

4. ALGEBRAS, σ - ALGEBRAS AND MEASURABILITY

4.1. Introduction: What are measures and why “measurable” sets.

Definition 4.1 (Preliminary). Suppose that X is a set and $\mathcal{P}(X)$ denotes the collection of all subsets of X . A measure μ on X is a function $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. If $\{A_i\}_{i=1}^N$ is a finite ($N < \infty$) or countable ($N = \infty$) collection of subsets of X which are pair-wise disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$\mu(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mu(A_i).$$

Example 4.2. Suppose that X is any set and $x \in X$ is a point. For $A \subset X$, let

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu = \delta_x$ is a measure on X called the Dirac delta function at x .

Example 4.3. Suppose that μ is a measure on X and $\lambda > 0$, then $\lambda \cdot \mu$ is also a measure on X . Moreover, if $\{\mu_\alpha\}_{\alpha \in J}$ are all measures on X , then $\mu = \sum_{\alpha \in J} \mu_\alpha$, i.e.

$$\mu(A) = \sum_{\alpha \in J} \mu_\alpha(A) \text{ for all } A \subset X$$

is a measure on X . (See Section 2 for the meaning of this sum.) To prove this we must show that μ is countably additive. Suppose that $\{A_i\}_{i=1}^\infty$ is a collection of pair-wise disjoint subsets of X , then

$$\begin{aligned} \mu(\cup_{i=1}^\infty A_i) &= \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \sum_{\alpha \in J} \mu_\alpha(A_i) \\ &= \sum_{\alpha \in J} \sum_{i=1}^\infty \mu_\alpha(A_i) = \sum_{\alpha \in J} \mu_\alpha(\cup_{i=1}^\infty A_i) \\ &= \mu(\cup_{i=1}^\infty A_i) \end{aligned}$$

wherein the third equality we used Theorem 2.21 and in the fourth we used that fact that μ_α is a measure.

Example 4.4. Suppose that X is a set $\lambda : X \rightarrow [0, \infty]$ is a function. Then

$$\mu := \sum_{x \in X} \lambda(x) \delta_x$$

is a measure, explicitly

$$\mu(A) = \sum_{x \in A} \lambda(x)$$

for all $A \subset X$.

4.2. The problem with Lebesgue “measure”.

Question 1. Does there exist a measure $\mu : \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$ such that

1. $\mu([a, b)) = (b - a)$ for all $a < b$ and
2. (Translation invariant) $\mu(A + x) = \mu(A)$ for all $x \in \mathbb{R}$? (Here $A + x := \{y + x : y \in A\} \subset \mathbb{R}$.)

The answer is no which we now demonstrate. In fact the answer is no even if we replace (1) by the condition that $0 < \mu((0, 1]) < \infty$.

Let us identify $[0, 1)$ with the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ by the map $\phi(t) = e^{i2\pi t} \in S^1$ for $t \in [0, 1)$. Using this identification we may use μ to define a function ν on $\mathcal{P}(S^1)$ by $\nu(\phi(A)) = \mu(A)$ for all $A \subset [0, 1)$. This new function is a measure on S^1 with the property that $0 < \nu((0, 1]) < \infty$. For $z \in S^1$ and $N \subset S^1$ let

$$(4.1) \quad zN := \{zn \in S^1 : n \in N\},$$

that is to say $e^{i\theta}N$ is N rotated counter clockwise by angle θ . We now claim that ν is invariant under these rotations, i.e.

$$(4.2) \quad \nu(zN) = \nu(N)$$

for all $z \in S^1$ and $N \subset S^1$. To verify this, write $N = \phi(A)$ and $z = \phi(t)$ for some $t \in [0, 1)$ and $A \subset [0, 1)$. Then

$$\phi(t)\phi(A) = \phi(t + A \bmod 1)$$

where for $A \subset [0, 1)$ and $\alpha \in [0, 1)$, let

$$\begin{aligned} t + A \bmod 1 &= \{a + t \bmod 1 \in [0, 1) : a \in A\} \\ &= (a + A \cap \{a < 1 - t\}) \cup ((t - 1) + A \cap \{a \geq 1 - t\}). \end{aligned}$$

Thus

$$\begin{aligned} \nu(\phi(t)\phi(A)) &= \mu(t + A \bmod 1) \\ &= \mu((a + A \cap \{a < 1 - t\}) \cup ((t - 1) + A \cap \{a \geq 1 - t\})) \\ &= \mu((a + A \cap \{a < 1 - t\})) + \mu(((t - 1) + A \cap \{a \geq 1 - t\})) \\ &= \mu(A \cap \{a < 1 - t\}) + \mu(A \cap \{a \geq 1 - t\}) \\ &= \mu((A \cap \{a < 1 - t\}) \cup (A \cap \{a \geq 1 - t\})) \\ &= \mu(A) = \nu(\phi(A)). \end{aligned}$$

Therefore it suffices to prove that no finite measure ν on S^1 such that Eq. (4.2) holds. To do this we will “construct” a non-measurable set $N = \phi(A)$ for some $A \subset [0, 1)$.

To do this let

$$R := \{z = e^{i2\pi t} : t \in \mathbb{Q}\} = \{z = e^{i2\pi t} : t \in [0, 1) \cap \mathbb{Q}\},$$

a countable subgroup of S^1 . As above R acts on S^1 by rotations and divides S^1 up into equivalence classes, where $z, w \in S^1$ are equivalent if $z = rw$ for some $r \in R$. Choose (using the axiom of choice) one representative point n from each of these equivalence classes and let $N \subset S^1$ be the set of these representative points. Then

every point $z \in S^1$ may be uniquely written as $z = nr$ with $n \in N$ and $r \in R$. That is to say

$$(4.3) \quad S^1 = \coprod_{r \in R} (rN)$$

where $\coprod_{\alpha} A_{\alpha}$ is used to denote the union of pair-wise disjoint sets $\{A_{\alpha}\}$. By Eqs. (4.2) and (4.3),

$$\nu(S^1) = \sum_{r \in R} \nu(rN) = \sum_{r \in R} \nu(N).$$

The right member from this equation is either 0 or ∞ , 0 if $\nu(N) = 0$ and ∞ if $\nu(N) > 0$. In either case it is not equal $\nu(S^1) \in (0, 1)$. Thus we have reached the desired contradiction.

Proof. (Second proof of Answer to Question 1) For $N \subset [0, 1)$ and $\alpha \in [0, 1)$, let

$$\begin{aligned} N^{\alpha} &= N + \alpha \bmod 1 \\ &= \{a + \alpha \bmod 1 \in [0, 1) : a \in N\} \\ &= (\alpha + N \cap \{a < 1 - \alpha\}) \cup ((\alpha - 1) + N \cap \{a \geq 1 - \alpha\}). \end{aligned}$$

If μ is a measure satisfying the properties of the Question we would have

$$\begin{aligned} \mu(N^{\alpha}) &= \mu(\alpha + N \cap \{a < 1 - \alpha\}) + \mu((\alpha - 1) + N \cap \{a \geq 1 - \alpha\}) \\ &= \mu(N \cap \{a < 1 - \alpha\}) + \mu(N \cap \{a \geq 1 - \alpha\}) \\ &= \mu(N \cap \{a < 1 - \alpha\} \cup (N \cap \{a \geq 1 - \alpha\})) \\ (4.4) \quad &= \mu(N). \end{aligned}$$

We will now construct a bad set N which coupled with Eq. (4.4) will lead to a contradiction.

