18. Fourier Transform

The underlying space in this section is $\mathbb{R}^n$ with Lebesgue measure. The Fourier inversion formula is going to state that

$$f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} d\xi e^{i\xi x} \int_{\mathbb{R}^n} dy f(y) e^{-iy\xi}. \tag{18.1}$$

If we let $\xi = 2\pi\eta$, this may be written as

$$f(x) = \int_{\mathbb{R}^n} d\eta e^{i2\pi\eta x} \int_{\mathbb{R}^n} dy f(y) e^{-iy2\pi\eta}$$

and we have removed the multiplicative factor of $\left(\frac{1}{2\pi}\right)^n$ in Eq. (18.1) at the expense of placing factors of 2π in the arguments of the exponential. Another way to avoid writing the 2π's altogether is to redefine dx and $d\xi$ and this is what we will do here.

Notation 18.1. Let m be Lebesgue measure on $\mathbb{R}^n$ and define:

$$\mathbf{d}x = \left(\frac{1}{\sqrt{2\pi}}\right)^n dm(x) \text{ and } \mathbf{d}\xi \equiv \left(\frac{1}{\sqrt{2\pi}}\right)^n dm(\xi).$$

To be consistent with this new normalization of Lebesgue measure we will redefine $\|f\|_p$ and $\langle f, g\rangle$ as

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathbf{d}x\right)^{1/p} = \left(\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} |f(x)|^p \, dm(x)\right)^{1/p}$$

and

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(x) g(x) \mathbf{d}x \text{ when } fg \in L^1.$$

Similarly we will define the convolution relative to these normalizations by $f \bigstar \mathbf{g} := \left(\frac{1}{2\pi}\right)^{n/2} f * g$, i.e.

$$f \bigstar g(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \mathbf{d}y = \int_{\mathbb{R}^n} f(x-y) g(y) \left(\frac{1}{2\pi}\right)^{n/2} dm(y).$$

The following notation will also be convenient; given a multi-index $\alpha \in \mathbb{Z}_+^n$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}, \; \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j} \text{ and}$$

$$D_x^\alpha = \left(\frac{1}{i}\right)^{|\alpha|} \left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha.$$

Also let

$$\langle x\rangle := (1 + |x|^2)^{1/2}$$

and for $s \in \mathbb{R}$ let

$$\nu_s(x) = (1 + |x|)^s.$$

18.1. **Fourier Transform.**

Definition 18.2 (Fourier Transform)**.** For $f \in L^1$, let

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathbf{d}x \tag{18.2}$$

$$g^\vee(x) = \mathcal{F}^{-1}g(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\mathbf{d}\xi = \mathcal{F}g(-x) \tag{18.3}$$

The next theorem summarizes some more basic properties of the Fourier transform.

Theorem 18.3. *Suppose that $f, g \in L^1$. Then*

(1) $\hat{f} \in C_0(\mathbb{R}^n)$ *and* $\left\|\hat{f}\right\|_u \le \|f\|_1$.
(2) *For* $y \in \mathbb{R}^n$, $(\tau_y f)\hat{}\,(\xi) = e^{-iy\cdot\xi}\hat{f}(\xi)$.
(3) *The Fourier transform takes convolution to products, i.e.* $(f \bigstar g)\hat{} = \hat{f}\hat{g}$.
(4) *For* $f, g \in L^1$, $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$.
(5) *If* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is an invertible linear transformation, then*

$$(f \circ T)^\wedge(\xi) = |\det T|^{-1} \hat{f}(\left(T^{-1}\right)^* \xi) \text{ and}$$
$$(f \circ T)^\vee(\xi) = |\det T|^{-1} f^\vee(\left(T^{-1}\right)^* \xi)$$

(6) *If* $(1+|x|)^k f(x) \in L^1$, *then* $\hat{f} \in C^k$ *and* $\partial^\alpha \hat{f} \in C_0$ *for all* $|\alpha| \le k$. *Moreover,*

$$\partial_\xi^\alpha \hat{f}(\xi) = \mathcal{F}\left[(-ix)^\alpha f(x)\right](\xi) \tag{18.4}$$

for all $|\alpha| \le k$.

(7) *If* $f \in C^k$ *and* $\partial^\alpha f \in L^1$ *for all* $|\alpha| \le k$, *then* $(1+|\xi|)^k \hat{f}(\xi) \in C_0$ *and*

$$(\partial^\alpha f)\hat{}\,(\xi) = (i\xi)^\alpha \hat{f}(\xi) \tag{18.5}$$

for all $|\alpha| \le k$.

(8) *Suppose* $g \in L^1(\mathbb{R}^k)$ *and* $h \in L^1(\mathbb{R}^{n-k})$ *and* $f = g \otimes h$, *i.e.*

$$f(x) = g(x_1, \dots, x_k)h(x_{k+1}, \dots, x_n),$$

then $\hat{f} = \hat{g} \otimes \hat{h}$.

Proof. Item 1. is the Riemann Lebesgue Lemma 9.26. Items 2. − 5. are proved by the following straight forward computations:

$$(\tau_y f)\hat{}\,(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x-y)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-i(x+y)\cdot\xi} f(x)\mathbf{d}x = e^{-iy\cdot\xi}\hat{f}(\xi),$$

$$\begin{aligned}
\langle \hat{f}, g\rangle &= \int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)\mathbf{d}\xi = \int_{\mathbb{R}^n} \mathbf{d}\xi g(\xi)\int_{\mathbb{R}^n} \mathbf{d}x e^{-ix\cdot\xi} f(x) \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{d}x\mathbf{d}\xi e^{-ix\cdot\xi} g(\xi) f(x) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{d}x \hat{g}(x) f(x) = \langle f, \hat{g}\rangle,
\end{aligned}$$

$$\begin{aligned}
(f \bigstar g)\hat{}\,(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f\bigstar g(x)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\left(\int_{\mathbb{R}^n} f(x-y)g(y)\mathbf{d}y\right)\mathbf{d}x \\
&= \int_{\mathbb{R}^n} \mathbf{d}y \int_{\mathbb{R}^n} \mathbf{d}x e^{-ix\cdot\xi} f(x-y)g(y) = \int_{\mathbb{R}^n} \mathbf{d}y \int_{\mathbb{R}^n} \mathbf{d}x e^{-i(x+y)\cdot\xi} f(x)g(y) \\
&= \int_{\mathbb{R}^n} \mathbf{d}y e^{-iy\cdot\xi} g(y) \int_{\mathbb{R}^n} \mathbf{d}x e^{-ix\cdot\xi} f(x) = \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

and letting $y = Tx$ so that $\mathbf{d}x = |\det T|^{-1}\,\mathbf{d}y$

$$\begin{aligned}(f\circ T)\hat{}\,(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(Tx)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-iT^{-1}y\cdot\xi} f(y)\,|\det T|^{-1}\,\mathbf{d}y \\ &= |\det T|^{-1}\,\hat{f}(\left(T^{-1}\right)^*\xi).\end{aligned}$$

Item 6. is simply a matter of differentiating under the integral sign which is easily justified because $(1+|x|)^k f(x) \in L^1$.

Item 7. follows by using Lemma 9.25 repeatedly (i.e. integration by parts) to find

$$\begin{aligned}(\partial^\alpha f)\hat{}\,(\xi) &= \int_{\mathbb{R}^n} \partial_x^\alpha f(x) e^{-ix\cdot\xi}\mathbf{d}x = (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)\partial_x^\alpha e^{-ix\cdot\xi}\mathbf{d}x \\ &= (-1)^{|\alpha|}\int_{\mathbb{R}^n} f(x)(-i\xi)^\alpha e^{-ix\cdot\xi}\mathbf{d}x = (i\xi)^\alpha \hat{f}(\xi).\end{aligned}$$

Since $\partial^\alpha f \in L^1$ for all $|\alpha| \le k$, it follows that $(i\xi)^\alpha \hat{f}(\xi) = (\partial^\alpha f)\hat{}\,(\xi) \in C_0$ for all $|\alpha| \le k$. Since

$$(1+|\xi|)^k \le \left(1+\sum_{i=1}^n |\xi_i|\right)^k = \sum_{|\alpha|\le k} c_\alpha\,|\xi^\alpha|$$

where $0 < c_\alpha < \infty$,

$$\left|(1+|\xi|)^k\,\hat{f}(\xi)\right| \le \sum_{|\alpha|\le k} c_\alpha \left|\xi^\alpha \hat{f}(\xi)\right| \to 0 \text{ as } \xi\to\infty.$$

Item 8. is a simple application of Fubini's theorem. ■

Example 18.4. If $f(x) = e^{-|x|^2/2}$ then $\hat{f}(\xi) = e^{-|\xi|^2/2}$, in short

$$\mathcal{F}e^{-|x|^2/2} = e^{-|\xi|^2/2} \text{ and } \mathcal{F}^{-1}e^{-|\xi|^2/2} = e^{-|x|^2/2}. \tag{18.6}$$

More generally, for $t > 0$ let

$$p_t(x) := t^{-n/2} e^{-\frac{1}{2t}|x|^2} \tag{18.7}$$

then

$$\widehat{p_t}(\xi) = e^{-\frac{t}{2}|\xi|^2} \text{ and } (\widehat{p_t})^\vee(x) = p_t(x). \tag{18.8}$$

By Item 8. of Theorem 18.3, to prove Eq. (18.6) it suffices to consider the 1 − dimensional case because $e^{-|x|^2/2} = \prod_{i=1}^n e^{-x_i^2/2}$. Let $g(\xi) := \left(\mathcal{F}e^{-x^2/2}\right)(\xi)$, then by Eq. (18.4) and Eq. (18.5),

$$g'(\xi) = \mathcal{F}\left[(-ix)\,e^{-x^2/2}\right](\xi) = i\mathcal{F}\left[\frac{d}{dx}e^{-x^2/2}\right](\xi) = i(i\xi)\mathcal{F}\left[e^{-x^2/2}\right](\xi) = -\xi g(\xi). \tag{18.9}$$

Lemma 6.36 implies

$$g(0) = \int_{\mathbb{R}} e^{-x^2/2}\mathbf{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}dm(x) = 1,$$

and so solving Eq. (18.9) with $g(0) = 1$ gives $\mathcal{F}\left[e^{-x^2/2}\right](\xi) = g(\xi) = e^{-\xi^2/2}$ as desired. The assertion that $\mathcal{F}^{-1}e^{-|\xi|^2/2} = e^{-|x|^2/2}$ follows similarly or by using Eq. (18.3) to conclude,

$$\mathcal{F}^{-1}\left[e^{-|\xi|^2/2}\right](x) = \mathcal{F}\left[e^{-|-\xi|^2/2}\right](x) = \mathcal{F}\left[e^{-|\xi|^2/2}\right](x) = e^{-|x|^2/2}.$$

The results in Eq. (18.8) now follow from Eq. (18.6) and item 5 of Theorem 18.3. For example, since $p_t(x) = t^{-n/2}p_1(x/\sqrt{t})$,

$$(\widehat{p}_t)(\xi) = t^{-n/2}\left(\sqrt{t}\right)^n \hat{p}_1(\sqrt{t}\xi) = e^{-\frac{t}{2}|\xi|^2}.$$

This may also be written as $(\widehat{p}_t)(\xi) = t^{-n/2}p_{\frac{1}{t}}(\xi)$. Using this and the fact that p_t is an even function,

$$(\widehat{p}_t)^{\vee}(x) = \mathcal{F}\widehat{p}_t(-x) = t^{-n/2}\mathcal{F}p_{\frac{1}{t}}(-x) = t^{-n/2}t^{n/2}p_t(-x) = p_t(x).$$

18.2. **Schwartz Test Functions.**

Definition 18.5. A function $f \in C(\mathbb{R}^n, \mathbb{C})$ is said to have **rapid decay** or **rapid decrease** if

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N |f(x)| < \infty \text{ for } N = 1, 2, \dots.$$

Equivalently, for each $N \in \mathbb{N}$ there exists constants $C_N < \infty$ such that $|f(x)| \le C_N(1+|x|)^{-N}$ for all $x \in \mathbb{R}^n$. A function $f \in C(\mathbb{R}^n, \mathbb{C})$ is said to have (at most) **polynomial growth** if there exists $N < \infty$ such

$$\sup\,(1+|x|)^{-N}|f(x)| < \infty,$$

i.e. there exists $N \in \mathbb{N}$ and $C < \infty$ such that $|f(x)| \le C(1+|x|)^N$ for all $x \in \mathbb{R}^n$.

