > 0$. Let $f_1, f_2 \in \mathbb{S}^+$. Then for any $g_i \in \mathbb{S}^+$ such that $g_i \le f_i$, we have $\mathbb{S}^+ \ni g_1 + g_2 \le f_1 + f_2$ and hence

$$I(g_1) + I(g_2) = I(g_1 + g_2) \le I_+(f_1 + f_2).$$

Therefore,

$$I_+(f_1) + I_+(f_2) = \sup\{I(g_1) + I(g_2) : \mathbb{S}^+ \ni g_i \le f_i\} \le I_+(f_1 + f_2). \tag{13.17}$$

For the opposite inequality, suppose $g \in \mathbb{S}^+$ and $g \le f_1 + f_2$. Let $g_1 = f_1 \wedge g$, then

$$\begin{aligned} 0 \le g_2 := g - g_1 = g - f_1 \wedge g &= \begin{cases} 0 & \text{if} \quad g \le f_1 \\ g - f_1 & \text{if} \quad g \ge f_1 \end{cases} \\ \le \begin{cases} 0 & \text{if} \quad g \le f_1 \\ f_1 + f_2 - f_1 & \text{if} \quad g \ge f_1 \end{cases} &\le f_2. \end{aligned}$$

Since $g = g_1 + g_2$ with $\mathbb{S}^+ \ni g_i \le f_i$,

$$I(g) = I(g_1) + I(g_2) \le I_+(f_1) + I_+(f_2)$$

and since $\mathbb{S}^+ \ni g \le f_1 + f_2$ was arbitrary, we may conclude

$$I_+(f_1 + f_2) \le I_+(f_1) + I_+(f_2). \tag{13.18}$$

Combining Eqs. (13.17) and (13.18) shows that

$$I_+(f_1 + f_2) = I_+(f_1) + I_+(f_2) \text{ for all } f_i \in \mathbb{S}^+. \tag{13.19}$$

We now extend I_+ to $\mathbb{S}$ by defining, for $f \in \mathbb{S}$,

$$I_+(f) = I_+(f_+) - I_+(f_-)$$

where $f_+ = f \vee 0$ and $f_- = -(f \wedge 0) = (-f) \vee 0$. (Notice that $f = f_+ - f_-$.) We will now shows that I_+ is linear.

If $c \ge 0$, we may use $(cf)_\pm = cf_\pm$ to conclude that

$$I_+(cf) = I_+(cf_+) - I_+(cf_-) = cI_+(f_+) - cI_+(f_-) = cI_+(f).$$

Similarly, using $(-f)_\pm = f_\mp$ it follows that $I_+(-f) = I_+(f_-) - I_+(f_+) = -I_+(f)$. Therefore we have shown

$$I_+(cf) = cI_+(f) \text{ for all } c \in \mathbb{R} \text{ and } f \in \mathbb{S}.$$

If $f = u - v$ with $u, v \in \mathbb{S}^+$ then

$$v + f_+ = u + f_- \in \mathbb{S}^+$$

and so by Eq. (13.19), $I_+(v) + I_+(f_+) = I_+(u) + I_+(f_-)$ or equivalently

$$I_+(f) = I_+(f_+) - I_+(f_-) = I_+(u) - I_+(v). \tag{13.20}$$

Now if $f, g \in \mathbb{S}$, then

$$\begin{aligned} I_+(f + g) &= I_+(f_+ + g_+ - (f_- + g_-)) \\ &= I_+(f_+ + g_+) - I_+(f_- + g_-) \\ &= I_+(f_+) + I_+(g_+) - I_+(f_-) - I_+(g_-) \\ &= I_+(f) + I_+(g), \end{aligned}$$

wherein the second equality we used Eq. (13.20).

The last two paragraphs show $I_+ : \mathbb{S} \to \mathbb{R}$ is linear. Moreover,

$$\begin{aligned}|I_+(f)| &= |I_+(f_+) - I_+(f_-)| \le \max(|I_+(f_+)|, |I_+(f_-)|) \\ &\le \|I\| \max(\|f_+\|, \|f_-\|) = \|I\| \|f\|\end{aligned}$$

which shows that $\|I_+\| \le \|I\|$. That is I_+ is a bounded positive linear functional on $\mathbb{S}$. Let $I_- = I_+ - I \in \mathbb{S}^*$. Then by definition of $I_+(f)$, $I_-(f) = I_+(f) - I(f) \ge 0$ for all $\mathbb{S} \ni f \ge 0$. Therefore $I = I_+ - I_-$ with $I_\pm$ being positive linear functionals on $\mathbb{S}$. ■

Corollary 13.41. *Suppose X is a second countable locally compact Hausdorff space and $I \in C_0(X,\mathbb{R})^*$, then there exists $\mu = \mu_+ - \mu_-$ where μ is a finite signed measure on $\mathcal{B}_\mathbb{R}$ such that $I(f) = \int_\mathbb{R} f d\mu$ for all $f \in C_0(X,\mathbb{R})$. Similalry if $I \in C_0(X,\mathbb{C})^*$ there exists a complex measure μ such that $I(f) = \int_\mathbb{R} f d\mu$ for all $f \in C_0(X,\mathbb{C})$.*

Proof. Let $I = I_+ - I_-$ be the decomposition given as above. Then we know there exists finite measure $\mu_\pm$ such that

$$I_\pm(f) = \int_X f d\mu_\pm \text{ for all } f \in C_0(X,\mathbb{R}).$$

and therefore $I(f) = \int_X f d\mu$ for all $f \in C_0(X,\mathbb{R})$ where $\mu = \mu_+ - \mu_-$. Moreover the measure μ is unique. Indeed if $I(f) = \int_X f d\mu$ for some finite signed measure μ, then the next result shows that $I_\pm(f) = \int_X f d\mu_\pm$ where $\mu_\pm$ is the Hahn decomposition of μ. Now the measures $\mu_\pm$ are uniquely determined by $I_\pm$. The complex case is a consequence of applying the real case just proved to $\operatorname{Re} I$ and $\operatorname{Im} I$. ■

Proposition 13.42. *Suppose that μ is a signed Radon measure and $I = I_\mu$. Let μ_+ and μ_- be the Radon measures associated to $I_\pm$, then $\mu = \mu_+ - \mu_-$ is the Jordan decomposition of μ.*

Proof. Let $X = P \cup P^c$ where P is a positive set for μ and P^c is a negative set. Then for $A \in \mathcal{B}_X$,

$$\mu(P \cap A) = \mu_+(P \cap A) - \mu_-(P \cap A) \le \mu_+(P \cap A) \le \mu_+(A). \tag{13.21}$$

To finish the proof we need only prove the reverse inequality. To this end let $\epsilon > 0$ and choose $K \sqsubset\sqsubset P \cap A \subset U \subset_o X$ such that $|\mu|(U \setminus K) < \epsilon$. Let $f, g \in C_c(U, [0,1])$ with $f \le g$, then

$$\begin{aligned}I(f) = \mu(f) &= \mu(f : K) + \mu(f : U \setminus K) \le \mu(g : K) + O(\epsilon) \\ &\le \mu(K) + O(\epsilon) \le \mu(P \cap A) + O(\epsilon).\end{aligned}$$