Set

$$Q_x \equiv \{x + r \in \mathbb{R} : r \in \mathbb{Q}\} = x + \mathbb{Q}.$$

Notice that $Q_x \cap Q_y \neq \emptyset$ implies that $Q_x = Q_y$. Let $\mathcal{O} = \{Q_x : x \in \mathbb{R}\}$ – the orbit space of the \mathbb{Q} action. For all $A \in \mathcal{O}$ choose $f(A) \in [0, 1/3) \cap A$.⁹ Define $N = f(\mathcal{O})$. Then observe:

1. $f(A) = f(B)$ implies that $A \cap B \neq \emptyset$ which implies that $A = B$ so that f is injective.
2. $\mathcal{O} = \{Q_n : n \in N\}$.

Let R be the countable set,

$$R \equiv \mathbb{Q} \cap [0, 1).$$

We now claim that

$$(4.5) \quad N^r \cap N^s = \emptyset \text{ if } r \neq s \text{ and}$$

$$(4.6) \quad [0, 1) = \cup_{r \in R} N^r.$$

Indeed, if $x \in N^r \cap N^s \neq \emptyset$ then $x = r + n \bmod 1$ and $x = s + n' \bmod 1$, then $n - n' \in \mathbb{Q}$, i.e. $Q_n = Q_{n'}$. That is to say, $n = f(Q_n) = f(Q_{n'}) = n'$ and hence that $s = r \bmod 1$, but $s, r \in [0, 1)$ implies that $s = r$. Furthermore, if $x \in [0, 1)$ and $n := f(Q_x)$, then $x - n = r \in \mathbb{Q}$ and $x \in N^r \bmod 1$.

⁹We have used the Axiom of choice here, i.e. $\prod_{A \in \mathcal{F}} (A \cap [0, 1/3)) \neq \emptyset$

Now that we have constructed N , we are ready for the contradiction. By Equations (4.4–4.6) we find

$$\begin{aligned} 1 = \mu([0, 1]) &= \sum_{r \in \mathbb{R}} \mu(N^r) = \sum_{r \in \mathbb{R}} \mu(N) \\ &= \begin{cases} \infty & \text{if } \mu(N) > 0 \\ 0 & \text{if } \mu(N) = 0 \end{cases} . \end{aligned}$$

which is certainly inconsistent. Incidentally we have just produced an example of so called “non – measurable” set. ■

Because of this example and our desire to have a measure μ on \mathbb{R} satisfying the properties in Question 1, we need to modify our definition of a measure. We will give up on trying to measure all subsets $A \subset \mathbb{R}$, i.e. we will only try to define μ on a smaller collection of “measurable” sets. Such collections will be called σ – algebras which we now introduce. The formal definition of a measure appears in Definition 5.1 of Section 5 below.

4.3. Algebras and σ – algebras.

Definition 4.5. A collection of subsets \mathcal{A} of X is an **Algebra** if

1. $\emptyset, X \in \mathcal{A}$
2. $A \in \mathcal{A}$ implies that $A^c \in \mathcal{A}$
3. \mathcal{A} is closed under finite unions, i.e. if $A_1, \dots, A_n \in \mathcal{A}$ then $A_1 \cup \dots \cup A_n \in \mathcal{A}$.
In view of conditions 1. and 2., 3. is equivalent to
- 3'. \mathcal{A} is closed under finite intersections.

Definition 4.6. A collection of subsets \mathcal{M} of X is a σ – algebra (σ – field) if \mathcal{M} is an algebra which also closed under countable unions, i.e. if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$. (Notice that since \mathcal{M} is also closed under taking complements, \mathcal{M} is also closed under taking countable intersections.) A pair (X, \mathcal{M}) , where X is a set and \mathcal{M} is a σ – algebra on X , is called a **measurable space**.

The reader should compare these definitions with that of a topology, see Definition 3.14. Recall that the elements of a topology are called open sets. Analogously, we will often refer to elements of an algebra \mathcal{A} or a σ – algebra \mathcal{M} as **measurable sets**.

Example 4.7. Here are some examples.

1. $\tau = \mathcal{M} = \mathcal{P}(X)$ in which case all subsets of X are open, closed, and measurable.
2. Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on X which is not an algebra.
3. $\tau = \mathcal{A} = \{\{1\}, \{2, 3\}, \emptyset, X\}$ is a topology, an algebra, and a σ – algebra on X . The sets $X, \{1\}, \{2, 3\}, \emptyset$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor **closed** and are not measurable.

Proposition 4.8. Let \mathcal{E} be any collection of subsets of X . Then there exists a unique smallest topology $\tau(\mathcal{E})$, algebra $\mathcal{A}(\mathcal{E})$ and σ -algebra $\sigma(\mathcal{E})$ which contains \mathcal{E} .

Proof. Note $\mathcal{P}(X)$ is a topology and an algebra and a σ -algebra and $\mathcal{E} \subset \mathcal{P}(X)$, so \mathcal{E} is always a subset of a topology, algebra, and σ – algebra. One may now easily check that

$$\tau(\mathcal{E}) \equiv \bigcap \{ \tau : \tau \text{ is a topology and } \mathcal{E} \subset \tau \}$$

is a topology which is clearly the smallest topology containing \mathcal{E} . The analogous construction works for the other cases as well. ■

We may give explicit descriptions of $\tau(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$. However $\sigma(\mathcal{E})$ typically does not admit a simple concrete description.

Proposition 4.9. *Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. For simplicity of notation, assume that $X, \emptyset \in \mathcal{E}$ (otherwise adjoin them to \mathcal{E} if necessary) and let $\mathcal{E}^c \equiv \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c = \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$. Then $\tau(\mathcal{E}) = \tau$ and $\mathcal{A}(\mathcal{E}) = \mathcal{A}$ where*

$$(4.7) \quad \tau := \{\text{arbitrary unions of finite intersections of elements from } \mathcal{E}\}$$

and

$$(4.8) \quad \mathcal{A} := \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\}.$$

Proof. From the definition of a topology and an algebra, it is clear that $\mathcal{E} \subset \tau \subset \tau(\mathcal{E})$ and $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$. Hence to finish that proof it suffices to show τ is a topology and \mathcal{A} is an algebra. The proof of these assertions are routine except for possibly showing that τ is closed under taking finite intersections and \mathcal{A} is closed under complementation.

To check \mathcal{A} is closed under complementation, let $Z \in \mathcal{A}$ be expressed as

$$Z = \bigcup_{i=1}^N \bigcap_{j=1}^K A_{ij}$$

where $A_{ij} \in \mathcal{E}_c$. Therefore, writing $B_{ij} = A_{ij}^c \in \mathcal{E}_c$, we find that

$$Z^c = \bigcap_{i=1}^N \bigcup_{j=1}^K B_{ij} = \bigcup_{j_1, \dots, j_N=1}^K (B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}) \in \mathcal{A}$$

wherein we have used the fact that $B_{1j_1} \cap B_{2j_2} \cap \dots \cap B_{Nj_N}$ is a finite intersection of sets from \mathcal{E}_c .

To show τ is closed under finite intersections it suffices to show for $V, W \in \tau$ that $V \cap W \in \tau$. Write

$$V = \bigcup_{\alpha \in A} V_\alpha \text{ and } W = \bigcup_{\beta \in B} W_\beta$$

where V_α and W_β are sets which are finite intersection of elements from \mathcal{E} . Then

$$V \cap W = (\bigcup_{\alpha \in A} V_\alpha) \cap (\bigcup_{\beta \in B} W_\beta) = \bigcup_{(\alpha, \beta) \in A \times B} V_\alpha \cap W_\beta \in \tau$$

since for each $(\alpha, \beta) \in A \times B$, $V_\alpha \cap W_\beta$ is still a finite intersection of elements from \mathcal{E} . ■

Remark 4.10. One might think that in general $\sigma(\mathcal{E})$ may be described as the countable unions of countable intersections of sets in \mathcal{E}^c . However this is **false**, since if

$$Z = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij}$$

with $A_{ij} \in \mathcal{E}_c$, then

$$Z^c = \bigcup_{j_1=1, j_2=1, \dots, j_N=1, \dots}^{\infty} \left(\bigcap_{\ell=1}^{\infty} A_{\ell, j_\ell}^c \right)$$

which is now an **uncountable** union. Thus the above description is not correct. In general it is complicated to explicitly describe $\sigma(\mathcal{E})$, see Proposition 1.23 on page 39 of Folland for details.