Definition 18.6 (Schwartz Test Functions)**.** Let $\mathcal{S}$ denote the space of functions $f \in C^\infty(\mathbb{R}^n)$ such that f and all of its partial derivatives have rapid decay and let

$$\|f\|_{N,\alpha} = \sup_{x\in\mathbb{R}^n}\left|(1+|x|)^N\partial^\alpha f(x)\right|$$

so that

$$\mathcal{S} = \left\{f \in C^\infty(\mathbb{R}^n) : \|f\|_{N,\alpha} < \infty \text{ for all } N \text{ and } \alpha\right\}.$$

Also let $\mathcal{P}$ denote those functions $g \in C^\infty(\mathbb{R}^n)$ such that g and all of its derivatives have at most polynomial growth, i.e. $g \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{P}$ iff for all multi-indices α, there exists $N_\alpha < \infty$ such

$$\sup\,(1+|x|)^{-N_\alpha}|\partial^\alpha g(x)| < \infty.$$

(Notice that any polynomial function on $\mathbb{R}^n$ is in $\mathcal{P}$.)

Remark 18.7. Since $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset L^2(\mathbb{R}^n)$, it follows that $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$.

Exercise 18.1. Let

$$L = \sum_{|\alpha|\le k} a_\alpha(x)\partial^\alpha \tag{18.10}$$

with $a_\alpha \in \mathcal{P}$. Show $L(\mathcal{S}) \subset \mathcal{S}$ and in particular $\partial^\alpha f$ and $x^\alpha f$ are back in $\mathcal{S}$ for all multi-indices α.

Suppose that $p(x,\xi) = \Sigma_{|\alpha|\le N}a_\alpha(x)\xi^\alpha$ where each function $a_\alpha(x)$ is a smooth function. We then set

$$p(x, D_x) := \Sigma_{|\alpha|\le N}a_\alpha(x)D_x^\alpha$$

and if each $a_\alpha(x)$ is also a polynomial in x we will let

$$p(-D_\xi, \xi) := \Sigma_{|\alpha|\le N}a_\alpha(-D_\xi)M_{\xi^\alpha}$$

where M_{ξ^α} is the operation of multiplication by ξ^α.

Proposition 18.8. *Suppose that each function $a_\alpha(x)$ is smooth and $f \in \mathcal{S}$, then*

$$p(x, D_x)f(x) = \int_{\mathbb{R}^n} p(x,\xi)\hat{f}(\xi)\, e^{ix\cdot\xi}\mathbf{d}\xi.$$

If we further assume that each function $a_\alpha(x)$ is a polynomial in x, then

$$(p(x, D_x)f)^\wedge(\xi) = p(-D_\xi, \xi)\hat{f}(\xi) \tag{18.11}$$

and

$$p(\xi, D_\xi)\hat{f}(\xi) = [p(D_x, -x)f(x)]^\wedge(\xi). \tag{18.12}$$

Alternatively we may write this last equation as

$$p(D_x, -x)f(x) = \left[p(\xi, D_\xi)\hat{f}(\xi)\right]^\vee(x). \tag{18.13}$$

Proof. For $f \in \mathcal{S}$, we have

$$\begin{aligned} p(x, D_x)f(x) &= p(x, D_x)\left(\mathcal{F}^{-1}\hat{f}\right)(x) = p(x, D_x)\int_{\mathbb{R}^n}\hat{f}(\xi)\, e^{ix\cdot\xi}\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n}\hat{f}(\xi)\, p(x, D_x)e^{ix\cdot\xi}\mathbf{d}\xi = \int_{\mathbb{R}^n}\hat{f}(\xi)\, p(x,\xi)e^{ix\cdot\xi}\mathbf{d}\xi \end{aligned}$$

wherein we have used the fact that $D_x^\alpha e^{ix\cdot\xi} = \xi^\alpha e^{ix\cdot\xi}$. Now if we further assume that each function $a_\alpha(x)$ is a polynomial in x, we may use the relation $x^\alpha e^{ix\cdot\xi} = D_\xi^\alpha e^{ix\cdot\xi}$ to write

$$\begin{aligned} \int_{\mathbb{R}^n}\hat{f}(\xi)\, p(x,\xi)e^{ix\cdot\xi}\mathbf{d}\xi &= \int_{\mathbb{R}^n}\hat{f}(\xi)\, \Sigma_{|\alpha|\le N}\xi^\alpha a_\alpha(D_\xi)e^{ix\cdot\xi}\mathbf{d}\xi \\ &= \Sigma_{|\alpha|\le N}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a_\alpha(-D_\xi)\left[\xi^\alpha\hat{f}(\xi)\right]\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}p(-D_\xi,\xi)\hat{f}(\xi)\,\mathbf{d}\xi \end{aligned}$$

wherein the second equality we have used repeated integration by parts. Combining the last two displayed equations gives

$$p(x, D_x)f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}p(-D_\xi,\xi)\hat{f}(\xi)\,\mathbf{d}\xi$$

which upon taking the Fourier Transform is equivalent to Eq. (18.11). The proof of Eq. (18.12) is similar:

$$\begin{aligned} p(\xi, D_\xi)\hat{f}(\xi) &= p(\xi, D_\xi)\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\mathbf{d}x = \int_{\mathbb{R}^n} f(x)p(\xi, -D_x)e^{-ix\cdot\xi}\mathbf{d}x \\ &= \sum_\alpha\int_{\mathbb{R}^n} f(x)(-x)^\alpha a_\alpha(\xi)e^{-ix\cdot\xi}\mathbf{d}x = \sum_\alpha\int_{\mathbb{R}^n} f(x)(-x)^\alpha a_\alpha(-D_x)e^{-ix\cdot\xi}\mathbf{d}x \\ &= \sum_\alpha\int_{\mathbb{R}^n} e^{-ix\cdot\xi}a_\alpha(D_x)\left[(-x)^\alpha f(x)\right]\mathbf{d}x = [p(D_x, -x)f(x)]^\wedge(\xi). \end{aligned}$$

■

Corollary 18.9. *The Fourier transform preserves the space $\mathcal{S}$, i.e. $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$.*

Proof. Let $p(x,\xi) = \Sigma_{|\alpha|\le N} a_\alpha(x)\xi^\alpha$ with $a_\alpha(x)$ being a polynomial function in x. If $f \in \mathcal{S}$, then because $p(D_x, -x)f \in \mathcal{S} \subset L^1$ we have by Eq. (18.12) that $p(\xi, D_\xi)\hat{f}(\xi)$ is bounded in ξ, i.e.

$$\sup_{\xi\in\mathbb{R}^n} |p(\xi, D_\xi)\hat{f}(\xi)| \le C(p,f) < \infty.$$

Since p is arbitrary it follows easily that all derivative of $\hat{f}(\xi)$ have fast decay and therefore $\hat{f}$ is in $\mathcal{S}$. ■

18.3. **Fourier Inversion Formula .**

Theorem 18.10 (Fourier Inversion Theorem)**.** *Suppose that $f \in L^1$ and $\hat{f} \in L^1$, then*

1. *there exists $f_0 \in C_0(\mathbb{R}^n)$ such that $f = f_0$ a.e.*
2. *$f_0 = \mathcal{F}^{-1}\mathcal{F} f$ and $f_0 = \mathcal{F}\mathcal{F}^{-1} f$,*
3. *f and $\hat{f}$ are in $L^1 \cap L^\infty$ and*
4. *$\|f\|_2 = \left\|\hat{f}\right\|_2$.*

In particular, $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a linear isomorphism of vector spaces.

Proof. First notice that $\hat{f} \in C_0(\mathbb{R}^n) \subset L^\infty$ and $\hat{f} \in L^1$ by assumption, so that $\hat{f} \in L^1 \cap L^\infty$. Let $p_t(x) \equiv t^{-n/2}e^{-\frac{1}{2t}|x|^2}$ be as in Example 18.4 so that $\widehat{p}_t(\xi) = e^{-\frac{t}{2}|\xi|^2}$ and $\widehat{p}_t^\vee = p_t$. Define $f_0 := \hat{f}^\vee \in C_0$ then

$$\begin{aligned} f_0(x) = (\hat{f})^\vee(x) &= \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}\mathbf{d}\xi = \lim_{t\downarrow 0}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}\widehat{p}_t(\xi)\mathbf{d}\xi \\ &= \lim_{t\downarrow 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)e^{i\xi\cdot(x-y)}\widehat{p}_t(\xi)\mathbf{d}\xi\ \mathbf{d}y \\ &= \lim_{t\downarrow 0}\int_{\mathbb{R}^n} f(y)p_t(y)\mathbf{d}y = f(x) \text{ a.e.} \end{aligned}$$

wherein we have used Theorem 9.20 in the last equality along with the observations that $p_t(y) = p_1(y/\sqrt{t})$ and $\int_{\mathbb{R}^n} p_1(y)\mathbf{d}y = 1$. In particular this shows that $f \in L^1 \cap L^\infty$. A similar argument shows that $\mathcal{F}^{-1}\mathcal{F} f = f_0$ as well.