Taking the supremum over all such $f \le g$, we learn that $I_+(g) \le \mu(P \cap A) + O(\epsilon)$ and then taking the supremum over all such g shows that

$$\mu_+(U) \le \mu(P \cap A) + O(\epsilon).$$

Taking the infimum over all $U \subset_o X$ such that $P \cap A \subset U$ shows that

$$\mu_+(P \cap A) \le \mu(P \cap A) + O(\epsilon) \tag{13.22}$$

From Eqs. (13.21) and (13.22) it follows that $\mu(P \cap A) = \mu_+(P \cap A)$. Since

$$I_-(f) = \sup_{0 \le g \le f} I(g) - I(f) = \sup_{0 \le g \le f} I(g - f) = \sup_{0 \le g \le f} -I(f - g) = \sup_{0 \le h \le f} -I(h)$$

the same argument applied to $-I$ shows that

$$-\mu(P^c \cap A) = \mu_-(P^c \cap A).$$

Since

$$\mu(A) = \mu(P \cap A) + \mu(P^c \cap A) = \mu_+(P \cap A) - \mu_-(P^c \cap A) \text{ and}$$
$$\mu(A) = \mu_+(A) - \mu_-(A)$$

it follows that

$$\mu_+(A \setminus P) = \mu_-(A \setminus P^c) = \mu_-(A \cap P).$$

Taking $A = P$ then shows that $\mu_-(P) = 0$ and taking $A = P^c$ shows that $\mu_+(P^c) = 0$ and hence

$$\mu(P \cap A) = \mu_+(P \cap A) = \mu_+(A) \text{ and}$$
$$-\mu(P^c \cap A) = \mu_-(P^c \cap A) = \mu_-(A)$$

as was to be proved. ■

13.7. **Exercises.**

Exercise 13.5. Let ν be a $\sigma -$ finite signed measure, $f \in L^1(|\nu|)$ and define

$$\int_X f d\nu = \int_X f d\nu_+ - \int_X f d\nu_-.$$

Suppose that μ is a $\sigma -$ finite measure and $\nu \ll \mu$. Show

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu. \tag{13.23}$$

Exercise 13.6. Suppose that ν is a signed or complex measure on $(X, \mathcal{M})$ and $A_n \in \mathcal{M}$ such that either $A_n \uparrow A$ or $A_n \downarrow A$ and $\nu(A_1) \in \mathbb{R}$, then show $\nu(A) = \lim_{n\to\infty} \nu(A_n)$.

Exercise 13.7. Suppose that μ and λ are positive measures and $\mu(X) < \infty$. Let $\nu := \lambda - \mu$, then show $\lambda \geq \nu_+$ and $\mu \geq \nu_-$.

Exercise 13.8. Folland Exercise 3.5 on p. 88 showing $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Exercise 13.9. Folland Exercise 3.7a on p. 88.

Exercise 13.10. Show Theorem 13.37 may fail if ν is not finite. (For a hint, see problem 3.10 on p. 92 of Folland.)

Exercise 13.11. Folland 3.14 on p. 92.

Exercise 13.12. Folland 3.15 on p. 92.

Exercise 13.13. Folland 3.20 on p. 94.

14. Lebesgue Differentiation and the Fundamental Theorem of Calculus

Notation 14.1. In this chapter, let $\mathcal{B} = \mathcal{B}_{\mathbb{R}^n}$ denote the Borel σ − algebra on $\mathbb{R}^n$ and m be Lebesgue measure on $\mathcal{B}$. If V is an open subset of $\mathbb{R}^n$, let $L^1_{loc}(V) := L^1_{loc}(V, m)$ and simply write L^1_{loc} for $L^1_{loc}(\mathbb{R}^n)$. We will also write $|A|$ for $m(A)$ when $A \in \mathcal{B}$.

Definition 14.2. A collection of measurable sets $\{E\}_{r>0} \subset \mathcal{B}$ is said to shrink nicely to $x \in \mathbb{R}^n$ if (i) $E_r \subset \overline{B_x(r)}$ for all $r > 0$ and (ii) there exists $\alpha > 0$ such that $m(E_r) \geq \alpha m(B_x(r))$. We will abbreviate this by writing $E_r \downarrow \{x\}$ nicely.

The main result of this chapter is the following theorem.

Theorem 14.3. *Suppose that ν is a complex measure on $(\mathbb{R}^n, \mathcal{B})$, then there exists $g \in L^1(\mathbb{R}^n, m)$ and a complex measure λ such that $\lambda \perp m$, $d\nu = gdm + d\lambda$, and for m - a.e. x,*

$$g(x) = \lim_{r \downarrow 0} \frac{\nu(E_r)}{m(E_r)} \tag{14.1}$$

for any collection of $\{E_r\}_{r>0} \subset \mathcal{B}$ which shrink nicely to $\{x\}$.

Proof. The existence of g and λ such that $\lambda \perp m$ and $d\nu = gdm + d\lambda$ is a consequence of the Radon-Nikodym theorem. Since

$$\frac{\nu(E_r)}{m(E_r)} = \frac{1}{m(E_r)} \int_{E_r} g(x) dm(x) + \frac{\lambda(E_r)}{m(E_r)}$$

Eq. (14.1) is a consequence of Theorem 14.13 and Corollary 14.15 below. ∎

The rest of this chapter will be devoted to filling in the details of the proof of this theorem.

14.1. **A Covering Lemma and Averaging Operators.**

Lemma 14.4 (Covering Lemma)**.** *Let $\mathcal{E}$ be a collection of open balls in $\mathbb{R}^n$ and $U = \cup_{B \in \mathcal{E}} B$. If $c < m(U)$, then there exists disjoint balls $B_1, \dots, B_k \in \mathcal{E}$ such that $\sum_{j=1}^{k} m(B_j) > 3^{-n} c$.*

Proof. Choose a compact set $K \subset U$ such that $m(K) > c$ and then let $\mathcal{E}_1 \subset \mathcal{E}$ be a finite subcover of K. Choose $B_1 \in \mathcal{E}_1$ to be a ball with largest diameter in $\mathcal{E}_1$. Let $\mathcal{E}_2 = \{A \in \mathcal{E}_1 : A \cap B_1 = \emptyset\}$. If $\mathcal{E}_2$ is not empty, choose $B_2 \in \mathcal{E}_2$ to be a ball with largest diameter in $\mathcal{E}_2$. Similarly Let $\mathcal{E}_3 = \{A \in \mathcal{E}_2 : A \cap B_2 = \emptyset\}$ and if $\mathcal{E}_3$ is not empty, choose $B_3 \in \mathcal{E}_3$ to be a ball with largest diameter in $\mathcal{E}_3$. Continue choosing $B_i \in \mathcal{E}$ for $i = 1, 2, \dots, k$ this way until $\mathcal{E}_{k+1}$ is empty.

If $B = B(x_0, r) \subset \mathbb{R}^n$, let $B^* = B(x_0, 3r) \subset \mathbb{R}^n$, that is B^* is the ball concentric with B which has three times the radius of B. We will now show $K \subset \cup_{i=1}^k B_i^*$. For each $A \in \mathcal{E}_1$ there exists a first i such that $B_i \cap A \neq \emptyset$. In this case $\mathrm{diam}(A) \leq \mathrm{diam}(B_i)$ and $A \subset B_i^*$. Therefore $A \subset \cup_{i=1}^k B_i^*$ for all j and hence $K \subset \cup\{A : A \in \mathcal{E}_1\} \subset \cup_{i=1}^k B_i^*$. Hence by subadditivity,

$$c < m(K) \leq \sum_{i=1}^{k} m(B_i^*) \leq 3^n \sum_{i=1}^{k} m(B_i).$$

∎

Definition 14.5. For $f \in L^1_{loc}$, $x \in \mathbb{R}^n$ and $r > 0$ let

$$(A_r f)(x) = \frac{1}{|B_x(r)|} \int\limits_{B_x(r)} f dm \tag{14.2}$$

where $B_x(r) = B(x, r) \subset \mathbb{R}^n$, and $|A| := m(A)$.