Exercise 4.1. Let τ be a topology on a set X and $\mathcal{A} = \mathcal{A}(\tau)$ be the algebra generated by τ . Show \mathcal{A} is the collection of subsets of X which may be written as finite union of sets of the form $F \cap V$ where F is closed and V is open.

The following notion will be useful in the sequel.

Definition 4.11. A set $\mathcal{E} \subset \mathcal{P}(X)$ is said to be an **elementary family or elementary class** provided that

- $\emptyset \in \mathcal{E}$
- \mathcal{E} is closed under finite intersections
- if $E \in \mathcal{E}$, then E^c is a finite disjoint union of sets from \mathcal{E} . (In particular $X = \emptyset^c$ is a disjoint union of elements from \mathcal{E} .)

Proposition 4.12. *Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family, then $\mathcal{A} = \mathcal{A}(\mathcal{E})$ consists of sets which may be written as finite disjoint unions of sets from \mathcal{E} .*

Proof. This could be proved making use of Proposition 4.12. However it is easier to give a direct proof.

Let \mathcal{A} denote the collection of sets which may be written as finite disjoint unions of sets from \mathcal{E} . Clearly $\mathcal{E} \subset \mathcal{A} \subset \mathcal{A}(\mathcal{E})$ so it suffices to show \mathcal{A} is an algebra since $\mathcal{A}(\mathcal{E})$ is the smallest algebra containing \mathcal{E} .

By the properties of \mathcal{E} , we know that $\emptyset, X \in \mathcal{A}$. Now suppose that $A_i = \coprod_{F \in \Lambda_i} F \in \mathcal{A}$ where, for $i = 1, 2, \dots, n$, Λ_i is a finite collection of disjoint sets from \mathcal{E} . Then

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \left(\coprod_{F \in \Lambda_i} F \right) = \bigcup_{(F_1, \dots, F_n) \in \Lambda_1 \times \dots \times \Lambda_n} (F_1 \cap F_2 \cap \dots \cap F_n)$$

and this is a disjoint (you check) union of elements from \mathcal{E} . Therefore \mathcal{A} is closed under finite intersections. Similarly, if $A = \coprod_{F \in \Lambda} F$ with Λ being a finite collection of disjoint sets from \mathcal{E} , then $A^c = \bigcap_{F \in \Lambda} F^c$. Since by assumption $F^c \in \mathcal{A}$ for $F \in \Lambda \subset \mathcal{E}$ and \mathcal{A} is closed under finite intersections, it follows that $A^c \in \mathcal{A}$. ■

Exercise 4.2. Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be elementary families. Show the collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is also an elementary family.

The analogous notion of elementary class \mathcal{E} for topologies is a basis \mathcal{V} defined below.

Definition 4.13. Let (X, τ) be a topological space. We say that $\mathcal{S} \subset \tau$ is a **sub-basis** for the topology τ iff $\tau = \tau(\mathcal{S})$ and $X = \cup \mathcal{S} := \cup_{V \in \mathcal{S}} V$. We say $\mathcal{V} \subset \tau$ is a **basis** for the topology τ iff \mathcal{V} is a sub-basis with the property that every element $V \in \tau$ may be written as

$$V = \cup \{B \in \mathcal{V} : B \subset V\}.$$

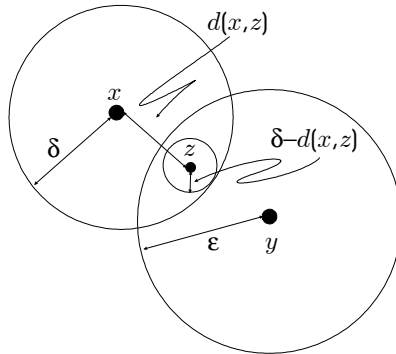


FIGURE 11. Fitting balls in the intersection.

Exercise 4.3. Suppose that \mathcal{S} is a sub-basis for a topology τ on a set X . Show $\mathcal{V} := \mathcal{S}_f$ consisting of finite intersections of elements from \mathcal{S} is a basis for τ . Moreover, \mathcal{S} is itself a basis for τ iff

$$V_1 \cap V_2 = \cup\{S \in \mathcal{S} : S \subset V_1 \cap V_2\}.$$

for every pair of sets $V_1, V_2 \in \mathcal{S}$.

Remark 4.14. Let (X, d) be a metric space, then $\mathcal{E} = \{B_x(\delta) : x \in X \text{ and } \delta > 0\}$ is a basis for τ_d – the topology associated to the metric d . This is the content of Exercise 3.3.

Let us check directly that \mathcal{E} is a basis for a topology. Suppose that $x, y \in X$ and $\epsilon, \delta > 0$. If $z \in B(x, \delta) \cap B(y, \epsilon)$, then

$$(4.9) \quad B(z, \alpha) \subset B(x, \delta) \cap B(y, \epsilon)$$

where $\alpha = \min\{\delta - d(x, z), \epsilon - d(y, z)\}$, see Figure 11. This is a formal consequence of the triangle inequality. For example let us show that $B(z, \alpha) \subset B(x, \delta)$. By the definition of α , we have that $\alpha \leq \delta - d(x, z)$ or that $d(x, z) \leq \delta - \alpha$. Hence if $w \in B(z, \alpha)$, then

$$d(x, w) \leq d(x, z) + d(z, w) \leq \delta - \alpha + d(z, w) < \delta - \alpha + \alpha = \delta$$

which shows that $w \in B(x, \delta)$. Similarly we show that $w \in B(y, \epsilon)$ as well.

Owing to Exercise 4.3, this shows \mathcal{E} is a basis for a topology. We do not need to use Exercise 4.3 here since in fact Equation (4.9) may be generalized to finite intersection of balls. Namely if $x_i \in X$, $\delta_i > 0$ and $z \in \cap_{i=1}^n B(x_i, \delta_i)$, then

$$(4.10) \quad B(z, \alpha) \subset \cap_{i=1}^n B(x_i, \delta_i)$$

where now $\alpha := \min\{\delta_i - d(x_i, z) : i = 1, 2, \dots, n\}$. By Eq. (4.10) it follows that any finite intersection of open balls may be written as a union of open balls.

Example 4.15. Suppose $X = \{1, 2, 3\}$ and $\mathcal{E} = \{\emptyset, X, \{1, 2\}, \{1, 3\}\}$, see Figure 12 below.

Then

$$\begin{aligned} \tau(\mathcal{E}) &= \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\} \\ \mathcal{A}(\mathcal{E}) &= \sigma(\mathcal{E}) = \mathcal{P}(X). \end{aligned}$$

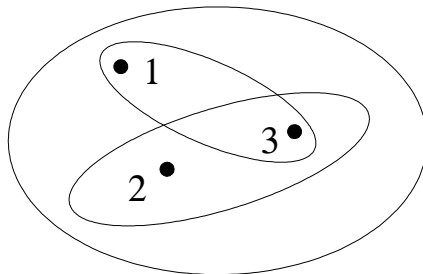


FIGURE 12. A collection of subsets.

Definition 4.16. Let X be a set. We say that a family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is a **partition** of X if X is the disjoint union of the sets in \mathcal{F} .