Let us now compute the L^2 – norm of $\hat{f}$,

$$\begin{aligned} \|\hat{f}\|_2^2 &= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi)}\mathbf{d}\xi = \int_{\mathbb{R}^n} \mathbf{d}\xi\hat{f}(\xi)\int_{\mathbb{R}^n}\mathbf{d}x\overline{f(x)}e^{ix\cdot\xi} \\ &= \int_{\mathbb{R}^n}\mathbf{d}x\,\overline{f(x)}\int_{\mathbb{R}^n}\mathbf{d}\xi\hat{f}(\xi)e^{ix\cdot\xi} \\ &= \int_{\mathbb{R}^n}\mathbf{d}x\,\overline{f(x)}f(x) = \|f\|_2^2 \end{aligned}$$

because $\int_{\mathbb{R}^n}\mathbf{d}\xi\hat{f}(\xi)e^{ix\cdot\xi} = \mathcal{F}^{-1}\hat{f}(x) = f(x)$ a.e. ■

Corollary 18.11. *By the B.L.T. Theorem 3.67, the maps $\mathcal{F}|_{\mathcal{S}}$ and $\mathcal{F}^{-1}|_{\mathcal{S}}$ extend to bounded linear maps $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^{-1}$ from $L^2 \to L^2$. These maps satisfy the following properties:*

1. *$\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^{-1}$ are unitary and are inverses to one another as the notation suggests.*

(2) *For $f \in L^2$ we may compute $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}^{-1}$ by*

$$\bar{\mathcal{F}} f(\xi) = L^2\text{-}\lim_{R\to\infty} \int_{|x|\le R} f(x) e^{-ix\cdot\xi} \mathbf{d}x \text{ and} \tag{18.14}$$

$$\bar{\mathcal{F}}^{-1} f(\xi) = L^2\text{-}\lim_{R\to\infty} \int_{|x|\le R} f(x) e^{ix\cdot\xi} \mathbf{d}x. \tag{18.15}$$

(3) *We may further extend $\bar{\mathcal{F}}$ to a map from $L^1 + L^2 \to C_0 + L^2$ (still denote by $\bar{\mathcal{F}}$) defined by $\bar{\mathcal{F}} f = \hat{h} + \bar{\mathcal{F}} g$ where $f = h + g \in L^1 + L^2$. For $f \in L^1 + L^2$, $\bar{\mathcal{F}} f$ may be characterized as the unique function $F \in L^1_{loc}(\mathbb{R}^n)$ such that*

$$\langle F, \phi\rangle = \langle f, \hat{\phi}\rangle \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n). \tag{18.16}$$

Moreover if Eq. (18.16) holds then $F \in C_0 + L^2 \subset L^1_{loc}(\mathbb{R}^n)$ and Eq.(18.16) is valid for all $\phi \in \mathcal{S}$.

Proof. Item 1., If $f \in L^2$ and $\phi_n \in \mathcal{S}$ such that $\phi_n \to f$ in L^2, then $\bar{\mathcal{F}} f := \lim_{n\to\infty} \hat{\phi}_n$. Since $\hat{\phi}_n \in \mathcal{S} \subset L^1$, we may concluded that $\left\|\hat{\phi}_n\right\|_2 = \|\phi_n\|_2$ for all n. Thus

$$\left\|\bar{\mathcal{F}} f\right\|_2 = \lim_{n\to\infty} \left\|\hat{\phi}_n\right\|_2 = \lim_{n\to\infty} \|\phi_n\|_2 = \|f\|_2$$

which shows that $\bar{\mathcal{F}}$ is an isometry from L^2 to L^2 and similarly $\bar{\mathcal{F}}^{-1}$ is an isometry. Since $\bar{\mathcal{F}}^{-1}\bar{\mathcal{F}} = \mathcal{F}^{-1}\mathcal{F} = id$ on the dense set $\mathcal{S}$, it follows by continuity that $\bar{\mathcal{F}}^{-1}\bar{\mathcal{F}} = id$ on all of L^2. Hence $\bar{\mathcal{F}}\bar{\mathcal{F}}^{-1} = id$, and thus $\bar{\mathcal{F}}^{-1}$ is the inverse of $\bar{\mathcal{F}}$. This proves item 1.

Item 2. Let $f \in L^2$ and $R < \infty$ and set $f_R(x) := f(x) 1_{|x|\le R}$. Then $f_R \in L^1 \cap L^2$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \phi(x)\mathbf{d}x = 1$ and set $\phi_k(x) = k^n \phi(kx)$. Then $f_R \bigstar \phi_k \to f_R \in L^1 \cap L^2$ with $f_R \bigstar \phi_k \in C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}$. Hence

$$\bar{\mathcal{F}} f_R = L^2\text{–}\lim_{k\to\infty} \mathcal{F}\left(f_R \bigstar \phi_k\right) = \mathcal{F} f_R \text{ a.e.}$$

where in the second equality we used the fact that $\mathcal{F}$ is continuous on L^1. Hence $\int_{|x|\le R} f(x) e^{-ix\cdot\xi}\mathbf{d}x$ represents $\bar{\mathcal{F}} f_R(\xi)$ in L^2. Since $f_R \to f$ in L^2, Eq. (18.14) follows by the continuity of $\bar{\mathcal{F}}$ on L^2.

Item 3. If $f = h + g \in L^1 + L^2$ and $\phi \in \mathcal{S}$, then

$$\begin{aligned}\langle \hat{h} + \bar{\mathcal{F}} g, \phi\rangle &= \langle h, \phi\rangle + \langle \bar{\mathcal{F}} g, \phi\rangle = \langle h, \hat{\phi}\rangle + \lim_{R\to\infty} \langle \mathcal{F}\left(g 1_{|\cdot|\le R}\right), \phi\rangle \\ &= \langle h, \hat{\phi}\rangle + \lim_{R\to\infty} \langle g 1_{|\cdot|\le R}, \hat{\phi}\rangle = \langle h + g, \hat{\phi}\rangle. \end{aligned} \tag{18.17}$$

In particular if $h + g = 0$ a.e., then $\langle \hat{h} + \bar{\mathcal{F}} g, \phi\rangle = 0$ for all $\phi \in \mathcal{S}$ and since $\hat{h} + \bar{\mathcal{F}} g \in L^1_{loc}$ it follows from Corollary 9.27 that $\hat{h} + \bar{\mathcal{F}} g = 0$ a.e. This shows that $\bar{\mathcal{F}} f$ is well defined independent of how $f \in L^1 + L^2$ is decomposed into the sum of an L^1 and an L^2 function. Moreover Eq. (18.17) shows Eq. (18.16) holds with $F = \hat{h} + \bar{\mathcal{F}} g \in C_0 + L^2$ and $\phi \in \mathcal{S}$. Now suppose $G \in L^1_{loc}$ and $\langle G, \phi\rangle = \langle f, \hat{\phi}\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Then by what we just proved, $\langle G, \phi\rangle = \langle F, \phi\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$ and so an application of Corollary 9.27 shows $G = F \in C_0 + L^2$. ■

Notation 18.12. Given the results of Corollary 18.11, there is little danger in writing $\hat{f}$ or $\mathcal{F} f$ for $\bar{\mathcal{F}} f$ when $f \in L^1 + L^2$.

Corollary 18.13. *If f and g are L^1 functions such that $\hat{f}, \hat{g} \in L^1$, then*

$$\mathcal{F}(fg) = \hat{f} \bigstar \hat{g} \text{ and } \mathcal{F}^{-1}(fg) = f^{\vee} \bigstar g^{\vee}.$$

Since $\mathcal{S}$ is closed under pointwise products and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism it follows that $\mathcal{S}$ is closed under convolution as well.

Proof. By Theorem 18.10, $f, g, \hat{f}, \hat{g} \in L^1 \cap L^\infty$ and hence $f \cdot g \in L^1 \cap L^\infty$ and $\hat{f} \bigstar \hat{g} \in L^1 \cap L^\infty$. Since

$$\mathcal{F}^{-1}\left(\hat{f} \bigstar \hat{g}\right) = \mathcal{F}^{-1}\left(\hat{f}\right) \cdot \mathcal{F}^{-1}\left(\hat{g}\right) = f \cdot g \in L^1$$

we may conclude from Theorem 18.10 that

$$\hat{f} \bigstar \hat{g} = \mathcal{F}\mathcal{F}^{-1}\left(\hat{f} \bigstar \hat{g}\right) = \mathcal{F}(f \cdot g).$$

Similarly one shows $\mathcal{F}^{-1}(fg) = f^{\vee} \bigstar g^{\vee}$. ■

Corollary 18.14. *Suppose α is a multi-index and $f \in L^2$. Then $\partial^\alpha f$ exists in L^2 iff $\xi \to (i\xi)^\alpha \hat{f}(\xi) \in L^2$ and if $\partial^\alpha f \in L^2$ then*

$$(\partial^\alpha f)^{\hat{}}(\xi) = (i\xi)^\alpha \hat{f}(\xi) \text{ for } \xi - a.e.\ \xi.$$

Proof. Suppose $f, \partial^\alpha f \in L^2$ and $\phi \in C_c^\infty(\mathbb{R}^n)$, then by item 2. of Lemma 17.12 and Eq. (18.4),

$$\begin{aligned}
\langle \widehat{\partial^\alpha f}, \phi \rangle &= \langle \partial^\alpha f, \hat{\phi} \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \hat{\phi} \rangle = (-1)^{|\alpha|} \langle f(\xi), \mathcal{F}\left[(-ix)^\alpha \phi(x)\right](\xi) \rangle \\
&= \langle \hat{f}(\xi), (i\xi)^\alpha \phi(\xi) \rangle = \langle (i\xi)^\alpha \hat{f}(\xi), \phi(\xi) \rangle.
\end{aligned}$$

Since this holds for all $\phi \in C_c^\infty(\mathbb{R}^n)$, we conclude that $(i\xi)^\alpha \hat{f}(\xi) = \widehat{\partial^\alpha f}(\xi) \in L^2$. Conversely if $(i\xi)^\alpha \hat{f}(\xi) \in L^2$, let $g(x) := \mathcal{F}^{-1}\left[(i\xi)^\alpha \hat{f}(\xi)\right](x) \in L^2$. Then for $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\langle g, \phi \rangle &= \langle \mathcal{F}^{-1}\left[(i\xi)^\alpha \hat{f}(\xi)\right](x), \phi(x) \rangle = \langle (i\xi)^\alpha \hat{f}(\xi), \phi^{\vee}(\xi) \rangle = \langle \hat{f}(\xi), (i\xi)^\alpha \phi^{\vee}(\xi) \rangle \\
&= \langle \hat{f}(\xi), \left((-\partial)^\alpha \phi\right)^{\vee}(\xi) \rangle = \langle f(x), \mathcal{F}\left[\left((-\partial)^\alpha \phi\right)^{\vee}(\xi)\right](x) \rangle \\
&= \langle f, (-\partial)^\alpha \phi \rangle.
\end{aligned}$$

Since this equation holds for all $\phi \in C_c^\infty(\mathbb{R}^n)$, it follows that $\partial^\alpha f$ exists and $\partial^\alpha f = g \in L^2$. ■

The following table summarizes some of the basic properties of the Fourier transform and its inverse.

f	$\longleftrightarrow$	$\hat{f}$ or $f^{\vee}$
Smoothness	$\longleftrightarrow$	Decay at infinity
∂^α	$\longleftrightarrow$	Multiplication by $(\pm i\xi)^\alpha$
$\mathcal{S}$	$\longleftrightarrow$	$\mathcal{S}$
$L^2(\mathbb{R}^n)$	$\longleftrightarrow$	$L^2(\mathbb{R}^n)$
Convolution	$\longleftrightarrow$	Products.