Lemma 14.6. *Let $f \in L^1_{loc}$, then for each $x \in \mathbb{R}^n$, $(0, \infty)$such that $r \to (A_r f)(x)$ is continuous and for each $r > 0$, $\mathbb{R}^n$ such that $x \to (A_r f)(x)$ is measurable.*

Proof. Recall that $|B_x(r)| = m(E_1)r^n$ which is continuous in r. Also $\lim_{r \to r_0} 1_{B_x(r)}(y) = 1_{B_x(r_0)}(y)$ if $|y| \neq r_0$ and since $m(\{y : |y| \neq r_0\}) = 0$ (you prove!), $\lim_{r \to r_0} 1_{B_x(r)}(y) = 1_{B_x(r_0)}(y)$ for m -a.e. y. So by the dominated convergence theorem,

$$\lim_{r \to r_0} \int\limits_{B_x(r)} f dm = \int\limits_{B_x(r_0)} f dm$$

and therefore

$$(A_r f)(x) = \frac{1}{m(E_1) r^n} \int\limits_{B_x(r)} f dm$$

is continuous in r. Let $g_r(x, y) := 1_{B_x(r)}(y) = 1_{|x-y|<r}$. Then g_r is $\mathcal{B} \otimes \mathcal{B}$ – measurable (for example write it as a limit of continuous functions or just notice that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $F(x, y) := |x - y|$ is continuous) and so that by Fubini's theorem

$$x \to \int\limits_{B_x(r)} f dm = \int\limits_{B_x(r)} g_r(x, y) f(y) dm(y)$$

is $\mathcal{B}$ – measurable and hence so is $x \to (A_r f)(x)$. ■

14.2. Maximal Functions.

Definition 14.7. For $f \in L^1(m)$, the Hardy - Littlewood maximal function Hf is defined by

$$(Hf)(x) = \sup_{r>0} A_r |f|(x).$$

Lemma 14.6 allows us to write

$$(Hf)(x) = \sup_{r \in \mathbb{Q},\, r>0} A_r |f|(x)$$

and then to concluded that Hf is measurable.

Theorem 14.8 (Maximal Inequality)**.** *If $f \in L^1(m)$ and $\alpha > 0$, then*

$$m(Hf > \alpha) \leq \frac{3^n}{\alpha} \|f\|_{L^1}.$$

This should be compared with Chebyshev's inequality which states that

$$m(|f| > \alpha) \leq \frac{\|f\|_{L^1}}{\alpha}.$$

Proof. Let $E_\alpha \equiv \{Hf > \alpha\}$. For all $x \in E_\alpha$ there exists r_x such that $A_{r_x}|f|(x) > \alpha$. Hence $E_\alpha \subset \cup_{x \in E_\alpha} B_x(r_x)$. By Lemma 14.4, if $c < m(E_\alpha) \leq$

$m(\cup_{x\in E_\alpha} B_x(r_x))$ then there exists $x_1, \dots, x_k \in E_\alpha$ and disjoint balls $B_i = B_{x_i}(r_{x_i})$ for $i = 1, 2, \dots, k$ such that $\sum |B_i| > 3^{-n}c$. Since

$$|B_i|^{-1} \int_{B_i} |f| dm = A_{r_{x_i}} |f|(x_i) > \alpha,$$

we have $|B_i| < \alpha^{-1} \int_{B_i} |f| dm$ and hence

$$3^{-n}c < \frac{1}{\alpha} \sum \int_{B_i} |f| dm \le \frac{1}{\alpha} \int_{\mathbb{R}^n} |f| dm = \frac{1}{\alpha} \|f\|_{L^1}.$$

This shows that $c < 3^n \alpha^{-1} \|f\|_{L^1}$ for all $c < m(E_\alpha)$ which proves $m(E_\alpha) \le 3^n \alpha^{-1} \|f\|$ ■

Theorem 14.9. *If $f \in L^1_{loc}$ then $\lim_{r\downarrow 0} (A_r f)(x) = f(x)$ for m – a.e. $x \in \mathbb{R}^n$.*

Proof. With out loss of generality we may assume $f \in L^1(m)$. We now begin with the special case where $f = g \in L^1(m)$ is also continuous. In this case we find:

$$\begin{aligned} |(A_r g)(x) - g(x)| &\le \frac{1}{|B_x(r)|} \int_{B_x(r)} |g(y) - g(x)| dm(y) \\ &\le \sup_{y \in B_x(r)} |g(y) - g(x)| \to 0 \text{ as } r \to 0. \end{aligned}$$

In fact we have shown that $(A_r g)(x) \to g(x)$ as $r \to 0$ uniformly for x in compact subsets of $\mathbb{R}^n$.

For general $f \in L^1(m)$,

$$\begin{aligned} |A_r f(x) - f(x)| &\le |A_r f(x) - A_r g(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)| \\ &= |A_r (f - g)(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)| \\ &\le H(f - g)(x) + |A_r g(x) - g(x)| + |g(x) - f(x)| \end{aligned}$$

and therefore,

$$\overline{\lim_{r\downarrow 0}} |A_r f(x) - f(x)| \le H(f - g)(x) + |g(x) - f(x)|.$$

So if $\alpha > 0$, then

$$E_\alpha \equiv \left\{ \overline{\lim_{r\downarrow 0}} |A_r f(x) - f(x)| > \alpha \right\} \subset \left\{ H(f - g) > \frac{\alpha}{2} \right\} \cup \left\{ |g - f| > \frac{\alpha}{2} \right\}$$

and thus

$$\begin{aligned} m(E_\alpha) &\le m\left(H(f - g) > \frac{\alpha}{2} \right) + m\left(|g - f| > \frac{\alpha}{2} \right) \\ &\le \frac{3^n}{\alpha/2} \|f - g\|_{L^1} + \frac{1}{\alpha/2} \|f - g\|_{L^1} \\ &\le 2(3^n + 1)\alpha^{-1} \|f - g\|_{L^1}, \end{aligned}$$

where in the second inequality we have used the Maximal inequality (Theorem 14.8) and Chebyshev's inequality. Since this is true for all continuous $g \in C(\mathbb{R}^n) \cap L^1(m)$ and this set is dense in $L^1(m)$, we may make $\|f - g\|_{L^1}$ as small as we please. This shows that

$$m\left(\left\{ x : \overline{\lim_{r\downarrow 0}} |A_r f(x) - f(x)| > 0 \right\} \right) = m(\cup_{n=1}^\infty E_{1/n}) \le \sum_{n=1}^\infty m(E_{1/n}) = 0.$$

■

Corollary 14.10. *If $d\mu = gdm$ with $g \in L^1_{loc}$ then*

$$\frac{\mu(B_x(r))}{|B_x(r)|} = A_r g(x) \to g(x) \text{ for } m \text{ – a.e. } x.$$

14.3. **Lebesque Set.**

Definition 14.11. For $f \in L^1_{loc}(m)$, the Lebesgue set of f is

$$\mathcal{L}_f \equiv \left\{ x \in \mathbb{R}^n : \lim_{r \downarrow 0} \frac{1}{|B_x(r)|} \int\limits_{B_x(r)} |f(y) - f(x)| dy = 0 \right\}.$$

Theorem 14.12. *For all $f \in L^1_{loc}(m)$, $0 = m(\mathcal{L}_f^c) = m(\mathbb{R}^n \setminus \mathcal{L}_f)$.*

Proof. For $w \in \mathbb{C}$ define $g_w(x) = |f(x) - w|$ and $E_w \equiv \{x : \lim_{r \downarrow 0} (A_r g_w)(x) \neq g_w(x)\}$. Then by Theorem 14.9 $m(E_w) = 0$ for all $w \in \mathbb{C}$ and therefore $m(E) = 0$ where

$$E = \bigcup_{w \in \mathbb{Q} + i\mathbb{Q}} E_w.$$

By definition of E, if $x \notin E$ then.