Example 4.17. Let X be a set and $\mathcal{E} = \{A_1, \dots, A_n\}$ where A_1, \dots, A_n is a partition of X . In this case

$$\mathcal{A}(\mathcal{E}) = \sigma(\mathcal{E}) = \tau(\mathcal{E}) = \{\cup_{i \in \Lambda} A_i : \Lambda \subset \{1, 2, \dots, n\}\}$$

where $\cup_{i \in \Lambda} A_i := \emptyset$ when $\Lambda = \emptyset$. Notice that

$$\#\mathcal{A}(\mathcal{E}) = \#(\mathcal{P}(\{1, 2, \dots, n\})) = 2^n.$$

Proposition 4.18. Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is a σ -algebra and \mathcal{M} is at most a countable set. Then there exists a unique **finite** partition \mathcal{F} of X such that $\mathcal{F} \subset \mathcal{M}$ and every element $A \in \mathcal{M}$ is of the form

$$(4.11) \quad A = \cup \{\alpha \in \mathcal{F} : \alpha \subset A\}.$$

In particular \mathcal{M} is actually a finite set.

Proof. For each $x \in X$ let

$$A_x = (\cap_{A \in \mathcal{M}, x \in A} A) \in \mathcal{M}.$$

That is, A_x is the smallest set in \mathcal{M} which contains x . Suppose that $C = A_x \cap A_y$ is non-empty. If $x \notin C$ then $x \in A_x \setminus C \in \mathcal{M}$ and hence $A_x \subset A_x \setminus C$ which shows that $A_x \cap C = \emptyset$ which is a contradiction. Hence $x \in C$ and similarly $y \in C$, therefore $A_x \subset C = A_x \cap A_y$ and $A_y \subset C = A_x \cap A_y$ which shows that $A_x = A_y$. Therefore, $\mathcal{F} = \{A_x : x \in X\}$ is a partition of X (which is necessarily countable) and Eq. (4.11) holds for all $A \in \mathcal{M}$. Let $\mathcal{F} = \{P_n\}_{n=1}^N$ where for the moment we allow $N = \infty$. If $N = \infty$, then \mathcal{M} is one to one correspondence with $\{0, 1\}^{\mathbb{N}}$. Indeed to each $a \in \{0, 1\}^{\mathbb{N}}$, let $A_a \in \mathcal{M}$ be defined by

$$A_a = \cup \{P_n : a_n = 1\}.$$

This shows that \mathcal{M} is uncountable since $\{0, 1\}^{\mathbb{N}}$ is uncountable; think of the base two expansion of numbers in $[0, 1]$ for example. Thus any countable σ -algebra is necessarily finite. This finishes the proof modulo the uniqueness assertion which is left as an exercise to the reader. ■

Example 4.19. Let $X = \mathbb{R}$ and

$$\mathcal{E} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\} = \{(a, \infty) \cap \mathbb{R} : a \in \bar{\mathbb{R}}\} \subset \mathcal{P}(\mathbb{R}).$$

Notice that $\mathcal{E}_f = \mathcal{E}$ and that \mathcal{E} is closed under unions, which shows that $\tau(\mathcal{E}) = \mathcal{E}$, i.e. \mathcal{E} is already a topology. Since $(a, \infty)^c = (-\infty, a]$ we find that $\mathcal{E}_c = \{(a, \infty), (-\infty, a], -\infty \leq a < \infty\} \cup \{\mathbb{R}, \emptyset\}$. Noting that

$$(a, \infty) \cap (-\infty, b] = (a, b]$$

it is easy to verify that the algebra $\mathcal{A}(\mathcal{E})$ generated by \mathcal{E} may be described as being those sets which are finite disjoint unions of sets from the following list

$$\tilde{\mathcal{E}} := \{(a, b] \cap \mathbb{R} : a, b \in \bar{\mathbb{R}}\}.$$

(This follows from Proposition 4.12 and the fact that $\tilde{\mathcal{E}}$ is an elementary family of subsets of \mathbb{R} .) The σ -algebra, $\sigma(\mathcal{E})$, generated by \mathcal{E} is **very complicated**. Here are some sets in $\sigma(\mathcal{E})$ – most of which are not in $\mathcal{A}(\mathcal{E})$.

- (a) $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] \in \sigma(\mathcal{E})$.
- (b) All of the standard open subsets of \mathbb{R} are in $\sigma(\mathcal{E})$.
- (c) $\{x\} = \bigcap_n (x - \frac{1}{n}, x] \in \sigma(\mathcal{E})$
- (d) $[a, b] = \{a\} \cup (a, b] \in \sigma(\mathcal{E})$
- (e) Any countable subset of \mathbb{R} is in $\sigma(\mathcal{E})$.

Remark 4.20. In the above example, one may replace \mathcal{E} by $\mathcal{E} = \{(a, \infty) : a \in \mathbb{Q}\} \cup \{\mathbb{R}, \emptyset\}$, in which case $\mathcal{A}(\mathcal{E})$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\{(a, \infty), (-\infty, a], (a, b] : a, b \in \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}.$$

This shows that $\mathcal{A}(\mathcal{E})$ is a countable set – a fact we will use later on.

Definition 4.21. A topological space, (X, τ) , is **second countable** if there exists a countable base \mathcal{V} for τ , i.e. $\mathcal{V} \subset \tau$ is a countable set such that for every $W \in \tau$,

$$W = \cup\{V : V \in \mathcal{V} \ni V \subset W\}.$$

Exercise 4.4. Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is a countable collection of subsets of X , then $\tau = \tau(\mathcal{E})$ is a second countable topology on X .

Proposition 4.22. *Every separable metric space, (X, ρ) is second countable.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X . Let $\mathcal{V} \equiv \{X, \emptyset\} \cup \bigcup_{m,n=1}^{\infty} \{B_{x_n}(r_m)\} \subset \tau_{\rho}$, where $\{r_m\}_{m=1}^{\infty}$ is dense in $(0, \infty)$. Then \mathcal{V} is a countable base for τ_{ρ} . To see this let $V \subset X$ be open and $x \in V$. Choose $\epsilon > 0$ such that $B_x(\epsilon) \subset V$ and then choose $x_n \in B_x(\epsilon/3)$. Choose r_m near $\epsilon/3$ such that $\rho(x, x_n) < r_m < \epsilon/3$ so that $x \in B_{x_n}(r_m) \subset V$. This shows $V = \cup\{B_{x_n}(r_m) : B_{x_n}(r_m) \subset V\}$. ■

Notation 4.23. For a general topological space (X, τ) , the **Borel σ -algebra** is the σ -algebra, $\mathcal{B}_X = \sigma(\tau)$. We will use $\mathcal{B}_{\mathbb{R}}$ to denote the Borel σ -algebra on \mathbb{R} .

Proposition 4.24. *If τ is a second countable topology on X and $\mathcal{E} \subset \mathcal{P}(X)$ is a countable set such that $\tau = \tau(\mathcal{E})$, then $\mathcal{B}_X := \sigma(\tau) = \sigma(\mathcal{E})$, i.e. $\sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E})$.*

Proof. Let \mathcal{E}_f denote the collection of subsets of X which are finite intersection of elements from \mathcal{E} along with X and \emptyset . Notice that \mathcal{E}_f is still countable (you prove). A set Z is in $\tau(\mathcal{E})$ iff Z is an arbitrary union of sets from \mathcal{E}_f . Therefore $Z = \bigcup_{A \in \mathcal{F}} A$ for some subset $\mathcal{F} \subset \mathcal{E}_f$ which is necessarily countable. Since $\mathcal{E}_f \subset \sigma(\mathcal{E})$ and $\sigma(\mathcal{E})$ is closed under countable unions it follows that $Z \in \sigma(\mathcal{E})$ and hence that $\tau(\mathcal{E}) \subset \sigma(\mathcal{E})$. For the last assertion, since $\mathcal{E} \subset \tau(\mathcal{E}) \subset \sigma(\mathcal{E})$ it follows that $\sigma(\mathcal{E}) \subset \sigma(\tau(\mathcal{E})) \subset \sigma(\mathcal{E})$.