18.4. **Constant Coefficient partial differential equations.** Suppose that $p(\xi) = \sum_{|\alpha| \le k} a_\alpha \xi^\alpha$ with $a_\alpha \in \mathbb{C}$ and

$$L = p(D_x) := \Sigma_{|\alpha| \le N} a_\alpha D_x^\alpha = \Sigma_{|\alpha| \le N} a_\alpha \left(\frac{1}{i}\partial_x\right)^\alpha.$$

Then for $f \in \mathcal{S}$

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

that is to say the Fourier transform takes a constant coefficient partial differential operator to multiplication by a polynomial. This fact can often be used to solve constant coefficient partial differential equation. For example suppose $g : \mathbb{R}^n \to \mathbb{C}$ is a given function and we want to find a solution to the equation $Lf = g$. Taking the Fourier transform of both sides of the equation $Lf = g$ would imply $p(\xi)\hat{f}(\xi) = \hat{g}(\xi)$ and therefore $\hat{f}(\xi) = \hat{g}(\xi)/p(\xi)$ provided $p(\xi)$ is never zero. (We will discuss what happens when $p(\xi)$ has zeros a bit more later on.) So we should expect

$$f(x) = \mathcal{F}^{-1}\left(\frac{1}{p(\xi)}\hat{g}(\xi)\right)(x) = \mathcal{F}^{-1}\left(\frac{1}{p(\xi)}\right) \bigstar g(x).$$

18.4.1. *Elliptic examples.* As a specific example consider the equation

$$\left(-\Delta + m^2\right) f = g$$

where $f, g : \mathbb{R}^n \to \mathbb{C}$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the usual Laplacian on $\mathbb{R}^n$. Taking the Fourier transform of this equation implies

$$\left(|\xi|^2 + m^2\right) \hat{f}(\xi) = \hat{g}(\xi)$$

and therefore,

$$\hat{f}(\xi) = \left(|\xi|^2 + m^2\right)^{-1} \hat{g}(\xi)$$

from which we deduce

$$f(x) = G_m \bigstar g(x) = \int_{\mathbb{R}^n} G_m(x - y) g(y) \mathbf{dy}$$

where

$$G_m(x) := \mathcal{F}^{-1}\left(|\xi|^2 + m^2\right)^{-1}(x) = \int_{\mathbb{R}^n} \frac{1}{m^2 + |\xi|^2} e^{i\xi\cdot x} \mathbf{d}\xi.$$

At the moment $\mathcal{F}^{-1}\left(|\xi|^2 + m^2\right)^{-1}$ only makes sense when $n = 1, 2$, or 3 because only then is $\left(|\xi|^2 + m^2\right)^{-1} \in L^1(\mathbb{R}^n)$. For now we will restrict our attention to the one dimensional case, $n = 1$, in which case

$$G_m(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{(\xi + mi)(\xi - mi)} e^{i\xi x} d\xi. \tag{18.18}$$

The function G_m may be computed using standard complex variable contour integration methods to find, for $x \ge 0$,

$$G_m(x) = \frac{1}{\sqrt{2\pi}} 2\pi i \frac{e^{i^2 mx}}{2im} = \frac{1}{2m}\sqrt{2\pi} e^{-mx}$$

and since G_m is an even function,

$$G_m(x) = \mathcal{F}^{-1}\left(|\xi|^2 + m^2\right)^{-1}(x) = \frac{\sqrt{2\pi}}{2m} e^{-m|x|}. \tag{18.19}$$

This result is easily verified to be correct, since

$$\begin{aligned}\mathcal{F}\left[\frac{\sqrt{2\pi}}{2m}e^{-m|x|}\right](\xi) &= \frac{\sqrt{2\pi}}{2m}\int_{\mathbb{R}} e^{-m|x|}e^{-ix\cdot\xi}\mathbf{d}x \\ &= \frac{1}{2m}\left(\int_0^\infty e^{-mx}e^{-ix\cdot\xi}dx + \int_{-\infty}^0 e^{mx}e^{-ix\cdot\xi}dx\right) \\ &= \frac{1}{2m}\left(\frac{1}{m+i\xi}+\frac{1}{m-i\xi}\right) = \frac{1}{m^2+\xi^2}.\end{aligned}$$

Hence in conclusion we find that $\left(-\Delta+m^2\right)f = g$ has solution given by

$$f(x) = G_m\bigstar g(x) = \frac{\sqrt{2\pi}}{2m}\int_{\mathbb{R}} e^{-m|x-y|}g(y)\mathbf{d}y = \frac{1}{2m}\int_{\mathbb{R}} e^{-m|x-y|}g(y)dy.$$

Question. Why do we get a unique answer here given that $f(x) = A\sinh(x) + B\cosh(x)$ solves

$$\left(-\Delta+m^2\right)f = 0?$$

The answer is that such an f is not in L^2 unless $f = 0$! More generally it is worth noting that $A\sinh(x) + B\cosh(x)$ is not in $\mathcal{P}$ unless $A = B = 0$.

What about when $m = 0$ in which case $m^2 + \xi^2$ becomes ξ^2 which has a zero at 0. Noting that constants are solutions to $\Delta f = 0$, we might look at

$$\lim_{m\downarrow 0}(G_m(x) - 1) = \lim_{m\downarrow 0}\frac{\sqrt{2\pi}}{2m}(e^{-m|x|}-1) = -\frac{\sqrt{2\pi}}{2}\left|x\right|.$$

as a solution, i.e. we might conjecture that

$$f(x) := -\frac{1}{2}\int_{\mathbb{R}}|x-y|\,g(y)dy$$

solves the equation $-f'' = g$. To verify this we have

$$f(x) := -\frac{1}{2}\int_{-\infty}^x (x-y)\,g(y)dy - \frac{1}{2}\int_x^\infty (y-x)\,g(y)dy$$

so that

$$\begin{aligned}f'(x) &= -\frac{1}{2}\int_{-\infty}^x g(y)dy + \frac{1}{2}\int_x^\infty g(y)dy \text{ and} \\ f''(x) &= -\frac{1}{2}g(x) - \frac{1}{2}g(x).\end{aligned}$$

18.4.2. *Heat Equation on* $\mathbb{R}^n$. The heat equation for a function $u : \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{C}$ is the partial differential equation

$$\left(\partial_t - \frac{1}{2}\Delta\right)u = 0 \text{ with } u(0,x) = f(x), \tag{18.20}$$

where f is a given function on $\mathbb{R}^n$. By Fourier transforming Eq. (18.20) in the $x-$ variables only, one finds that (18.20) implies that

$$\left(\partial_t + \frac{1}{2}\left|\xi\right|^2\right)\hat{u}(t,\xi) = 0 \text{ with } \hat{u}(0,\xi) = \hat{f}(\xi). \tag{18.21}$$

and hence that $\hat{u}(t,\xi) = e^{-t|\xi|^2/2}\hat{f}(\xi)$. Inverting the Fourier transform then shows that

$$u(t,x) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\hat{f}(\xi)\right)(x) = \left(\mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\right)\bigstar f\right)(x).$$

From Example 18.4,

$$\mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\right)(x) = p_t(x) = t^{-n/2}e^{-\frac{1}{2t}|x|^2}$$

and therefore,

$$u(t,x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)\mathbf{d}y.$$

This suggests the following theorem.

Theorem 18.15. *Let*

$$\rho(t,x,y) := (2\pi t)^{-n/2}\, e^{-|x-y|^2/2t} \tag{18.22}$$

be the ***heat kernel*** *on* $\mathbb{R}^n$. *Then*

$$\left(\partial_t - \frac{1}{2}\Delta_x\right)\rho(t,x,y) = 0 \text{ and } \lim_{t\downarrow 0}\rho(t,x,y) = \delta_x(y), \tag{18.23}$$

where δ_x *is the* δ *– function at* x *in* $\mathbb{R}^n$. *More precisely, if* f *is a continuous bounded (can be relaxed considerably) function on* $\mathbb{R}^n$, *then* $u(t,x) = \int_{\mathbb{R}^n}\rho(t,x,y)f(y)dy$ *is a solution to Eq. (18.20) where* $u(0,x) := \lim_{t\downarrow 0}u(t,x)$.

Proof. Direct computations show that $\left(\partial_t - \frac{1}{2}\Delta_x\right)\rho(t,x,y) = 0$ and an application of Theorem 9.20 shows $\lim_{t\downarrow 0}\rho(t,x,y) = \delta_x(y)$ or equivalently that $\lim_{t\downarrow 0}\int_{\mathbb{R}^n}\rho(t,x,y)f(y)dy = f(x)$ uniformly on compact subsets of $\mathbb{R}^n$. This shows that $\lim_{t\downarrow 0}u(t,x) = f(x)$ uniformly on compact subsets of $\mathbb{R}^n$. ■

Notation 18.16. We will write $\left(e^{t\Delta/2}f\right)(x)$ for $\int_{\mathbb{R}^n}\rho(t,x,y)f(y)dy = p_t\bigstar f$.

This notation suggests that we should be able to compute the solution to g to $(\Delta - m^2)g = f$ using

$$g(x) = \left(m^2 - \Delta\right)^{-1}f(x) = \int_0^\infty \left(e^{\left(\Delta - m^2\right)t}f\right)(x)dt = \int_0^\infty \left(e^{-m^2t}p_{2t}\bigstar f\right)(x)dt,$$

as may be easily verified using the Fourier transform. This gives us a method to compute $G_m(x)$ from the previous section, namely $t^{-n/2}e^{-\frac{1}{2t}|x|^2}$

$$G_m(x) = \int_0^\infty e^{-m^2t}p_{2t}(x)dt = \int_0^\infty (2t)^{-n/2}e^{-m^2t-\frac{1}{4t}|x|^2}dt.$$

We make the change of variables, $\lambda = |x|^2/4t$ ($t = |x|^2/4\lambda$, $dt = -\frac{|x|^2}{4\lambda^2}d\lambda$) to find

$$G_m(x) = \int_0^\infty (2t)^{-n/2}e^{-m^2t-\frac{1}{4t}|x|^2}dt = \int_0^\infty \left(\frac{|x|^2}{2\lambda}\right)^{-n/2} e^{-m^2|x|^2/4\lambda-\lambda}\frac{|x|^2}{(2\lambda)^2}d\lambda$$

$$= \frac{2^{(n/2-2)}}{|x|^{n-2}}\int_0^\infty \lambda^{n/2-2}e^{-\lambda}e^{-m^2|x|^2/4\lambda}d\lambda. \tag{18.24}$$

In case $n = 3$, Eq. (18.24) becomes

$$G_m(x) = \frac{\sqrt{\pi}}{\sqrt{2}\,|x|}\int_0^\infty \frac{1}{\sqrt{\pi\lambda}}e^{-\lambda}e^{-m^2|x|^2/4\lambda}d\lambda = \frac{\sqrt{\pi}}{\sqrt{2}\,|x|}e^{-m|x|}$$

where the last equality follows from Exercise 18.4. Hence when $n = 3$ we have found

$$\left(m^2 - \Delta\right)^{-1} f(x) = G_m \bigstar f(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \frac{\sqrt{\pi}}{\sqrt{2}\,|x|} e^{-m|x-y|} f(y) dy$$

$$= \int_{\mathbb{R}^3} \frac{1}{4\pi\,|x-y|} e^{-m|x-y|} f(y) dy. \tag{18.25}$$

The function $\frac{1}{4\pi|x|} e^{-m|x|}$ is called the Yukawa potential.