$$\lim_{r \downarrow 0} (A_r |f(\cdot) - w|)(x) = |f(x) - w|$$

for all $w \in \mathbb{Q} + i\mathbb{Q}$. Since

$$|f(\cdot) - f(x)| \leq |f(\cdot) - w| + |w - f(x)|,$$

$$\begin{aligned}
(A_r |f(\cdot) - f(x)|)(x) &\leq (A_r |f(\cdot) - w|)(x) + (A_r |w - f(x)|)(x) \\
&= (A_r |f(\cdot) - w|)(x) + |w - f(x)|
\end{aligned}$$

and hence for $x \notin E$,

$$\begin{aligned}
\overline{\lim_{r \downarrow 0}} (A_r |f(\cdot) - f(x)|)(x) &\leq |f(x) - w| + |w - f(x)| \\
&\leq 2|f(x) - w|.
\end{aligned}$$

Since this is true for all $w \in \mathbb{Q} + i\mathbb{Q}$, we see that

$$\overline{\lim_{r \downarrow 0}} (A_r |f(\cdot) - f(x)|)(x) = 0 \text{ for all } x \notin E,$$

i.e. $E^c \subset \mathcal{L}_f$ or equivalently $\mathcal{L}_f^c \subset E$. So $m(\mathcal{L}_f^c) \leq m(E) = 0$. ■

Theorem 14.13 (Lebesque Differentiation Theorem)**.** *Suppose $f \in L^1_{loc}$ for all $x \in \mathcal{L}_f$ (so in particular for m – a.e. x)*

$$\lim_{r \downarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0$$

and

$$\lim_{r \downarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$$

when $E_r \downarrow \{x\}$ nicely.

Proof. For all $x \in \mathcal{L}_f$,

$$\begin{aligned}\left|\frac{1}{m(E_r)}\int_{E_r} f(y)dy - f(x)\right| &= \left|\frac{1}{m(E_r)}\int_{E_r} (f(y)-f(x))\,dy\right| \\ &\le \frac{1}{m(E_r)}\int_{E_r} |f(y)-f(x)|dy \\ &\le \frac{1}{\alpha m(B_x(r))}\int\limits_{B_x(r)} |f(y)-f(x)|dy\end{aligned}$$

which tends to zero as $r \downarrow 0$ by Theorem 14.12. In the second inequality we have used the fact that $m(\overline{B_x(r)} \setminus B_x(r)) = 0$. ■

Lemma 14.14. *Suppose λ is positive σ – finite measure on $\mathcal{B} \equiv \mathcal{B}_{\mathbb{R}^n}$ such that $\lambda \perp m$. Then for m – a.e. x,*

$$\lim_{r\downarrow 0}\frac{\lambda(B_x(r))}{m(B_x(r))} = 0.$$

Proof. Let $A \in \mathcal{B}$ such that $\lambda(A) = 0$ and $m(A^c) = 0$. By the regularity theorem (Exercise 6.4), for all $\epsilon > 0$ there exists an open set $V_\epsilon \subset \mathbb{R}^n$ such that $A \subset V_\epsilon$ and $\lambda(V_\epsilon) < \epsilon$. Let

$$F_k \equiv \left\{x \in A : \overline{\lim_{r\downarrow 0}}\frac{\lambda(B_x(r))}{m(B_x(r))} > \frac{1}{k}\right\}$$

the for $x \in F_k$ choose $r_x > 0$ such that $B_x(r_x) \subset V_\epsilon$ (see Figure 27)and $\frac{\lambda(B_x(r_x))}{m(B_x(r_x))} > \frac{1}{k}$, i.e.

$$m(B_x(r_x)) < k\ \lambda(B_x(r_x)).$$

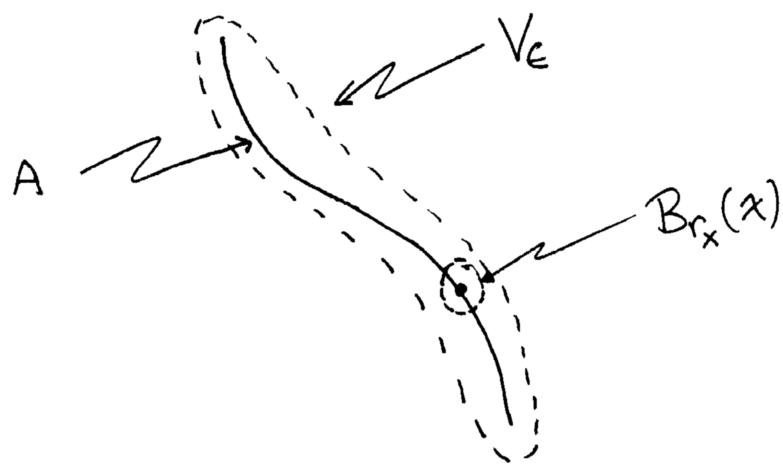

Figure 27. Covering a small set with balls.

Let $\mathcal{E} = \{B_x(r_x)\}_{x\in F_k}$ and $U \equiv \bigcup_{x\in F_k} B_x(r_x) \subset V_\epsilon$. Heuristically if all the balls in $\mathcal{E}$ were disjoint and $\mathcal{E}$ were countable, then

$$\begin{aligned} m(F_k) &\le \sum_{x\in F_k} m(B_x(r_x)) < k \sum_{x\in F_k} \lambda(B_x(r_x)) \\ &= k\lambda(U) \le k\ \lambda(V_\epsilon) \le k\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary this would imply that $m(F_k) = 0$.

To fix the above argument, suppose that $c < m(U)$ and use the covering lemma to find disjoint balls $B_1, \dots, B_N \in \mathcal{E}$ such that

$$\begin{aligned} c &< 3^n \sum_{i=1}^N m(B_i) < k3^n \sum_{i=1}^n \lambda(B_i) \\ &\le k3^n \lambda(U) \le k3^n \lambda(V_\epsilon) \le k3^n \epsilon. \end{aligned}$$

Since $c < m(U)$ is arbitrary we learn that $m(F_k) \le m(U) \le k3^n\epsilon$ and in particular that $m(F_k) \le k3^n\epsilon$. Since $\epsilon > 0$ is arbitrary, this shows that $m(F_k) = 0$. This implies, letting

$$F_\infty \equiv \left\{x \in A : \overline{\lim_{r\downarrow 0}} \frac{\lambda(B_x(r))}{m(B_x(r))} > 0\right\},$$

that $m(F_\infty) = \lim_{k\to\infty} m(F_k) = 0$. Since $m(A^c) = 0$, this shows that

$$m(\{x \in \mathbb{R}^n : \overline{\lim_{r\downarrow 0}} \frac{\lambda(B_x(r))}{m(B_x(r))} > 0\}) = 0.$$

■

Corollary 14.15. *Let λ be a complex or a σ – finite signed measure such that $\lambda \perp m$. Then for m – a.e. x,*

$$\lim_{r\downarrow 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

whenever $E_r \downarrow \{x\}$ nicely.

Proof. Recalling the $\lambda \perp m$ implies $|\lambda| \perp m$, Lemma 14.14 and the inequalities,

$$\frac{|\lambda(E_r)|}{m(E_r)} \le \frac{|\lambda|(E_r)}{\alpha m(B_x(r))} \le \frac{|\lambda|(\overline{B_x(r)})}{\alpha m(B_x(r))} \le \frac{|\lambda|(B_x(2r))}{\alpha 2^{-n} m(B_x(2r))}$$

proves the result. ■

14.4. **The Fundamental Theorem of Calculus.** In this section we will restrict the results above to the one dimensional setting. So for the rest of this chapter, $n = 1$ and m denotes one dimensional Lebesgue measure on $\mathcal{B} := \mathcal{B}_\mathbb{R}$.