■

Exercise 4.5. Verify the following identities

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a, \infty) : a \in \mathbb{R}\}) = \sigma(\{(a, \infty) : a \in \mathbb{Q}\}) = \sigma(\{[a, \infty) : a \in \mathbb{Q}\}).$$

4.4. Continuous and Measurable Functions. Our notion of a “measurable” function will be analogous to that for a continuous function. For motivational purposes, suppose (X, \mathcal{M}, μ) is a measure space and $f : X \rightarrow \mathbb{R}_+$. Roughly speaking, in the next section we are going to define $\int_X f d\mu$ by

$$\int_X f d\mu = \lim_{\text{mesh} \rightarrow 0} \sum_{0 < a_1 < a_2 < a_3 < \dots}^{\infty} a_i \mu(f^{-1}(a_i, a_{i+1}]).$$

For this to make sense we will need to require $f^{-1}((a, b]) \in \mathcal{M}$ for all $a < b$. Because of Lemma 4.30 below, this last condition is equivalent to the condition

$$f^{-1}(\mathcal{B}_{\mathbb{R}}) \subset \mathcal{M},$$

where we are using the following notation.

Notation 4.25. If $f : X \rightarrow Y$ is a function and $\mathcal{E} \subset \mathcal{P}(Y)$ let

$$f^{-1}\mathcal{E} \equiv f^{-1}(\mathcal{E}) \equiv \{f^{-1}(E) | E \in \mathcal{E}\}.$$

If $\mathcal{G} \subset \mathcal{P}(X)$, let

$$f_*\mathcal{G} \equiv \{A \in \mathcal{P}(Y) | f^{-1}(A) \in \mathcal{G}\}.$$

Exercise 4.6. Show $f^{-1}\mathcal{E}$ and $f_*\mathcal{G}$ are σ -algebras (topologies) provided \mathcal{E} and \mathcal{G} are σ -algebras (topologies).

Definition 4.26. Let (X, \mathcal{M}) and (Y, \mathcal{F}) be measurable (topological) spaces. A function $f : X \rightarrow Y$ is **measurable (continuous)** if $f^{-1}(\mathcal{F}) \subset \mathcal{M}$. We will also say that f is \mathcal{M}/\mathcal{F} -measurable (continuous) or $(\mathcal{M}, \mathcal{F})$ -measurable (continuous).

Example 4.27 (Characteristic Functions). Let (X, \mathcal{M}) be a measurable space and $A \subset X$. We define the characteristic function $1_A : X \rightarrow \mathbb{R}$ by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

If $A \in \mathcal{M}$, then 1_A is $(\mathcal{M}, \mathcal{P}(\mathbb{R}))$ -measurable because $1_A^{-1}(W)$ is either \emptyset , X , A or A^c for any $W \subset \mathbb{R}$. Conversely, if \mathcal{F} is any σ -algebra on \mathbb{R} containing a set $W \subset \mathbb{R}$ such that $1 \in W$ and $0 \in W^c$, then $A \in \mathcal{M}$ if 1_A is $(\mathcal{M}, \mathcal{F})$ -measurable. This is because $A = 1_A^{-1}(W) \in \mathcal{M}$.

Remark 4.28. Let $f : X \rightarrow Y$ be a function. Given a σ -algebra (topology) $\mathcal{F} \subset \mathcal{P}(Y)$, the σ -algebra (topology) $\mathcal{M} := f^{-1}(\mathcal{F})$ is the smallest σ -algebra (topology) on X such that f is $(\mathcal{M}, \mathcal{F})$ -measurable (continuous). Similarly, if \mathcal{M} is a σ -algebra (topology) on X then $\mathcal{F} = f_*\mathcal{M}$ is the largest σ -algebra (topology) on Y such that f is $(\mathcal{M}, \mathcal{F})$ -measurable (continuous).

Lemma 4.29. *Suppose that (X, \mathcal{M}) , (Y, \mathcal{F}) and (Z, \mathcal{G}) are measurable (topological) spaces. If $f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{F})$ and $g : (Y, \mathcal{F}) \rightarrow (Z, \mathcal{G})$ are measurable (continuous) functions then $g \circ f : (X, \mathcal{M}) \rightarrow (Z, \mathcal{G})$ is measurable (continuous) as well.*

Proof. This is easy since by assumption $g^{-1}(\mathcal{G}) \subset \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subset \mathcal{M}$ so that

$$(g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) \subset f^{-1}(\mathcal{F}) \subset \mathcal{M}.$$

■

Lemma 4.30. *Suppose that $f : X \rightarrow Y$ is a function and $\mathcal{E} \subset \mathcal{P}(Y)$, then*

$$(4.12) \quad \sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E})) \text{ and}$$

$$(4.13) \quad \tau(f^{-1}(\mathcal{E})) = f^{-1}(\tau(\mathcal{E})).$$

Moreover, if $\mathcal{F} = \sigma(\mathcal{E})$ (or $\mathcal{F} = \tau(\mathcal{E})$) and \mathcal{M} is a σ -algebra (topology) on X , then f is $(\mathcal{M}, \mathcal{F})$ -measurable (continuous) iff $f^{-1}(\mathcal{E}) \subset \mathcal{M}$.

Proof. We will prove Eq. (4.12), the proof of Eq. (4.13) being analogous. If $\mathcal{E} \subset \mathcal{F}$, then $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ and therefore, (because $f^{-1}(\sigma(\mathcal{E}))$ is a σ -algebra)

$$\mathcal{G} := \sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$$

which proves half of Eq. (4.12). For the reverse inclusion notice that

$$f_*\mathcal{G} = \{B \subset Y : f^{-1}(B) \in \mathcal{G}\}$$

is a σ -algebra which contains \mathcal{E} and thus $\sigma(\mathcal{E}) \subset f_*\mathcal{G}$. Hence if $B \in \sigma(\mathcal{E})$ we know that $f^{-1}(B) \in \mathcal{G}$, i.e. $f^{-1}(\sigma(\mathcal{E})) \subset \mathcal{G}$. The last assertion of the Lemma is an easy consequence of Eqs. (4.12) and (4.13). For example, if $f^{-1}\mathcal{E} \subset \mathcal{M}$, then $f^{-1}\sigma(\mathcal{E}) = \sigma(f^{-1}\mathcal{E}) \subset \mathcal{M}$ which shows f is $(\mathcal{M}, \mathcal{F})$ -measurable. ■

Definition 4.31. A function $f : X \rightarrow Y$ between topological spaces is **Borel measurable** if $f^{-1}(\mathcal{B}_Y) \subset \mathcal{B}_X$.