When $m = 0$, Eq. (18.24) becomes

$$G_0(x) = \frac{2^{(n/2-2)}}{|x|^{n-2}} \int_0^\infty \lambda^{n/2-1} e^{-\lambda} \frac{d\lambda}{\lambda} = \frac{2^{(n/2-2)}}{|x|^{n-2}} \Gamma(n/2-1)$$

where $\Gamma(x)$ in the gamma function defined in Eq. (6.30). Hence for "reasonable" functions f (and $n \neq 2$)

$$(-\Delta)^{-1} f(x) = G_0 \bigstar f(x) = 2^{(n/2-2)} \Gamma(n/2-1) (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y) dy$$

$$= \frac{1}{4\pi^{n/2}} \Gamma(n/2-1) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y) dy.$$

The function

$$\tilde{G}_0(x,y) := \frac{1}{4\pi^{n/2}} \Gamma(n/2-1) \frac{1}{|x-y|^{n-2}}$$

is a "Green's function" for $-\Delta$. Recall from Exercise 6.16 that, for $n = 2k$, $\Gamma(\frac{n}{2} - 1) = \Gamma(k-1) = (k-2)!$, and for $n = 2k+1$,

$$\Gamma(\frac{n}{2} - 1) = \Gamma(k - 1/2) = \Gamma(k - 1 + 1/2) = \sqrt{\pi} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-3)}{2^{k-1}}$$

$$= \sqrt{\pi} \frac{(2k-3)!!}{2^{k-1}} \text{ where } (-1)!! \equiv 1.$$

Hence

$$\tilde{G}_0(x,y) = \frac{1}{4} \frac{1}{|x-y|^{n-2}} \begin{cases} \frac{1}{\pi^k} (k-2)! & \text{if} \quad n = 2k \\ \frac{1}{\pi^k} \frac{(2k-3)!!}{2^{k-1}} & \text{if} \quad n = 2k+1 \end{cases}$$

and in particular when $n = 3$,

$$\tilde{G}_0(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|}$$

which is consistent with Eq. (18.25) with $m = 0$.

18.4.3. *Wave Equation on* $\mathbb{R}^n$. Let us now consider the wave equation on $\mathbb{R}^n$,

$$0 = \left(\partial_t^2 - \Delta\right) u(t,x) \text{ with}$$
$$u(0,x) = f(x) \text{ and } u_t(0,x) = g(x).$$

Taking the Fourier transform in the x variables gives the following equation

$$0 = \hat{u}_{tt}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) \text{ with}$$
$$\hat{u}(0,\xi) = \hat{f}(\xi) \text{ and } \hat{u}_t(0,\xi) = \hat{g}(\xi).$$

The solution to these equations is

$$\hat{u}(t,\xi) = \hat{f}(\xi) \cos(t\,|\xi|) + \hat{g}(\xi) \frac{\sin t|\xi|}{|\xi|}$$

and hence we should have

$$\begin{aligned}
u(t,x) &= \mathcal{F}^{-1}\left(\hat{f}(\xi)\cos(t\,|\xi|) + \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|}\right)(x) \\
&= \mathcal{F}^{-1}\cos(t\,|\xi|)\bigstar f(x) + \mathcal{F}^{-1}\frac{\sin t|\xi|}{|\xi|}\bigstar g\,(x) \\
&= \frac{d}{dt}\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]\bigstar f(x) + \mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]\bigstar g\,(x). \qquad (18.26)
\end{aligned}$$

The question now is how interpret this equation. In particular what are the inverse Fourier transforms of $\mathcal{F}^{-1}\cos(t\,|\xi|)$ and $\mathcal{F}^{-1}\frac{\sin t|\xi|}{|\xi|}$. Since $\frac{d}{dt}\mathcal{F}^{-1}\frac{\sin t|\xi|}{|\xi|}\bigstar f(x) = \mathcal{F}^{-1}\cos(t\,|\xi|)\bigstar f(x)$, it really suffices to understand $\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]$. The problem we immediately run into here is that $\frac{\sin t|\xi|}{|\xi|} \in L^2(\mathbb{R}^n)$ iff $n = 1$ so that is the case we should start with.

Again by complex contour integration methods one can show

$$\begin{aligned}
\left(\mathcal{F}^{-1}\xi^{-1}\sin t\xi\right)(x) &= \frac{\pi}{\sqrt{2\pi}}\left(1_{x+t>0} - 1_{(x-t)>0}\right) \\
&= \frac{\pi}{\sqrt{2\pi}}\left(1_{x>-t} - 1_{x>t}\right) = \frac{\pi}{\sqrt{2\pi}}1_{[-t,t]}(x)
\end{aligned}$$

where in writing the last line we have assume that $t \geq 0$. Again this easily seen to be correct because

$$\begin{aligned}
\mathcal{F}\left[\frac{\pi}{\sqrt{2\pi}}1_{[-t,t]}(x)\right](\xi) &= \frac{1}{2}\int_{\mathbb{R}} 1_{[-t,t]}(x)e^{-i\xi\cdot x}dx = \frac{1}{-2i\xi}e^{-i\xi\cdot x}|_{-t}^{t} \\
&= \frac{1}{2i\xi}\left[e^{i\xi t} - e^{-i\xi t}\right] = \xi^{-1}\sin t\xi.
\end{aligned}$$

Therefore,

$$\left(\mathcal{F}^{-1}\xi^{-1}\sin t\xi\right)\bigstar f(x) = \frac{1}{2}\int_{-t}^{t} f(x-y)dy$$

and the solution to the one dimensional wave equation is

$$\begin{aligned}
u(t,x) &= \frac{d}{dt}\frac{1}{2}\int_{-t}^{t} f(x-y)dy + \frac{1}{2}\int_{-t}^{t} g(x-y)dy \\
&= \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{-t}^{t} g(x-y)dy \\
&= \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy.
\end{aligned}$$

We can arrive at this same solution by more elementary means as follows. We first note in the one dimensional case that wave operator factors, namely

$$0 = \left(\partial_t^2 - \partial_x^2\right)u(t,x) = \left(\partial_t - \partial_x\right)\left(\partial_t + \partial_x\right)u(t,x).$$

Let $U(t,x) := \left(\partial_t + \partial_x\right)u(t,x)$, then the wave equation states $\left(\partial_t - \partial_x\right)U = 0$ and hence by the chain rule $\frac{d}{dt}U(t,x-t) = 0$. So

$$U(t,x-t) = U(0,x) = g(x) + f'(x)$$

and replacing x by $x+t$ in this equation shows

$$\left(\partial_t + \partial_x\right)u(t,x) = U(t,x) = g(x+t) + f'(x+t).$$

Working similarly, we learn that

$$\frac{d}{dt}u(t,x+t) = g(x+2t) + f'(x+2t)$$

which upon integration implies

$$\begin{aligned} u(t,x+t) &= u(0,x) + \int_0^t \{g(x+2\tau) + f'(x+2\tau)\}\, d\tau \\ &= f(x) + \int_0^t g(x+2\tau)d\tau + \frac{1}{2}f(x+2\tau)|_0^t \\ &= \frac{1}{2}\left(f(x) + f(x+2t)\right) + \int_0^t g(x+2\tau)d\tau. \end{aligned}$$

Replacing $x \to x - t$ in this equation gives

$$u(t,x) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \int_0^t g(x-t+2\tau)d\tau$$

and then letting $y = x - t + 2\tau$ in the last integral shows again that

$$u(t,x) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

Let us now go to the $n = 3$ case where it turns out that we should interpret $\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]$ as a measure. When $n > 3$ it is necessary to treat $\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]$ as a "distribution" or "generalized function," see Section 24 below. To motivate the next definition suppose that μ is a finite measure on $\mathbb{R}^n$ which is absolutely continuous relative to Lebesgue measure, $d\mu(x) = \rho(x)\mathbf{d}x$. Then it is reasonable to require

$$\hat{\mu}(\xi) := \hat{\rho}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\rho(x)\mathbf{d}x = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}d\mu(x)$$

and

$$(\mu\bigstar g)(x) := \rho\bigstar g(x) = \int_{\mathbb{R}^n} g(x-y)\rho(x)dx = \int_{\mathbb{R}^n} g(x-y)d\mu(y)$$

when $g : \mathbb{R}^n \to \mathbb{C}$ is a function such that the latter integral is defined, for example assume g is bounded. This considerations lead to the following definitions.

Definition 18.17. The Fourier transform, $\hat{\mu}$, of a complex measure μ on $\mathcal{B}_{\mathbb{R}^n}$ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}d\mu(x) \tag{18.27}$$

and the convolution with a function g is defined by

$$(\mu\bigstar g)(x) = \int_{\mathbb{R}^n} g(x-y)d\mu(y)$$

when the integral is defined.

It follows from the dominated convergence theorem that $\hat{\mu}$ is continuous. Also by a variant of Exercise 9.11, if μ and ν are two complex measure on $\mathcal{B}_{\mathbb{R}^n}$ such that $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$. The reader is asked to give another proof of this fact in Exercise 18.3 below.