Notation 14.16. Given a function $F : \mathbb{R} \to \bar{\mathbb{R}}$ or $F : \mathbb{R} \to \mathbb{C}$, let $F(x-) = \lim_{y\uparrow x} F(y)$, $F(x+) = \lim_{y\downarrow x} F(y)$ and $F(\pm\infty) = \lim_{x\to\pm\infty} F(x)$ whenever the limits exist. Notice that if F is a monotone functions then $F(\pm\infty)$ and $F(x\pm)$ exist for all x.

Theorem 14.17. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and define $G(x) = F(x+)$. Then*
(a) $\{x \in \mathbb{R} : F(x+) > F(x-)\}$ is countable.
(b) The function G increasing and right continuous.
(c) For m – a.e. x, $F'(x)$ and $G'(x)$ exists and $F'(x) = G'(x)$.

Proof. Properties (a) and (b) have already been proved in Theorem 11.34.

(c) Let μ_G denote the unique measure on $\mathcal{B}$ such that $\mu_G((a,b]) = G(b) - G(a)$ for all $a < b$. By Theorem 14.3, for m - a.e. x, for all sequences $\{E_r\}_{r>0}$ which shrink nicely to $\{x\}$, $\lim_{r\downarrow 0}(\mu_G(E_r)/m(E_r))$ exists and is independent of the choice of sequence $\{E_r\}_{r>0}$ shrinking to $\{x\}$. Since $(x, x+r] \downarrow \{x\}$ and $(x-r, x] \downarrow \{x\}$ nicely,

$$\lim_{r\downarrow 0} \frac{\mu_G(x, x+r])}{m((x, x+r])} = \lim_{r\downarrow 0} \frac{G(x+r) - G(x)}{r} = \frac{d}{dx^+}G(x)$$

and

$$\lim_{r\downarrow 0} \frac{\mu_G((x-r, x])}{m((x-r, x])} = \lim_{r\downarrow 0} \frac{G(x) - G(x-r)}{r} = \lim_{r\downarrow 0} \frac{G(x-r) - G(x)}{-r} = \frac{d}{dx^-}G(x)$$

exist and are equal for m - a.e. x, i.e. $G'(x)$ exists for m -a.e. x.

For $x \in \mathbb{R}$, let

$$H(x) \equiv G(x) - F(x) = F(x+) - F(x) \geq 0.$$

The proof will be completed by showing that $H'(x) = 0$ for m – a.e. x. Let

$$\Lambda \equiv \{x \in \mathbb{R} : F(x+) > F(x)\} \subset \Gamma.$$

Then $\Lambda \subset \mathbb{R}$ is a countable set and $H(x) = 0$ if $x \notin \Lambda$. Let λ be the measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\lambda = \sum_{x\in\mathbb{R}} H(x)\delta_x = \sum_{x\in\Lambda} H(x)\delta_x.$$

Since

$$\begin{aligned}\lambda((-N, N)) = \sum_{x\in(-N,N)} H(x) &= \sum_{x\in\Lambda\cap(-N,N)} (F(x+) - F(x)) \\ &\leq \sum_{x\in(-N,N)} (F(x+) - F(x-)),\end{aligned}$$

Eq. (11.26) guarantees that λ is finite on bounded sets. Since $\lambda(\Lambda^c) = 0$ and $m(\Lambda) = 0$, $\lambda \perp m$ and so by Corollary 14.15 for m - a.e. x,

$$\left|\frac{H(x+r) - H(x)}{r}\right| \leq 2\frac{H(x+|r|) + H(x-|r|) + H(x)}{|r|} \leq 2\frac{\lambda([x-|r|, x+|r|])}{|r|}$$

and the last term goes to zero as $r \to 0$ because $\{[x-r, x+r]\}_{r>0}$ shrinks nicely to $\{x\}$. Hence we conclude for m – a.e. x that $H'(x) = 0$. ∎

Definition 14.18. For $-\infty \leq a < b < \infty$, a partition $\mathbb{P}$ of $(a,b]$ is a finite subset of $[a,b] \cap \mathbb{R}$ such that $\{a,b\} \cap \mathbb{R} \subset \mathbb{P}$. For $x \in \mathbb{P}\setminus\{b\}$, let $x_+ = \min\{y \in \mathbb{P} : y > x\}$ and if $x = b$ let $x_+ = b$.

Proposition 14.19. *Let μ be a complex measure on $\mathcal{B}$ and let F be a function such that*

$$F(b) - F(a) = \mu((a,b]) \text{ for all } a < b,$$

for example let $F(x) = \mu((-\infty, x])$ in which case $F(-\infty) = 0$. The function F is right continuous and for $-\infty < a < b < \infty$,

$$|\mu|(a,b] = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |\mu(x, x_+]| = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |F(x_+) - F(x)| \tag{14.3}$$

where supremum is over all partitions $\mathbb{P}$ of $(a,b]$. Moreover $\mu \ll m$ iff for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |\mu((a_i, b_i])| = \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon \tag{14.4}$$

whenever $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Proof. Eq. (14.3) follows from Proposition 13.35 and the fact that $\mathcal{B} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is the algebra generated by $(a,b]\cap\mathbb{R}$ with $a, b \in \bar{\mathbb{R}}$. Suppose that Eq. (14.4) holds under the stronger condition that $\{(a_i, b_i]\}_{i=1}^n$ are disjoint intervals in $(a,b]$. If $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then for all $\rho > 0$, $\{(a_i + \rho, b_i]\}_{i=1}^n$ are disjoint intervals in $(a,b]$ and $\sum_{i=1}^{n}(b_i - (a_i + \rho)) < \delta$ so that by assumption,

$$\sum_{i=1}^{n} |F(b_i) - F(a_i + \rho)| < \epsilon.$$

Since $\rho > 0$ is arbitrary in this equation and F is right continuous, we conclude that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| \le \epsilon$$

whenever $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$. So it suffices to prove Eq. (14.4) under the stronger condition that $\{(a_i, b_i]\}_{i=1}^n$ are disjoint intervals in $(a,b]$. But this last assertion follows directly from Theorem 13.39 and the fact that $\mathcal{B} = \sigma(\mathcal{A})$. ■

Definition 14.20. A function $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous** if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon$$

whenever $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Definition 14.21. Given a function $F : \mathbb{R} \to \mathbb{C}$ let μ_F be the unique additive measure on $\mathcal{A}$ (the algebra of half open intervals) such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a < b$. For $a \in \mathbb{R}$ define

$$T_F(a) \equiv \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |F(x_+) - F(x)| = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |\mu_F(x, x_+]|$$

where the supremum is taken over all partitions of $(-\infty, a]$. More generally if $-\infty \le a < b$, let

$$T_F(a,b] = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |\mu_F(x, x_+]| = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |F(x_+) - F(x)|$$

where supremum is over all partitions $\mathbb{P}$ of $(a,b]$. A function $F : \mathbb{R} \to \mathbb{C}$ is said to be of **bounded variation** if $T_F(\infty) < \infty$ and we write $F \in BV$. More generally we will let $BV((a,b])$ denote the functions, $F : [a,b]\cap\mathbb{R} \to \mathbb{C}$, such that $T_F(a,b] < \infty$.