Proposition 4.32. *Let X and Y be two topological spaces and $f : X \rightarrow Y$ be a continuous function. Then f is Borel measurable.*

Proof. Using Lemma 4.30 and $\mathcal{B}_Y = \sigma(\tau_Y)$,

$$f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\tau_Y)) = \sigma(f^{-1}(\tau_Y)) \subset \sigma(\tau_X) = \mathcal{B}_X.$$

■

Corollary 4.33. *Suppose that (X, \mathcal{M}) is a measurable space. Then $f : X \rightarrow \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable iff $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$ iff $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$ iff $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$, etc. Similarly, if (X, \mathcal{M}) is a topological space, then $f : X \rightarrow \mathbb{R}$ is $(\mathcal{M}, \tau_{\mathbb{R}})$ -continuous iff $f^{-1}((a, b)) \in \mathcal{M}$ for all $-\infty < a < b < \infty$ iff $f^{-1}((a, \infty)) \in \mathcal{M}$ and $f^{-1}((-\infty, b)) \in \mathcal{M}$ for all $a, b \in \mathbb{Q}$. (We are using $\tau_{\mathbb{R}}$ to denote the standard topology on \mathbb{R} induced by the metric $d(x, y) = |x - y|$.)*

Proof. This is an exercise (Exercise 4.7) in using Lemma 4.30. ■

We will often deal with functions $f : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let

$$(4.14) \quad \mathcal{B}_{\overline{\mathbb{R}}} := \sigma(\{[a, \infty] : a \in \mathbb{R}\}).$$

The following Corollary of Lemma 4.30 is a direct analogue of Corollary 4.33.

Corollary 4.34. $f : X \rightarrow \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ - measurable iff $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$ iff $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$, etc.

Proposition 4.35. Let $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\bar{\mathbb{R}}}$ be as above, then

$$(4.15) \quad \mathcal{B}_{\bar{\mathbb{R}}} = \{A \subset \bar{\mathbb{R}} : A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

In particular $\{\infty\}, \{-\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}$ and $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\bar{\mathbb{R}}}$.

Proof. Let us first observe that

$$\begin{aligned} \{-\infty\} &= \bigcap_{n=1}^{\infty} [-\infty, -n] = \bigcap_{n=1}^{\infty} [-n, \infty]^c \in \mathcal{B}_{\bar{\mathbb{R}}}, \\ \{\infty\} &= \bigcap_{n=1}^{\infty} [n, \infty] \in \mathcal{B}_{\bar{\mathbb{R}}} \text{ and } \mathbb{R} = \bar{\mathbb{R}} \setminus \{\pm\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}. \end{aligned}$$

Letting $i : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ be the inclusion map,

$$\begin{aligned} i^{-1}(\mathcal{B}_{\bar{\mathbb{R}}}) &= \sigma(i^{-1}(\{[a, \infty] : a \in \bar{\mathbb{R}}\})) = \sigma(\{i^{-1}([a, \infty]) : a \in \bar{\mathbb{R}}\}) \\ &= \sigma(\{[a, \infty] \cap \mathbb{R} : a \in \bar{\mathbb{R}}\}) = \sigma(\{[a, \infty] : a \in \mathbb{R}\}) = \mathcal{B}_{\mathbb{R}}. \end{aligned}$$

Thus we have shown

$$\mathcal{B}_{\mathbb{R}} = i^{-1}(\mathcal{B}_{\bar{\mathbb{R}}}) = \{A \cap \mathbb{R} : A \in \mathcal{B}_{\bar{\mathbb{R}}}\}.$$

This implies:

1. $A \in \mathcal{B}_{\bar{\mathbb{R}}} \implies A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ and
2. if $A \subset \bar{\mathbb{R}}$ is such that $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ there exists $B \in \mathcal{B}_{\bar{\mathbb{R}}}$ such that $A \cap \mathbb{R} = B \cap \mathbb{R}$.
Because $A \Delta B \subset \{\pm\infty\}$ and $\{\infty\}, \{-\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}$ we may conclude that $A \in \mathcal{B}_{\bar{\mathbb{R}}}$ as well.

This proves Eq. (4.15). ■

Proposition 4.36 (Closure under sups, infs and limits). *Suppose that (X, \mathcal{M}) is a measurable space and $f_j : (X, \mathcal{M}) \rightarrow \bar{\mathbb{R}}$ is a sequence of $\mathcal{M}/\mathcal{B}_{\bar{\mathbb{R}}}$ - measurable functions. Then*

$$\sup_j f_j, \quad \inf_j f_j, \quad \limsup_{j \rightarrow \infty} f_j \text{ and } \liminf_{j \rightarrow \infty} f_j$$

are all $\mathcal{M}/\mathcal{B}_{\bar{\mathbb{R}}}$ - measurable functions. (Note that this result is in general false when (X, \mathcal{M}) is a topological space and measurable is replaced by continuous in the statement.)

Proof. Define $g_+(x) := \sup_j f_j(x)$, then

$$\begin{aligned} \{x : g_+(x) \leq a\} &= \{x : f_j(x) \leq a \forall j\} \\ &= \bigcap_j \{x : f_j(x) \leq a\} \in \mathcal{M} \end{aligned}$$

so that g_+ is measurable. Similarly if $g_-(x) = \inf_j f_j(x)$ then

$$\{x : g_-(x) \geq a\} = \bigcap_j \{x : f_j(x) \geq a\} \in \mathcal{M}.$$

Since

$$\begin{aligned} \limsup_{j \rightarrow \infty} f_j &= \inf_n \sup \{f_j : j \geq n\} \text{ and} \\ \liminf_{j \rightarrow \infty} f_j &= \sup_n \inf \{f_j : j \geq n\} \end{aligned}$$

we are done by what we have already proved. ■

4.4.1. *More general pointwise limits.*

Lemma 4.37. *Suppose that (X, \mathcal{M}) is a measurable space, (Y, d) is a metric space and $f_j : X \rightarrow Y$ is $(\mathcal{M}, \mathcal{B}_Y)$ – measurable for all j . Also assume that for each $x \in X$, $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ exists. Then $f : X \rightarrow Y$ is also $(\mathcal{M}, \mathcal{B}_Y)$ – measurable.*

Proof. Let $V \in \tau_d$ and $W_m := \{y \in Y : d_{V^c}(y) > 1/m\}$ for $m = 1, 2, \dots$. Then $W_m \in \tau_d$,

$$W_m \subset \bar{W}_m \subset \{y \in Y : d_{V^c}(y) \geq 1/m\} \subset V$$

for all m and $W_m \uparrow V$ as $m \rightarrow \infty$. The proof will be completed by verifying the identity,

$$f^{-1}(V) = \cup_{m=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n \geq N} f_n^{-1}(W_m) \in \mathcal{M}.$$

If $x \in f^{-1}(V)$ then $f(x) \in V$ and hence $f(x) \in W_m$ for some m . Since $f_n(x) \rightarrow f(x)$, $f_n(x) \in W_m$ for almost all n . That is $x \in \cup_{m=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n \geq N} f_n^{-1}(W_m)$. Conversely when $x \in \cup_{m=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n \geq N} f_n^{-1}(W_m)$ there exists an m such that $f_n(x) \in W_m \subset \bar{W}_m$ for almost all n . Since $f_n(x) \rightarrow f(x) \in \bar{W}_m \subset V$, it follows that $x \in f^{-1}(V)$. ■

Remark 4.38. In the previous Lemma 4.37 it is possible to let (Y, τ) be any topological space which has the “regularity” property that if $V \in \tau$ there exists $W_m \in \tau$ such that $W_m \subset \bar{W}_m \subset V$ and $V = \cup_{m=1}^{\infty} W_m$. Moreover, some extra condition is necessary on the topology τ in order for Lemma 4.37 to be correct. For example if $Y = \{1, 2, 3\}$ and $\tau = \{Y, \emptyset, \{1, 2\}, \{2, 3\}, \{2\}\}$ as in Example 3.28 and $X = \{a, b\}$ with the trivial σ – algebra. Let $f_j(a) = f_j(b) = 2$ for all j , then f_j is constant and hence measurable. Let $f(a) = 1$ and $f(b) = 2$, then $f_j \rightarrow f$ as $j \rightarrow \infty$ with f being non-measurable. Notice that the Borel σ – algebra on Y is $\mathcal{P}(Y)$.