Example 18.18. Let σ_t be the surface measure on the sphere S_t of radius t centered at zero in $\mathbb{R}^3$. Then

$$\hat{\sigma}_t(\xi) = 4\pi t \frac{\sin t\,|\xi|}{|\xi|}.$$

Indeed,

$$\begin{aligned}
\hat{\sigma}_t(\xi) &= \int_{tS^2} e^{-ix\cdot\xi} d\sigma(x) = t^2 \int_{S^2} e^{-itx\cdot\xi} d\sigma(x) \\
&= t^2 \int_{S^2} e^{-itx_3|\xi|} d\sigma(x) = t^2 \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \sin\phi e^{-it\cos\phi|\xi|} \\
&= 2\pi t^2 \int_{-1}^{1} e^{-itu|\xi|} du = 2\pi t^2 \frac{1}{-it\,|\xi|} e^{-itu|\xi|}|_{u=-1}^{u=1} = 4\pi t^2 \frac{\sin t\,|\xi|}{t\,|\xi|}.
\end{aligned}$$

From this example we should expect

$$\mathcal{F}^{-1}\left[\frac{\sin t\,|\xi|}{|\xi|}\right] = \frac{t}{4\pi t^2}\sigma_t = t\bar{\sigma}_t$$

where $\bar{\sigma}_t$ is $\frac{1}{4\pi t^2}\sigma_t$, the surface measure on S_t normalized to have total measure one. Hence from Eq. (18.26) the solution to the three dimensional wave equation should be given by

$$u(t,x) = \frac{d}{dt}\left(t\bar{\sigma}_t \bigstar f(x)\right) + t\bar{\sigma}_t \bigstar g\,(x)\,. \tag{18.28}$$

Using this definition in Eq. (18.28) gives

$$\begin{aligned}
u(t,x) &= \frac{d}{dt}\left\{t\int_{S_t} f(x-y) d\bar{\sigma}_t(y)\right\} + t\int_{S_t} g(x-y) d\bar{\sigma}_t(y) \\
&= \frac{d}{dt}\left\{t\int_{S_1} f(x-t\omega) d\omega\right\} + t\int_{S_1} g(x-t\omega) d\omega \\
&= \frac{d}{dt}\left\{t\int_{S_1} f(x+t\omega) d\omega\right\} + t\int_{S_1} g(x+t\omega) d\omega \qquad (18.29)
\end{aligned}$$

where $d\omega := d\bar{\sigma}_1(\omega)$. It is possible to verify directly that this formula does solve the wave equation when $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ but we will not pause to do this now.

Rather let us simply point out that solution exhibits a basic property of wave equations, namely finite propagation speed. To exhibit the finite propagation speed, suppose that $f = 0$ (for simplicity) and g has compact support near the origin, for example think of $g = \delta_0(x)$. Then $x + tw = 0$ for some w iff $|x| = t$. Hence the "wave front" propagates at unit speed and the wave front is sharp. See Figure 36 below.

The solution of the two dimensional wave equation may be found using "Hadamard's method of decent" which we now describe. Suppose now that f and g are functions on $\mathbb{R}^2$ which we may view as functions on $\mathbb{R}^3$ which happen not to depend on the third coordinate. We now go ahead and solve the three dimensional wave equation using Eq. (18.29) and f and g as initial conditions. It is easily seen that the solution $u(t,x,y,z)$ is again independent of z and hence is a solution to the two dimensional wave equation. See figure 37 below.

Notice that we still have finite speed of propagation but no longer sharp propagation. In fact we can work out the solution analytically as follows. Again for

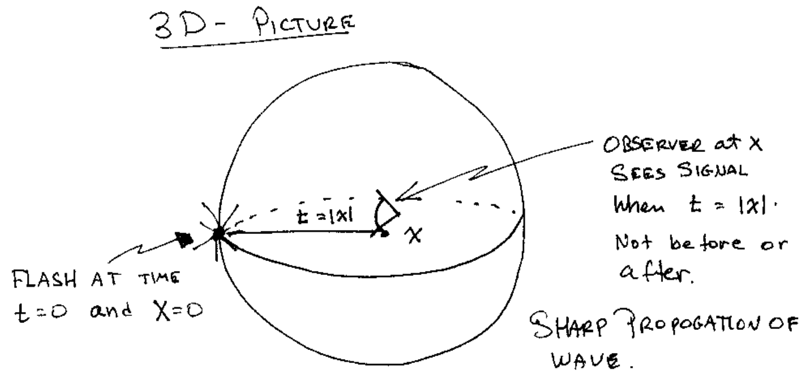

FIGURE 36. The geometry of the solution to the wave equation in three dimensions.

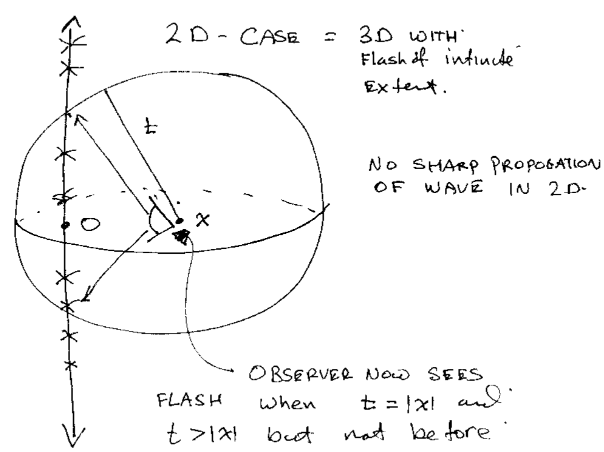

FIGURE 37. The geometry of the solution to the wave equation in two dimensions.

simplicity assume that $f \equiv 0$. Then

$$\begin{aligned} u(t,x,y) &= \frac{t}{4\pi}\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \sin\phi g((x,y)+t(\sin\phi\cos\theta,\sin\phi\sin\theta)) \\ &= \frac{t}{2\pi}\int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \sin\phi g((x,y)+t(\sin\phi\cos\theta,\sin\phi\sin\theta)) \end{aligned}$$

and letting $u=\sin\phi$, so that $du=\cos\phi d\phi=\sqrt{1-u^2}d\phi$ we find

$$u(t,x,y)=\frac{t}{2\pi}\int_0^{2\pi} d\theta\int_0^1 \frac{du}{\sqrt{1-u^2}} ug((x,y)+ut(\cos\theta,\sin\theta))$$

and then letting $r = ut$ we learn,

$$\begin{aligned} u(t,x,y) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^t \frac{dr}{\sqrt{1-r^2/t^2}}\frac{r}{t} g((x,y)+r(\cos\theta,\sin\theta)) \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^t \frac{dr}{\sqrt{t^2-r^2}} r g((x,y)+r(\cos\theta,\sin\theta)) \\ &= \frac{1}{2\pi}\iint_{D_t} \frac{g((x,y)+w))}{\sqrt{t^2-|w|^2}} dm(w). \end{aligned}$$

Here is a better alternative derivation of this result. We begin by using symmetry to find

$$u(t,x) = 2t\int_{S_t^+} g(x-y)d\bar{\sigma}_t(y) = 2t\int_{S_t^+} g(x+y)d\bar{\sigma}_t(y)$$

where S_t^+ is the portion of S_t with $z \ge 0$. This sphere is parametrized by $R(u,v) = (u,v,\sqrt{t^2-u^2-v^2})$ with $(u,v) \in D_t := \left\{(u,v) : u^2+v^2 \le t^2\right\}$. In these coordinates we have

$$\begin{aligned} 4\pi t^2 d\bar{\sigma}_t &= \left|\left(-\partial_u\sqrt{t^2-u^2-v^2}, -\partial_v\sqrt{t^2-u^2-v^2}, 1\right)\right| dudv \\ &= \left|\left(\frac{u}{\sqrt{t^2-u^2-v^2}}, \frac{v}{\sqrt{t^2-u^2-v^2}}, 1\right)\right| dudv \\ &= \sqrt{\frac{u^2+v^2}{t^2-u^2-v^2}+1} dudv = \frac{|t|}{\sqrt{t^2-u^2-v^2}} dudv \end{aligned}$$

and therefore,

$$\begin{aligned} u(t,x) &= \frac{2t}{4\pi t^2}\int_{S_t^+} g(x+(u,v,\sqrt{t^2-u^2-v^2}))\frac{|t|}{\sqrt{t^2-u^2-v^2}} dudv \\ &= \frac{1}{2\pi}\operatorname{sgn}(t)\int_{S_t^+} \frac{g(x+(u,v))}{\sqrt{t^2-u^2-v^2}} dudv. \end{aligned}$$

This may be written as

$$u(t,x) = \frac{1}{2\pi}\operatorname{sgn}(t)\iint_{D_t} \frac{g((x,y)+w))}{\sqrt{t^2-|w|^2}} dm(w)$$

as before. (I should check on the sgn(t) term.)

18.5. **Bochner's Theorem.** Let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^n}$ and $\hat{\mu}(\xi)$ be the Fourier transform of μ as in Definition 18.17 above. If $\lambda \in \mathbb{C}^m$, then

$$\begin{aligned} \sum_{k,j=1}^m \hat{\mu}(\xi_k-\xi_j)\lambda_k\bar{\lambda}_j &= \int_{\mathbb{R}^n}\sum_{k,j=1}^m e^{-i(\xi_k-\xi_j)\cdot x}\lambda_k\bar{\lambda}_j d\mu(x) = \int_{\mathbb{R}^n}\sum_{k,j=1}^m e^{-i\xi_k\cdot x}\lambda_k\overline{e^{-i\xi_j\cdot x}\lambda_j} d\mu(x) \\ &= \int_{\mathbb{R}^n}\left|\sum_{k=1}^m e^{-i\xi_k\cdot x}\lambda_k\right|^2 d\mu(x) \ge 0 \end{aligned}$$

that is to say $\hat{\mu}$ is a positive definite function. Since $\hat{\mu}(0)\left|\lambda\right|^2 \ge 0$, it follows that $\hat{\mu}(0) \ge 0$ and since

$$A := \begin{bmatrix} \hat{\mu}(0) & \hat{\mu}(\xi-\eta) \\ \hat{\mu}(\eta-\xi) & \hat{\mu}(0) \end{bmatrix}$$

is positive definite of all $\xi, \eta \in \mathbb{R}^d$ it follows that $\hat{\mu}(-\eta) = \overline{\hat{\mu}(\eta)}$ (since $A^* = A$) and

$$0 \le \det \begin{bmatrix} \hat{\mu}(0) & \hat{\mu}(\xi-\eta) \\ \hat{\mu}(\eta-\xi) & \hat{\mu}(0) \end{bmatrix} = |\hat{\mu}(0)|^2 - |\hat{\mu}(\xi-\eta)|^2$$

and hence $|\hat{\mu}(\xi)| \le \hat{\mu}(0)$ for all ξ. In particular $\hat{\mu}$ is continuous. Moreover, if $f \in \mathbb{S}(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} \hat{\mu}(\xi-\eta) f(\xi) \overline{f(\eta)} d\xi d\eta = \lim_{mesh \to 0} \sum \hat{\mu}(\xi_k - \xi_j) f(\xi_j) \overline{f(\xi_k)} \ge 0.$$

Theorem 18.19. *Suppose $\chi \in C(\mathbb{R}^n, \mathbb{C})$ is positive definite function, then there exists a unique measure μ on $\mathcal{B}_{\mathbb{R}^n}$ such that $\chi = \hat{\mu}$.*

Proof. For $f \in \mathcal{S}$, define

$$I(f) := \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi) \mathbf{d}\xi.$$

Notice that

$$\begin{aligned} I(|f|^2) &= \int_{\mathbb{R}^n} \chi(\xi) \left(|f|^2\right)^\vee (\xi) \mathbf{d}\xi = \int_{\mathbb{R}^n} \chi(\xi) \left(f^\vee \bigstar \bar{f}^\vee\right)(\xi) \mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi-\eta) \bar{f}^\vee(\eta) \mathbf{d}\eta \mathbf{d}\xi = \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi-\eta) \overline{f^\vee(-\eta)} \mathbf{d}\eta \mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} \chi(\xi-\eta) f^\vee(\xi) \overline{f^\vee(\eta)} \mathbf{d}\eta \mathbf{d}\xi \ge 0 \end{aligned}$$

for all $f \in \mathcal{S}$. Let $p_t(x - \cdot)$ be the usual heat kernel, then $\sqrt{p_t(x-\cdot)}$ is still a Gaussian function and hence $\sqrt{p_t(x-\cdot)} \in \mathcal{S}$. Therefore,

$$\langle I, p_t(x-\cdot)\rangle = \langle I, \left|\sqrt{p_t(x-\cdot)}\right|^2\rangle \ge 0 \text{ for all } x \in \mathbb{R} \text{ and } t > 0.$$

Hence we have shown, $I \bigstar p_t \ge 0$ for all t.