Lemma 14.22. *Let $F : \mathbb{R} \to \mathbb{C}$ be any funtion and $-\infty \le a < b < c$, then*

(1)

$$T_F(a, c] = T_F(a, b] + T_F(b, c]. \tag{14.5}$$

(2) *Letting $a = -\infty$ in this expression implies*

$$T_F(c) = T_F(b) + T_F(b, c] \tag{14.6}$$

and in particular T_F is monotone increasing.

(3) *If $T_F(b) < \infty$ for some $b \in \mathbb{R}$ then $T_F(-\infty) = 0$ and*

$$T_F(a+) - T_F(a) \le \limsup_{y \downarrow a} |F(y) - F(a)| \tag{14.7}$$

for all $a \in (-\infty, b)$. In particular T_F is right continuous if F is right continuous.

Proof. By the triangle inequality, if $\mathbb{P}$ and $\mathbb{P}'$ are partition of $(a, c]$ such that $\mathbb{P} \subset \mathbb{P}'$, then

$$\sum_{x \in \mathbb{P}} |F(x_+) - F(x)| \le \sum_{x \in \mathbb{P}'} |F(x_+) - F(x)|.$$

So if $\mathbb{P}$ is a partition of $(a, c]$, then $\mathbb{P} \subset \mathbb{P}' := \mathbb{P} \cup \{b\}$ implies

$$\begin{aligned} \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| &\le \sum_{x \in \mathbb{P}'} |F(x_+) - F(x)| \\ &= \sum_{x \in \mathbb{P}' \cap [a,b]} |F(x_+) - F(x)| + \sum_{x \in \mathbb{P}' \cap [b,c]} |F(x_+) - F(x)| \\ &\le T_F(a, b] + T_F(b, c]. \end{aligned}$$

Thus we see that $T_F(a, c] \le T_F(a, b] + T_F(b, c]$. Similarly if $\mathbb{P}_1$ is a partition of $(a, b]$ and $\mathbb{P}_2$ is a partition of $(b, c]$, then $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2$ is a partition of $(a, c]$ and

$$\sum_{x \in \mathbb{P}_1} |F(x_+) - F(x)| + \sum_{x \in \mathbb{P}_2} |F(x_+) - F(x)| = \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| \le T_F(a, c].$$

From this we conclude $T_F(a, b] + T_F(b, c] \le T_F(a, c]$ which finishes the proof of Eqs. (14.5) and (14.6).

Suppose that $T_F(b) < \infty$ and given $\epsilon > 0$ let $\mathbb{P}$ be a partition of $(-\infty, b]$ such that

$$T_F(b) \le \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| + \epsilon.$$

Let $x_0 = \min \mathbb{P}$ then $T_F(b) = T_F(x_0) + T_F(x_0, b]$ and by the previous equation

$$T_F(x_0) + T_F(x_0, b] \le \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| + \epsilon \le T_F(x_0, b] + \epsilon$$

which shows that $T_F(x_0) \le \epsilon$. Since T_F is monotone increasing and $\epsilon > 0$, we conclude that $T_F(-\infty) = 0$.

Finally let $a \in (-\infty, b)$ and given $\epsilon > 0$ let $\mathbb{P}$ be a partition of $(a, b]$ such that

$$T_F(b) - T_F(a) = T_F(a, b] \le \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| + \epsilon. \tag{14.8}$$

Let $y \in (a, a_+)$, then

$$\sum_{x\in\mathbb{P}} |F(x_+) - F(x)| + \epsilon \le \sum_{x\in\mathbb{P}\cup\{y\}} |F(x_+) - F(x)| + \epsilon$$
$$= |F(y) - F(a)| + \sum_{x\in\mathbb{P}\setminus\{y\}} |F(x_+) - F(x)| + \epsilon$$
$$\le |F(y) - F(a)| + T_F(y, b] + \epsilon. \tag{14.9}$$

Combining Eqs. (14.8) and (14.9) shows

$$T_F(y) - T_F(a) + T_F(y, b] = T_F(b) - T_F(a)$$
$$\le |F(y) - F(a)| + T_F(y, b] + \epsilon.$$

Since $y \in (a, a_+)$ is arbitrary we conclude that

$$T_F(a+) - T_F(a) = \limsup_{y\downarrow a} T_F(y) - T_F(a) \le \limsup_{y\downarrow a} |F(y) - F(a)| + \epsilon$$

which proves Eq. (14.7) since $\epsilon > 0$ is arbitrary. ■

The following lemma should help to clarify Proposition 14.19 and Definition 14.20.

Lemma 14.23. *Let μ and F be as in Proposition 14.19 and $\mathcal{A}$ be the algebra generated by $(a, b] \cap \mathbb{R}$ with $a, b \in \bar{\mathbb{R}}$.. Then the following are equivalent:*

1. $\mu_F \ll m$
2. $|\mu_F| \ll m$
3. *For all $\epsilon > 0$ there exists a $\delta > 0$ such that $T_F(A) < \epsilon$ whenever $m(A) < \delta$.*
4. *For all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\mu_F(A)| < \epsilon$ whenever $m(A) < \delta$.*

Moreover, condition 4. shows that we could replace the last statement in Proposition 14.19 by: $\mu \ll m$ iff for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\sum_{i=1}^n \mu\left((a_i, b_i]\right)\right| = \left|\sum_{i=1}^n [F(b_i) - F(a_i)]\right| < \epsilon$$

whenever $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a, b]$ such that $\sum_{i=1}^n (b_i - a_i) < \delta$.

Proof. This follows directly from Lemma 13.36 and Theorem 13.39. ■

Lemma 14.24.

1. *Monotone functions $F : \mathbb{R} \to \mathbb{R}$ are in $BV(a, b]$ for all $-\infty < a < b < \infty$.*
2. *Linear combinations of functions in BV are in BV, i.e. BV is a vector space.*
3. *If $F : \mathbb{R} \to \mathbb{C}$ is absolutely continuous then F is continuous and $F \in BV(a, b]$ for all $-\infty < a < b < \infty$.*
4. *If $F : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $\sup_{x\in\mathbb{R}} |F'(x)| = M < \infty$, then F is absolutely continuous and $T_F(a, b] \le M(b - a)$ for all $-\infty < a < b < \infty$.*
5. *Let $f \in L^1((a, b], m)$ and set*
$$F(x) = \int_{(a,x]} f dm \tag{14.10}$$
for $x \in (a, b]$. Then F is absolutely continuous.

Proof.

(1) If F is monotone increasing and $\mathbb{P}$ is a partition of $(a,b]$ then
$$\sum_{x\in\mathbb{P}}|F(x_+)-F(x)|=\sum_{x\in\mathbb{P}}(F(x_+)-F(x))=F(b)-F(a)$$
so that $T_F(a,b]=F(b)-F(a)$. Also note that $F\in BV$ iff $F(\infty)-F(-\infty)<\infty$.

(2) Item 2. follows from the triangle inequality.

(3) Since F is absolutely continuous, there exists $\delta>0$ such that whenever $a<b<a+\delta$ and $\mathbb{P}$ is a partition of $(a,b]$,
$$\sum_{x\in\mathbb{P}}|F(x_+)-F(x)|\le 1.$$
This shows that $T_F(a,b]\le 1$ for all $a<b$ with $b-a<\delta$. Thus using Eq. (14.5), it follows that $T_F(a,b]\le N<\infty$ if $b-a<N\delta$ for an $N\in\mathbb{N}$.

(4) Suppose that $\{(a_i,b_i)\}_{i=1}^n\subset(a,b]$ are disjoint intervals, then by the mean value theorem,
$$\begin{aligned}\sum_{i=1}^n|F(b_i)-F(a_i)|&\le\sum_{i=1}^n|F'(c_i)|(b_i-a_i)\le Mm\left(\cup_{i=1}^n(a_i,b_i)\right)\\&\le M\sum_{i=1}^n(b_i-a_i)\le M(b-a)\end{aligned}$$
form which it clearly follows that F is absolutely continuous. Moreover we may conclude that $T_F(a,b]\le M(b-a)$.