4.5. **Topologies and σ – Algebras Generated by Functions.**

Definition 4.39. Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets, $A \subset X$, $i_A : A \rightarrow X$ be the inclusion map ($i_A(x) = x$) for all $x \in A$, and

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}.$$

When $\mathcal{E} = \tau$ is a topology or $\mathcal{E} = \mathcal{M}$ is a σ – algebra we call τ_A the relative topology and \mathcal{M}_A the relative σ – algebra on A .

Proposition 4.40. *Suppose that $A \subset X$, $\mathcal{M} \subset \mathcal{P}(X)$ is a σ – algebra and $\tau \subset \mathcal{P}(X)$ is a topology, then $\mathcal{M}_A \subset \mathcal{P}(A)$ is a σ – algebra and $\tau_A \subset \mathcal{P}(A)$ is a topology. Moreover if $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\mathcal{M} = \sigma(\mathcal{E})$ ($\tau = \tau(\mathcal{E})$) then $\mathcal{M}_A = \sigma(\mathcal{E}_A)$ ($\tau_A = \tau(\mathcal{E}_A)$).*

Proof. The first assertion is Exercise 4.6 and the second assertion is a consequence of Lemma 4.30. Indeed,

$$\mathcal{M}_A = i_A^{-1}(\mathcal{M}) = i_A^{-1}(\sigma(\mathcal{E})) = \sigma(i_A^{-1}(\mathcal{E})) = \sigma(\mathcal{E}_A)$$

and similarly

$$\tau_A = i_A^{-1}(\tau) = i_A^{-1}(\tau(\mathcal{E})) = \tau(i_A^{-1}(\mathcal{E})) = \tau(\mathcal{E}_A).$$

■

Example 4.41. Suppose that (X, d) is a metric space and $A \subset X$ is a set. Let $\tau = \tau_d$ and $d_A := d|_{A \times A}$ be the metric d restricted to A . Then $\tau_A = \tau_{d_A}$, i.e. the relative topology, τ_A , of τ_d on A is the same as the topology induced by the

restriction of the metric d to A . Indeed, if $V \in \tau_A$ there exists $W \in \tau$ such that $V \cap A = W$. Therefore for all $x \in A$ there exists $\epsilon > 0$ such that $B_x(\epsilon) \subset W$ and hence $B_x(\epsilon) \cap A \subset V$. Since $B_x(\epsilon) \cap A = B_x^{d_A}(\epsilon)$ is a d_A -ball in A , this shows V is d_A -open, i.e. $\tau_A \subset \tau_{d_A}$. Conversely, if $V \in \tau_{d_A}$, then for each $x \in A$ there exists $\epsilon_x > 0$ such that $B_x^{d_A}(\epsilon) = B_x(\epsilon) \cap A \subset V$. Therefore $V = A \cap W$ with $W := \cup_{x \in A} B_x(\epsilon) \in \tau$. This shows $\tau_{d_A} \subset \tau_A$.

Definition 4.42. Let $A \subset X$, $f : A \rightarrow \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a σ -algebra and $\tau \subset \mathcal{P}(X)$ be a topology, then we say that $f|_A$ is measurable (continuous) if $f|_A$ is \mathcal{M}_A -measurable (τ_A -continuous).

Proposition 4.43. Let $A \subset X$, $f : X \rightarrow \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a σ -algebra and $\tau \subset \mathcal{P}(X)$ be a topology. If f is \mathcal{M} -measurable (τ -continuous) then $f|_A$ is \mathcal{M}_A -measurable (τ_A -continuous). Moreover if $A_n \in \mathcal{M}$ ($A_n \in \tau$) such that $X = \cup_{n=1}^{\infty} A_n$ and $f|_{A_n}$ is \mathcal{M}_{A_n} -measurable (τ_{A_n} -continuous) for all n , then f is \mathcal{M} -measurable (τ -continuous).

Proof. Notice that i_A is $(\mathcal{M}_A, \mathcal{M})$ -measurable (τ_A, τ -continuous) hence $f|_A = f \circ i_A$ is \mathcal{M}_A -measurable (τ_A -continuous). Let $B \subset \mathbb{C}$ be a Borel set and consider

$$f^{-1}(B) = \cup_{n=1}^{\infty} (f^{-1}(B) \cap A_n) = \cup_{n=1}^{\infty} f|_{A_n}^{-1}(B).$$

If $A \in \mathcal{M}$ ($A \in \tau$), then it is easy to check that

$$\begin{aligned} \mathcal{M}_A &= \{B \in \mathcal{M} : B \subset A\} \subset \mathcal{M} \text{ and} \\ \tau_A &= \{B \in \tau : B \subset A\} \subset \tau. \end{aligned}$$

The second assertion is now an easy consequence of the previous three equations.

■

Definition 4.44. Let X and A be sets, and suppose for $\alpha \in A$ we are given a measurable (topological) space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \rightarrow Y_\alpha$. We will write $\sigma(f_\alpha : \alpha \in A)$ ($\tau(f_\alpha : \alpha \in A)$) for the smallest σ -algebra (topology) on X such that each f_α is measurable (continuous), i.e.

$$\begin{aligned} \sigma(f_\alpha : \alpha \in A) &= \sigma(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)) \text{ and} \\ \tau(f_\alpha : \alpha \in A) &= \tau(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)). \end{aligned}$$

Proposition 4.45. Assuming the notation in Definition 4.44 and additionally let (Z, \mathcal{M}) be a measurable (topological) space and $g : Z \rightarrow X$ be a function. Then g is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ -measurable ($(\mathcal{M}, \tau(f_\alpha : \alpha \in A))$ -continuous) iff $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ -measurable (continuous) for all $\alpha \in A$.

Proof. (\Rightarrow) If g is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ -measurable, then the composition $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ -measurable by Lemma 4.29.

(\Leftarrow) Let

$$\mathcal{G} = \sigma(f_\alpha : \alpha \in A) = \sigma(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)).$$

If $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ -measurable for all α , then

$$g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{M} \forall \alpha \in A$$

and therefore

$$g^{-1}(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)) = \cup_{\alpha \in A} g^{-1} f_\alpha^{-1}(\mathcal{F}_\alpha) \subset \mathcal{M}.$$

Hence

$$g^{-1}(\mathcal{G}) = g^{-1}(\sigma(\cup_{\alpha \in A} f_{\alpha}^{-1}(\mathcal{F}_{\alpha}))) = \sigma(g^{-1}(\cup_{\alpha \in A} f_{\alpha}^{-1}(\mathcal{F}_{\alpha}))) \subset \mathcal{M}$$

which shows that g is $(\mathcal{M}, \mathcal{G})$ -measurable.

The topological case is proved in the same way. ■

4.6. Product Spaces. In this section we consider product topologies and σ -algebras. We will start with a finite number of factors first and then later mention what happens for an infinite number of factors.

4.6.1. Products with a Finite Number of Factors. Let $\{X_i\}_{i=1}^n$ be a collection of sets, $X := X_1 \times X_2 \times \cdots \times X_n$ and $\pi_i : X \rightarrow X_i$ be the projection map $\pi(x_1, x_2, \dots, x_n) = x_i$ for each $1 \leq i \leq n$. Let us also suppose that τ_i is a topology on X_i and \mathcal{M}_i is a σ -algebra on X_i for each i .

Notation 4.46. Let $\mathcal{E}_i \subset \mathcal{P}(X_i)$ be a collection of subsets of X_i for $i = 1, 2, \dots, n$ we will write, by abuse of notation, $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ for the collection of subsets of $X_1 \times \cdots \times X_n$ of the form $A_1 \times A_2 \times \cdots \times A_n$ with $A_i \in \mathcal{E}_i$ for all i . That is we are identifying (A_1, A_2, \dots, A_n) with $A_1 \times A_2 \times \cdots \times A_n$.