We will now show[37] for $\psi \in \mathcal{S}$ that $\langle I \bigstar p_t, \psi\rangle \to \langle I, \psi\rangle$ as $t \downarrow 0$. Since

$$\begin{aligned} I \bigstar p_t(x) = \langle I, p_t(x-\cdot)\rangle &= \int_{\mathbb{R}^n} \left(e^{i\xi A} v, v\right) \widehat{p_t(x-\cdot)}(\xi) \mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} \left(e^{i\xi A} v, v\right) e^{-ix\cdot\xi} \hat{p}_t(\xi) \mathbf{d}\xi, \end{aligned}$$

$$\begin{aligned} \langle I \bigstar p_t, \psi\rangle &= \int_{\mathbb{R}^n} \mathbf{d}x \psi(x) \int_{\mathbb{R}^n} \mathbf{d}\xi \left(e^{i\xi A} v, v\right) e^{-ix\cdot\xi} \hat{p}_t(\xi) \\ &= \int_{\mathbb{R}^n} \mathbf{d}\xi \int_{\mathbb{R}^n} \mathbf{d}x \psi(x) \left(e^{i\xi A} v, v\right) e^{-ix\cdot\xi} \hat{p}_t(\xi) \\ &= \int_{\mathbb{R}^n} \mathbf{d}\xi \hat{\psi}(\xi) \left(e^{i\xi A} v, v\right) \hat{p}_t(\xi) \to \int_{\mathbb{R}^n} \mathbf{d}\xi \hat{\psi}(\xi) \left(e^{i\xi A} v, v\right) = \langle I, \psi\rangle \end{aligned}$$

since $\hat{p}_t(\xi) = e^{-t|\xi|^2/2} \to 1$ as $t \downarrow 0$. Hence if $\psi \ge 0$, we find

$$\langle I, \psi\rangle = \lim_{t \downarrow 0} \langle I \bigstar p_t, \psi\rangle \ge 0.$$

[37]It is known more generally that if $T \in \mathbb{S}'$ then $T * p_t \to T$ in $\mathcal{S}'$ as $t \downarrow 0$.

Let $K \subset \mathbb{R}$ be a compact set and $\psi \in C_c(\mathbb{R},[0,\infty))$ be a function such that $\psi = 1$ on K. If $f \in C_c^\infty(\mathbb{R},\mathbb{R})$ is a smooth function with $\mathrm{supp}(f) \subset K$, then $0 \le \|f\|_\infty \psi - f \in \mathcal{S}$ and hence

$$0 \le \langle I, \|f\|_\infty \psi - f\rangle = \|f\|_\infty \langle I,\psi\rangle - \langle I,f\rangle$$

and therefore $\langle I,f\rangle \le \|f\|_\infty \langle I,\psi\rangle$. Replacing f by $-f$ implies, $-\langle I,f\rangle \le \|f\|_\infty \langle I,\psi\rangle$ and hence we have proved

$$|\langle I,f\rangle| \le C(\mathrm{supp}(f))\,\|f\|_\infty \tag{18.30}$$

for all $f \in \mathcal{D}_{\mathbb{R}^n} := C_c^\infty(\mathbb{R}^n,\mathbb{R})$ where $C(K)$ is a finite constant for each compact subset of $\mathbb{R}^n$. Because of the estimate in Eq. (18.30), it follows that $I|_{\mathcal{D}_{\mathbb{R}^n}}$ has a unique extension I to $C_c(\mathbb{R}^n,\mathbb{R})$ still satisfying the estimates in Eq. (18.30) and moreover this extension is still positive. So by the Riesz – Markov theorem, there exists a unique Radon – measure μ on $\mathbb{R}^n$ such that such that $\langle I,f\rangle = \mu(f)$ for all $f \in C_c(\mathbb{R}^n,\mathbb{R})$.

To finish the proof we must show $\hat{\mu}(\eta) = \chi(\eta)$ for all $\eta \in \mathbb{R}^n$ given

$$\mu(f) = \int_{\mathbb{R}^n} \chi(\xi) f^\vee(\xi)\mathbf{d}\xi \text{ for all } f \in C_c^\infty(\mathbb{R}^n,\mathbb{R}).$$

Let $f \in C_c^\infty(\mathbb{R}^n,\mathbb{R}_+)$ be a radial function such $f(0)=1$ and $f(x)$ is decreasing as $|x|$ increases. Let $f_\epsilon(x) := f(\epsilon x)$, then by Theorem 18.3,

$$\mathcal{F}^{-1}\left[e^{-i\eta x} f_\epsilon(x)\right](\xi) = \epsilon^{-n} f^\vee(\frac{\xi-\eta}{\epsilon})$$

and therefore

$$\mu(e^{-i\eta x} f_\epsilon(x)) = \int_{\mathbb{R}^n} \chi(\xi)\epsilon^{-n} f^\vee(\frac{\xi-\eta}{\epsilon})\mathbf{d}\xi. \tag{18.31}$$

Because $\int_{\mathbb{R}^n} f^\vee(\xi)\mathbf{d}\xi = \mathcal{F}f^\vee(0) = f(0) = 1$, we may apply the approximate δ – function Theorem 9.20 to Eq. (18.31) to find

$$\mu(e^{-i\eta x} f_\epsilon(x)) \to \chi(\eta) \text{ as } \epsilon \downarrow 0. \tag{18.32}$$

On the the other hand, when $\eta = 0$, the monotone convergence theorem implies $\mu(f_\epsilon) \uparrow \mu(1) = \mu(\mathbb{R}^n)$ and therefore $\mu(\mathbb{R}^n) = \mu(1) = \chi(0) < \infty$. Now knowing the μ is a finite measure we may use the dominated convergence theorem to concluded

$$\mu(e^{-i\eta x} f_\epsilon(x)) \to \mu(e^{-i\eta x}) = \hat{\mu}(\eta) \text{ as } \epsilon \downarrow 0$$

for all η. Combining this equation with Eq. (18.32) shows $\hat{\mu}(\eta) = \chi(\eta)$ for all $\eta \in \mathbb{R}^n$. ■

18.6. **Supplement: Heisenberg Uncertainty Principle Exercise.** Suppose that H is a Hilbert space and A, B are two densely defined symmetric operators on H. More explicitly, A is a densely defined symmetric linear operator on H means there is a dense subspace $\mathcal{D}_A \subset H$ and a linear map $A : \mathcal{D}_A \to H$ such that $(A\phi,\psi) = (\phi, A\psi)$ for all $\phi,\psi \in \mathcal{D}_A$. Let $\mathcal{D}_{AB} := \{\phi \in H : \phi \in \mathcal{D}_B \text{ and } B\phi \in \mathcal{D}_A\}$ and for $\phi \in \mathcal{D}_{AB}$ let $(AB)\phi = A(B\phi)$ with a similar definition of $\mathcal{D}_{BA}$ and BA. Moreover, let $\mathcal{D}_C := \mathcal{D}_{AB} \cap \mathcal{D}_{BA}$ and for $\phi \in \mathcal{D}_C$, let

$$C\phi = \frac{1}{i}[A,B]\phi = \frac{1}{i}(AB - BA)\phi.$$

Notice that for $\phi, \psi \in \mathcal{D}_C$ we have

$$\begin{aligned}(C\phi, \psi) &= \frac{1}{i}\{(AB\phi, \psi) - (BA\phi, \psi)\} = \frac{1}{i}\{(B\phi, A\psi) - (A\phi, B\psi)\} \\ &= \frac{1}{i}\{(\phi, BA\psi) - (\phi, AB\psi)\} = (\phi, C\psi),\end{aligned}$$

so that C is symmetric as well.

Theorem 18.20 (Heisenberg Uncertainty Principle)**.** *Continue the above notation and assumptions,*

$$\frac{1}{2}|(\psi, C\psi)| \le \sqrt{\|A\psi\|^2 - (\psi, A\psi)} \cdot \sqrt{\|B\psi\|^2 - (\psi, B\psi)} \tag{18.33}$$

for all $\psi \in \mathcal{D}_C$. Moreover if $\|\psi\| = 1$ and equality holds in Eq. (18.33), then

$$\begin{aligned}&(A - (\psi, A\psi))\psi = i\lambda(B - (\psi, B\psi))\psi \text{ or} \\ &(B - (\psi, B\psi)) = i\lambda\psi(A - (\psi, A\psi))\psi \end{aligned} \tag{18.34}$$

for some $\lambda \in \mathbb{R}$.

Proof. By homogeneity (18.33) we may assume that $\|\psi\| = 1$. Let $a := (\psi, A\psi)$, $b = (\psi, B\psi)$, $\tilde{A} = A - aI$, and $\tilde{B} = B - bI$. Then we have still have

$$[\tilde{A}, \tilde{B}] = [A - aI, B - bI] = iC.$$

Now

$$\begin{aligned}i(\psi, C\psi) &= (\psi, iC\psi) = (\psi, [\tilde{A}, \tilde{B}]\psi) = (\psi, \tilde{A}\tilde{B}\psi) - (\psi, \tilde{B}\tilde{A}\psi) \\ &= (\tilde{A}\psi, \tilde{B}\psi) - (\tilde{B}\psi, \tilde{A}\psi) = 2i\,\mathrm{Im}(\tilde{A}\psi, \tilde{B}\psi)\end{aligned}$$

from which we learn

$$|(\psi, C\psi)| = 2\left|\mathrm{Im}(\tilde{A}\psi, \tilde{B}\psi)\right| \le 2\left|(\tilde{A}\psi, \tilde{B}\psi)\right| \le 2\left\|\tilde{A}\psi\right\|\left\|\tilde{B}\psi\right\|$$

with equality iff $\mathrm{Re}(\tilde{A}\psi, \tilde{B}\psi) = 0$ and $\tilde{A}\psi$ and $\tilde{B}\psi$ are linearly dependent, i.e. iff Eq. (18.34) holds.