(5) Let μ be the positive measure $d\mu=|f|\,dm$ on $(a,b]$. Let $\{(a_i,b_i)\}_{i=1}^n\subset(a,b]$ be disjoint intervals as above, then
$$\begin{aligned}\sum_{i=1}^n|F(b_i)-F(a_i)|&=\sum_{i=1}^n\left|\int_{(a_i,b_i]}f\,dm\right|\\&\le\sum_{i=1}^n\int_{(a_i,b_i]}|f|\,dm\\&=\int_{\cup_{i=1}^n(a_i,b_i]}|f|\,dm=\mu(\cup_{i=1}^n(a_i,b_i]).\end{aligned}\tag{14.11}$$
Since μ is absolutely continuous relative to m for all $\epsilon>0$ there exist $\delta>0$ such that $\mu(A)<\epsilon$ if $m(A)<\delta$. Taking $A=\cup_{i=1}^n(a_i,b_i]$ in Eq. (14.11) shows that F is absolutely continuous. It is also easy to see from Eq. (14.11) that $T_F(a,b]\le\int_{(a,b]}|f|\,dm$.

■

Theorem 14.25. *Let $F:\mathbb{R}\to\mathbb{C}$ be a function, then*

(1) *$F\in BV$ iff $\operatorname{Re}F\in BV$ and $\operatorname{Im}F\in BV$.*

(2) *If $F:\mathbb{R}\to\mathbb{R}$ is in BV then the functions $F_\pm:=(T_F\pm F)/2$ are bounded and increasing functions.*

(3) *$F:\mathbb{R}\to\mathbb{R}$ is in BV iff $F=F_+-F_-$ where $F_\pm$ are bounded increasing functions.*

(4) *If $F\in BV$ then $F(x\pm)$ exist for all $x\in\bar{\mathbb{R}}$. Let $G(x):=F(x+)$.*

(5) *$F \in BV$ then $\{x : \lim_{y \to x} F(y) \neq F(x)\}$ is a countable set and in particular $G(x) = F(x+)$ for all but a countable number of $x \in \mathbb{R}$.*
(6) *If $F \in BV$, then for m – a.e. x, $F'(x)$ and $G'(x)$ exist and $F'(x) = G'(x)$.*

Proof.

(1) Item 1. is a consequence of the inequalities

$$|F(b) - F(a)| \leq |\operatorname{Re} F(b) - \operatorname{Re} F(a)| + |\operatorname{Im} F(b) - \operatorname{Im} F(a)| \leq 2\,|F(b) - F(a)|\,.$$

(2) By Lemma 14.22, for all $a < b$,

$$T_F(b) - T_F(a) = T_F(a, b] \geq |F(b) - F(a)| \tag{14.12}$$

and therefore

$$T_F(b) \pm F(b) \geq T_F(a) \pm F(a)$$

which shows that $F_\pm$ are increasing. Moreover from Eq. (14.12), for $b \geq 0$ and $a \leq 0$,

$$\begin{aligned}|F(b)| &\leq |F(b) - F(0)| + |F(0)| \leq T_F(0, b] + |F(0)| \\ &\leq T_F(0, \infty) + |F(0)|\end{aligned}$$

and similarly

$$|F(a)| \leq |F(0)| + T_F(-\infty, 0)$$

which shows that F is bounded by $|F(0)| + T_F(\infty)$. Therefore $F_\pm$ is bounded as well.

(3) By Lemma 14.24 if $F = F_+ - F_-$, then

$$T_F(a, b] \leq T_{F_+}(a, b] + T_{F_-}(a, b] = |F_+(b) - F_+(a)| + |F_-(b) - F_-(a)|$$

which is bounded showing that $F \in BV$. Conversely if F is bounded variation, then $F = F_+ - F_-$ where $F_\pm$ are defined as in Item 2.

Items 4. – 6. follow from Items 1. – 3. and Theorem 14.17. ∎

Theorem 14.26. *Suppose that $F : \mathbb{R} \to \mathbb{C}$ is in BV, then*

$$|T_F(x+) - T_F(x)| \leq |F(x+) - F(x)| \tag{14.13}$$

for all $x \in \mathbb{R}$. If we further assume that F is right continuous then there exists a unique measure μ on $\mathcal{B} = \mathcal{B}_\mathbb{R}$. such that

$$\mu((-\infty, x]) = F(x) - F(-\infty) \text{ for all } x \in \mathbb{R}. \tag{14.14}$$

Proof. Since $F \in BV$, $F(x+)$ exists for all $x \in \mathbb{R}$ and hence Eq. (14.13) is a consequence of Eq. (14.7). Now assume that F is right continuous. In this case Eq. (14.13) shows that $T_F(x)$ is also right continuous. By considering the real and imaginary parts of F separately it suffices to prove there exists a unique finite signed measure μ satisfying Eq. (14.14) in the case that F is real valued. Now let $F_\pm = (T_F \pm F)/2$, then $F_\pm$ are increasing right continuous bounded functions. Hence there exists unique measure $\mu_\pm$ on $\mathcal{B}$ such that

$$\mu_\pm((-\infty, x]) = F_\pm(x) - F_\pm(-\infty)\ \forall x \in \mathbb{R}.$$

The finite signed measure $\mu \equiv \mu_+ - \mu_-$ satisfies Eq. (14.14). So it only remains to prove that μ is unique.

Suppose that $\tilde{\mu}$ is another such measure such that (14.14) holds with μ replaced by $\tilde{\mu}$. Then for $(a, b]$,

$$|\mu|(a,b] = \sup_{\mathbb{P}} \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| = |\tilde{\mu}|(a,b]$$

where the supremum is over all partition of $(a, b]$. This shows that $|\mu| = |\tilde{\mu}|$ on $\mathcal{A} \subset \mathcal{B}$ – the algebra generated by half open intervals and hence $|\mu| = |\tilde{\mu}|$. It now follows that $|\mu| + \mu$ and $|\tilde{\mu}| + \tilde{\mu}$ are finite positive measure on $\mathcal{B}$ such that

$$\begin{aligned}
(|\mu| + \mu)((a,b]) &= |\mu|((a,b]) + (F(b) - F(a)) \\
&= |\tilde{\mu}|((a,b]) + (F(b) - F(a)) \\
&= (|\tilde{\mu}| + \tilde{\mu})((a,b])
\end{aligned}$$

from which we infer that $|\mu| + \mu = |\tilde{\mu}| + \tilde{\mu} = |\mu| + \tilde{\mu}$ on $\mathcal{B}$. Thus $\mu = \tilde{\mu}$.

Alternatively, one may prove the uniqueness by showing that $\mathcal{C} := \{A \in \mathcal{B} : \mu(A) = \widetilde{\mu}(A)\}$ is a monotone class which contains $\mathcal{A}$ or using the $\pi - \lambda$ theorem. ■

Remark 14.27. One may also construct the measure μ_F by appealing to the complex Riesz Theorem (Corollary 13.41). Indeed suppose that F has bounded variation and let $I(f) := \int_{\mathbb{R}} f dF$ be defined analogously to the real incresing case in Notation 11.6 above. Then one easily shows that $|I(f)| \le T_F(\infty) \cdot \|f\|_u$ and therefore $I \in C_0(\mathbb{R}, \mathbb{C})^*$. So there exists a unique complex measure μ_F such that

$$\int_{\mathbb{R}} f dF = \int_{\mathbb{R}} f d\mu \text{ for all } f \in C_0(\mathbb{R}, \mathbb{C}).$$

Letting ϕ_ϵ be as in the proof of Theorem 11.35, then one may show

$$\left| \int_{\mathbb{R}} \phi_\epsilon dF - (F(b+\epsilon) - F(a+2\epsilon)) \right| \le T_F((a, a+2\epsilon]) + T_F((b+\epsilon, ba+2\epsilon]) \to 0 \text{ as } \epsilon \downarrow 0$$

and hence

$$\mu((a,b]) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \phi_\epsilon d\mu = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \phi_\epsilon dF = F(b) - F(a).$$

Definition 14.28. A function $F : \mathbb{R} \to \mathbb{C}$ is said to be of normalized bounded variation if $F \in BV$, F is right continuous and $F(-\infty) = 0$. We will abbreviate this by saying $F \in NBV$. (The condition: $F(-\infty) = 0$ is not essential and plays no role in the discussion below.)

Theorem 14.29. *Suppose that $F \in NBV$ and μ_F is the measure defined by Eq. (14.14), then*

$$d\mu_F = F' dm + d\lambda \tag{14.15}$$

where $\lambda \perp m$ and in particular for $-\infty < a < b < \infty$,

$$F(b) - F(a) = \int_a^b F' dm + \lambda((a,b]). \tag{14.16}$$

Proof. By Theorem 14.3, there exists $f \in L^1(m)$ and a complex measure λ such that for m -a.e. x,

$$f(x) = \lim_{r \downarrow 0} \frac{\mu(E_r)}{m(E_r)}, \tag{14.17}$$

for any collection of $\{E_r\}_{r>0} \subset \mathcal{B}$ which shrink nicely to $\{x\}$, $\lambda \perp m$ and

$$d\mu_F = f dm + d\lambda.$$

From Eq. (14.17) it follows that

$$\lim_{h\downarrow 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\downarrow 0}\frac{\mu_F((x,x+h])}{h}=f(x)\text{ and}$$
$$\lim_{h\downarrow 0}\frac{F(x-h)-F(x)}{-h}=\lim_{h\downarrow 0}\frac{\mu_F((x-h,x])}{h}=f(x)$$

for $m-$ a.e. x, i.e. $\frac{d}{dx^+}F(x)=\frac{d}{dx^-}F(x)=f(x)$ for m –a.e. x. This implies that F is $m-$ a.e. differentiable and $F'(x)=f(x)$ for $m-$ a.e. x. ■

Corollary 14.30. *Let $F:\mathbb{R}\to\mathbb{C}$ be in NBV, then*
(1) *$\mu_F\perp m$ iff $F'=0\quad m$ a.e.*
(2) *$\mu_F\ll m$ iff $\lambda=0$ iff*

$$\mu_F((a,b])=\int\limits_{(a,b]}F'(x)dm(x)\text{ for all }a<b. \tag{14.18}$$

Proof.
(1) If $F'(x)=0$ for m a.e. x, then by Eq. (14.15), $\mu_F=\lambda\perp m$. If $\mu_F\perp m$, then by Eq. (14.15), $F'dm=d\mu_F-d\lambda\perp dm$ and by Remark 13.8 $F'dm=0$, i.e. $F'=0$ m -a.e.
(2) If $\mu_F\ll m$, then $d\lambda=d\mu_F-F'dm\ll dm$ which implies by Lemma 13.28 that $\lambda=0$. Therefore Eq. (14.16) becomes (14.18). Now let

$$\rho(A):=\int\limits_A F'(x)dm(x)\text{ for all }A\in\mathcal{B}.$$

Recall by the Radon - Nikodym theorem that $\int_{\mathbb{R}}|F'(x)|\,dm(x)<\infty$ so that ρ is a complex measure on $\mathcal{B}$. So if Eq. (14.18) holds, then $\rho=\mu_F$ on the algebra generated by half open intervals. Therefore $\rho=\mu_F$ as in the uniqueness part of the proof of Theorem 14.26. Therefore $d\mu_F=F'dm$ and hence $\lambda=0$.

■

Theorem 14.31. *Suppose that $F:[a,b]\to\mathbb{C}$ is a measurable function. Then the following are equivalent:*
(1) *F is absolutely continuous on $[a,b]$.*
(2) *There exists $f\in L^1([a,b]),dm)$ such that*

$$F(x)-F(a)=\int_a^x fdm\ \forall x\in[a,b] \tag{14.19}$$

(3) *F' exists a.e., $F'\in L^1([a,b],dm)$ and*

$$F(x)-F(a)=\int_a^x F'dm\,\forall x\in[a,b]. \tag{14.20}$$

Proof. In order to apply the previous results, extend F to $\mathbb{R}$ by $F(x)=F(b)$ if $x\geq b$ and $F(x)=F(a)$ if $x\leq a$.

1. $\implies$ 3. If F is absolutely continuous then F is continuous on $[a,b]$ and $F-F(a)=F-F(-\infty)\in NBV$ by Lemma 14.24. By Proposition 14.19, $\mu_F\ll m$ and hence Item 3. is now a consequence of Item 2. of Corollary 14.30. The assertion 3. $\implies$ 2. is trivial.

2. $\Longrightarrow$ 1. If 2. holds then F is absolutely continuous on $[a, b]$ by Lemma 14.24. ■

14.5. **Counter Examples:** These are taken from I. P. Natanson,"Theory of functions of a real variable," p.269. Note it is proved in Natanson or in Rudin that the fundamental theorem of calculus holds for $f \in C([0,1])$ such that $f'(x)$ exists for all $x \in [0,1]$ and $f' \in L^1$. Now we give a couple of examples.

Example 14.32. In each case $f \in C([-1,1])$.

(1) Let $f(x) = |x|^{3/2} \sin \frac{1}{x}$ with $f(0) = 0$, then f is everywhere differentiable but f' is not bounded near zero. However, the function $f' \in L^1([-1,1])$.
(2) Let $f(x) = x^2 \cos \frac{\pi}{x^2}$ with $f(0) = 0$, then f is everywhere differentiable but $f' \notin L^1_{loc}(-\epsilon, \epsilon)$. Indeed, if $0 \notin (\alpha, \beta)$ then
$$\int_\alpha^\beta f'(x)dx = f(\beta) - f(\alpha) = \beta^2 \cos \frac{\pi}{\beta^2} - \alpha^2 \cos \frac{\pi}{\alpha^2}.$$
Now take $\alpha_n := \sqrt{\frac{2}{4n+1}}$ and $\beta_n = 1/\sqrt{2n}$. Then
$$\int_{\alpha_n}^{\beta_n} f'(x)dx = \frac{2}{4n+1} \cos \frac{\pi(4n+1)}{2} - \frac{1}{2n} \cos 2n\pi = \frac{1}{2n}$$
and noting that $\{(\alpha_n, \beta_n)\}_{n=1}^\infty$ are all disjoint, we find $\int_0^\epsilon |f'(x)|\, dx = \infty$.

14.6. **Exercises.**

Exercise 14.1. Folland 3.22 on p. 100.

Exercise 14.2. Folland 3.24 on p. 100.

Exercise 14.3. Folland 3.25 on p. 100.

Exercise 14.4. Folland 3.27 on p. 107.

Exercise 14.5. Folland 3.29 on p. 107.

Exercise 14.6. Folland 3.30 on p. 107.

Exercise 14.7. Folland 3.33 on p. 108.

Exercise 14.8. Folland 3.35 on p. 108.

Exercise 14.9. Folland 3.37 on p. 108.

Exercise 14.10. Folland 3.39 on p. 108.