The result follows from this equality and the identities

$$\begin{aligned}\left\|\tilde{A}\psi\right\|^2 &= \|A\psi - a\psi\|^2 = \|A\psi\|^2 + a^2\|\psi\|^2 - 2a\,\mathrm{Re}(A\psi, \psi) \\ &= \|A\psi\|^2 + a^2 - 2a^2 = \|A\psi\|^2 - (A\psi, \psi)\end{aligned}$$

and

$$\left\|\tilde{B}\psi\right\| = \|B\psi\|^2 - (B\psi, \psi).$$

■

Example 18.21. As an example, take $H = L^2(\mathbb{R})$, $A = \frac{1}{i}\partial_x$ and $B = M_x$ with $\mathcal{D}_A := \{f \in H : f' \in H\}$ (f' is the weak derivative) and $\mathcal{D}_B := \left\{f \in H : \int_{\mathbb{R}} |xf(x)|^2\, dx < \infty\right\}$. In this case,

$$\mathcal{D}_C = \{f \in H : f', xf \text{ and } xf' \text{ are in } H\}$$

and $C = -I$ on $\mathcal{D}_C$. Therefore for a **unit** vector $\psi \in \mathcal{D}_C$,

$$\frac{1}{2} \le \left\|\frac{1}{i}\psi' - a\psi\right\|_2 \cdot \|x\psi - b\psi\|_2$$

where $a = i\int_{\mathbb{R}} \psi\bar{\psi}' dm$ [38] and $b = \int_{\mathbb{R}} x\,|\psi(x)|^2\,dm(x)$. Thus we have

$$\frac{1}{4} = \frac{1}{4}\int_{\mathbb{R}} |\psi|^2\,dm \le \int_{\mathbb{R}} (k-a)^2 \left|\hat{\psi}(k)\right|^2 dk \cdot \int_{\mathbb{R}} (x-b)^2\,|\psi(x)|^2\,dx. \tag{18.35}$$

Equality occurs if there exists $\lambda \in \mathbb{R}$ such that

$$i\lambda\,(x-b)\,\psi(x) = (\frac{1}{i}\partial_x - a)\psi(x) \text{ a.e.}$$

Working formally, this gives rise to the ordinary differential equation (in weak form),

$$\psi_x = [-\lambda(x-b) + ia]\,\psi \tag{18.36}$$

which has solutions (see Exercise 18.5 below)

$$\psi = C\exp\left(\int_{\mathbb{R}} [-\lambda(x-b) + ia]\,dx\right) = C\exp\left(-\frac{\lambda}{2}(x-b)^2 + iax\right). \tag{18.37}$$

Let $\lambda = \frac{1}{2t}$ and choose C so that $\|\psi\|_2 = 1$ to find

$$\psi_{t,a,b}(x) = \left(\frac{1}{2t}\right)^{1/4} \exp\left(-\frac{1}{4t}(x-b)^2 + iax\right)$$

are the functions which saturate the Heisenberg uncertainty principle in Eq. (18.35).

18.6.1. *Exercises.*

Exercise 18.2. Let $f \in L^2(\mathbb{R}^n)$ and α be a multi-index. If $\partial^\alpha f$ exists in $L^2(\mathbb{R}^n)$ then $\mathcal{F}(\partial^\alpha f) = (i\xi)^\alpha\,\hat{f}(\xi)$ in $L^2(\mathbb{R}^n)$ and conversely if $\left(\xi \to \xi^\alpha \hat{f}(\xi)\right) \in L^2(\mathbb{R}^n)$ then $\partial^\alpha f$ exists.

Exercise 18.3. Suppose μ is a complex measure on $\mathbb{R}^n$ and $\hat{\mu}(\xi)$ is its Fourier transform as defined in Definition 18.17. Show μ satisfies,

$$\langle \hat{\mu}, \phi\rangle := \int_{\mathbb{R}^n} \hat{\mu}(\xi)\phi(\xi)d\xi = \mu(\hat{\phi}) := \int_{\mathbb{R}^n} \hat{\phi}\,d\mu \text{ for all } \phi \in \mathcal{S}$$

and use this to show if μ is a complex measure such that $\hat{\mu} \equiv 0$, then $\mu \equiv 0$.

Exercise 18.4. Using

$$\frac{1}{|\xi|^2 + m^2} = \int_0^\infty e^{-\lambda(|\xi|^2 + m^2)} d\lambda$$

the identity in Eq. (18.19) and Example 18.4, show for $m > 0$ and $x \ge 0$ that

$$e^{-mx} = \frac{m}{\sqrt{\pi}} \int_0^\infty d\lambda \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{4\lambda}x^2} e^{-\lambda m^2} \text{ (let } \lambda \to \lambda/m^2)$$

$$= \int_0^\infty d\lambda \frac{1}{\sqrt{\pi\lambda}} e^{-\lambda} e^{-\frac{m^2}{4\lambda}x^2}. \tag{18.38}$$

Use this to formula and and Example 18.4 again to show, in dimension n, that

$$\mathcal{F}\left[e^{-m|x|}\right](\xi) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi}2^{n/2}} \frac{m}{(m^2 + |\xi|^2)^{(n+1)/2}}$$

where $\Gamma(x)$ in the gamma function defined in Eq. (6.30). (I am not absolutely positive I have got all the constants exactly right, but they should be close.)

[38] We will see in later that a may be described using the Fourier transform as: $a = \int k\left|\hat{\psi}(k)\right|^2 dm(k)$.

Exercise 18.5. Show that ψ described in Eq. (18.37) is the general solution to Eq. (18.36). **Hint:** Suppose that ϕ is any solution to Eq. (18.36) and ψ is given as in Eq. (18.37) with $C = 1$. Consider the weak – differential equation solved by ϕ/ψ.

18.6.2. *More Proofs of the Fourier Inversion Theorem.*

Exercise 18.6. Suppose that $f \in L^1(\mathbb{R})$ and assume that f continuously differentiable in a neighborhood of 0, show

$$\lim_{M\to\infty}\int_{-\infty}^{\infty}\frac{\sin Mx}{x}f(x)dx = \pi f(0) \tag{18.39}$$

using the following steps.

(1) Use Example 6.26 to deduce,
$$\lim_{M\to\infty}\int_{-1}^{1}\frac{\sin Mx}{x}dx = \lim_{M\to\infty}\int_{-M}^{M}\frac{\sin x}{x}dx = \pi.$$
(2) Explain why
$$0 = \lim_{M\to\infty}\int_{|x|\geq 1}\sin Mx\cdot\frac{f(x)}{x}dx \text{ and}$$
$$0 = \lim_{M\to\infty}\int_{|x|\leq 1}\sin Mx\cdot\frac{f(x)-f(0)}{x}dx.$$
(3) Add the previous two equations and use part (1) to prove Eq. (18.39).

Exercise 18.7 (Fourier Inversion Formula)**.** Suppose that $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$.

(1) Further assume that f is continuously differentiable in a neighborhood of 0. Show that
$$\Lambda := \int_{\mathbb{R}}\hat{f}(\xi)\mathbf{d}\xi = f(0).$$
Hint: by the dominated convergence theorem, $\Lambda := \lim_{M\to\infty}\int_{|\xi|\leq M}\hat{f}(\xi)\mathbf{d}\xi$. Now use the definition of $\hat{f}(\xi)$, Fubini's theorem and Exercise 18.6.
(2) Apply part 1. of this exercise with f replace by $\tau_y f$ for some $y \in \mathbb{R}$ to prove
$$f(y) = \int_{\mathbb{R}}\hat{f}(\xi)e^{iy\cdot\xi}\mathbf{d}\xi \tag{18.40}$$
provided f is now continuously differentiable near y.

The goal of the next exercises is to give yet another proof of the Fourier inversion formula.

Notation 18.22. For $L > 0$, let $C_L^k(\mathbb{R})$ denote the space of C^k – $2\pi L$ periodic functions:
$$C_L^k(\mathbb{R}) := \left\{f \in C^k(\mathbb{R}) : f(x+2\pi L) = f(x) \text{ for all } x \in \mathbb{R}\right\}.$$
Also let $\langle\cdot,\cdot\rangle_L$ denote the inner product on the Hilbert space $H_L := L^2([-\pi L, \pi L])$ given by
$$(f,g)_L := \frac{1}{2\pi L}\int_{[-\pi L,\pi L]}f(x)\bar{g}(x)dx.$$

Exercise 18.8. Recall that $\left\{\chi_k^L(x) := e^{ikx/L} : k \in \mathbb{Z}\right\}$ is an orthonormal basis for H_L and in particular for $f \in H_L$,

$$f = \sum_{k\in\mathbb{Z}} \langle f, \chi_k^L\rangle_L \chi_k^L \tag{18.41}$$

where the convergence takes place in $L^2([-\pi L, \pi L])$. Suppose now that $f \in C_L^2(\mathbb{R})$[39]. Show (by two integration by parts)

$$\left|(f_L, \chi_k^L)_L\right| \le \frac{L^2}{k^2} \left\|f''\right\|_u$$

where $\|g\|_u$ denote the uniform norm of a function g. Use this to conclude that the sum in Eq. (18.41) is uniformly convergent and from this conclude that Eq. (18.41) holds pointwise.

Exercise 18.9 (Fourier Inversion Formula on $\mathcal{S}$)**.** Let $f \in \mathcal{S}(\mathbb{R})$, $L > 0$ and

$$f_L(x) := \sum_{k\in\mathbb{Z}} f(x + 2\pi k L). \tag{18.42}$$

Show:

1. The sum defining f_L is convergent and moreover that $f_L \in C_L^\infty(\mathbb{R})$.
2. Show $(f_L, \chi_k^L)_L = \frac{1}{\sqrt{2\pi}L}\hat{f}(k/L)$.
3. Conclude from Exercise 18.8 that

$$f_L(x) = \frac{1}{\sqrt{2\pi}L}\sum_{k\in\mathbb{Z}} \hat{f}(k/L)e^{ikx/L} \text{ for all } x \in \mathbb{R}. \tag{18.43}$$

4. Show, by passing to the limit, $L \to \infty$, in Eq. (18.43) that Eq. (18.40) holds for all $x \in \mathbb{R}$. **Hint:** Recall that $\hat{f} \in \mathcal{S}$.

Exercise 18.10. Folland 8.13 on p. 254.

Exercise 18.11. Folland 8.14 on p. 254. (Wirtinger's inequality.)

Exercise 18.12. Folland 8.15 on p. 255. (The sampling Theorem. Modify to agree with notation in notes, see Solution F.19 below.)

Exercise 18.13. Folland 8.16 on p. 255.

Exercise 18.14. Folland 8.17 on p. 255.

Exercise 18.15. .Folland 8.19 on p. 256. (The Fourier transform of a function whose support has finite measure.)

Exercise 18.16. Folland 8.22 on p. 256. (Bessel functions.)

Exercise 18.17. Folland 8.23 on p. 256. (Hermite Polynomial problems and Harmonic oscillators.)

Exercise 18.18. Folland 8.31 on p. 263. (Poisson Summation formula problem.)

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[39] We view $C_L^2(\mathbb{R})$ as a subspace of H_L by identifying $f \in C_L^2(\mathbb{R})$ with $f|_{[-\pi L, \pi L]} \in H_L$.