Definition 4.47. The **product topology** on X , denoted by $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ is the smallest topology on X so that each map $\pi_i : X \rightarrow X_i$ is continuous. Similarly, the **product σ -algebra** on X , denoted by $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n$, is the smallest σ -algebra on X so that each map $\pi_i : X \rightarrow X_i$ is measurable.

Remark 4.48. The product topology may also be described as the smallest topology containing sets from $\tau_1 \times \cdots \times \tau_n$, i.e.

$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n = \tau(\tau_1 \times \cdots \times \tau_n).$$

Indeed,

$$\begin{aligned} \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n &= \tau(\pi_1, \pi_2, \dots, \pi_n) \\ &= \tau(\{\cap_{i=1}^n \pi_i^{-1}(V_i) : V_i \in \tau_i \text{ for } i = 1, 2, \dots, n\}) \\ &= \tau(\{V_1 \times V_2 \times \cdots \times V_n : V_i \in \tau_i \text{ for } i = 1, 2, \dots, n\}). \end{aligned}$$

Similarly,

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n = \sigma(\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n).$$

Furthermore if $\mathcal{B}_i \subset \tau_i$ is a basis for the topology τ_i for each i , then $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is a basis for $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$. Indeed, $\tau_1 \times \cdots \times \tau_n$ is closed under finite intersections and generates $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$, therefore $\tau_1 \times \cdots \times \tau_n$ is a basis for the product topology. Hence for $W \in \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ and $x = (x_1, \dots, x_n) \in W$, there exists $V_1 \times V_2 \times \cdots \times V_n \in \tau_1 \times \cdots \times \tau_n$ such that

$$x \in V_1 \times V_2 \times \cdots \times V_n \subset W.$$

Since \mathcal{B}_i is a basis for τ_i , we may now choose $U_i \in \mathcal{B}_i$ such that $x_i \in U_i \subset V_i$ for each i . Thus

$$x \in U_1 \times U_2 \times \cdots \times U_n \subset W$$

and we have shown W may be written as a union of sets from $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$. Since

$$\mathcal{B}_1 \times \cdots \times \mathcal{B}_n \subset \tau_1 \times \cdots \times \tau_n \subset \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n,$$

this shows $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is a basis for $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$.

Lemma 4.49. *Let (X_i, d_i) for $i = 1, \dots, n$ be metric spaces, $X := X_1 \times \dots \times X_n$ and for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in X let*

$$(4.16) \quad d(x, y) = \sum_{i=1}^n d_i(x_i, y_i).$$

Then the topology, τ_d , associated to the metric d is the product topology on X , i.e.

$$\tau_d = \tau_{d_1} \otimes \tau_{d_2} \otimes \dots \otimes \tau_{d_n}.$$

Proof. Let $\rho(x, y) = \max\{d_i(x_i, y_i) : i = 1, 2, \dots, n\}$. Then ρ is equivalent to d and hence $\tau_\rho = \tau_d$. Moreover if $\epsilon > 0$ and $x = (x_1, x_2, \dots, x_n) \in X$, then

$$B_x^\rho(\epsilon) = B_{x_1}^{d_1}(\epsilon) \times \dots \times B_{x_n}^{d_n}(\epsilon).$$

By Remark 4.14,

$$\mathcal{E} := \{B_x^\rho(\epsilon) : x \in X \text{ and } \epsilon > 0\}$$

is a basis for τ_ρ and by Remark 4.48 \mathcal{E} is also a basis for $\tau_{d_1} \otimes \tau_{d_2} \otimes \dots \otimes \tau_{d_n}$. Therefore,

$$\tau_{d_1} \otimes \tau_{d_2} \otimes \dots \otimes \tau_{d_n} = \tau(\mathcal{E}) = \tau_\rho = \tau_d.$$

■

Remark 4.50. Let (Z, \mathcal{M}) be a measurable (topological) space, then by Proposition 4.45, a function $f : Z \rightarrow X$ is measurable (continuous) iff $\pi_i \circ f : Z \rightarrow X_i$ is $(\mathcal{M}, \mathcal{M}_i)$ -measurable ((τ, τ_i) -continuous) for $i = 1, 2, \dots, n$. So if we write

$$f(z) = (f_1(z), f_2(z), \dots, f_n(z)) \in X_1 \times X_2 \times \dots \times X_n,$$

then $f : Z \rightarrow X$ is measurable (continuous) iff $f_i : Z \rightarrow X_i$ is measurable (continuous) for all i .

Theorem 4.51. *For $i = 1, 2, \dots, n$, let $\mathcal{E}_i \subset \mathcal{P}(X_i)$ be a collection of subsets of X_i such that $X_i \in \mathcal{E}_i$ and $\mathcal{M}_i = \sigma(\mathcal{E}_i)$ (or $\tau_i = \tau(\mathcal{E}_i)$) for $i = 1, 2, \dots, n$, then*

$$\begin{aligned} \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \dots \otimes \mathcal{M}_n &= \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \dots \times \mathcal{E}_n) \text{ and} \\ \tau_1 \otimes \tau_2 \otimes \dots \otimes \tau_n &= \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \dots \times \mathcal{E}_n). \end{aligned}$$

Written out more explicitly, these equations state

$$(4.17) \quad \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \dots \times \sigma(\mathcal{E}_n)) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \dots \times \mathcal{E}_n) \text{ and}$$

$$(4.18) \quad \tau(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2) \times \dots \times \tau(\mathcal{E}_n)) = \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \dots \times \mathcal{E}_n).$$

Moreover if $\{(X_i, \tau_i)\}_{i=1}^n$ is a sequence of second countable topological spaces, $\tau = \tau_1 \otimes \tau_2 \otimes \dots \otimes \tau_n$ is the product topology on $X = X_1 \times \dots \times X_n$, then

$$\mathcal{B}_X := \sigma(\tau_1 \otimes \tau_2 \otimes \dots \otimes \tau_n) = \sigma(\mathcal{B}_{X_1} \times \dots \times \mathcal{B}_{X_n}) =: \mathcal{B}_{X_1} \otimes \dots \otimes \mathcal{B}_{X_n}.$$

That is to say the Borel σ -algebra and the product σ -algebra on X are the same.

Proof. We will prove Eq. (4.17). The proof of Eq. (4.18) is completely analogous. Let us first do the case of two factors. Since

$$\mathcal{E}_1 \times \mathcal{E}_2 \subset \sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)$$

it follows that

$$\sigma(\mathcal{E}_1 \times \mathcal{E}_2) \subset \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = \sigma(\pi_1, \pi_2).$$

To prove the reverse inequality it suffices to show $\pi_i : X_1 \times X_2 \rightarrow X_i$ is $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) - \mathcal{M}_i = \sigma(\mathcal{E}_i)$ measurable for $i = 1, 2$. To prove this suppose that $E \in \mathcal{E}_1$, then

$$\pi_1^{-1}(E) = E \times X_2 \in \mathcal{E}_1 \times \mathcal{E}_2 \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$$

wherein we have used the fact that $X_2 \in \mathcal{E}_2$. Similarly, for $E \in \mathcal{E}_2$ we have

$$\pi_2^{-1}(E) = X_1 \times E \in \mathcal{E}_1 \times \mathcal{E}_2 \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2).$$

This proves the desired measurability, and hence

$$\sigma(\pi_1, \pi_2) \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2) \subset \sigma(\pi_1, \pi_2).$$

To prove the last assertion we may assume each \mathcal{E}_i is countable for $i = 1, 2$. Since $\mathcal{E}_1 \times \mathcal{E}_2$ is countable, a couple of applications of Proposition 4.24 along with the first two assertions of the theorems gives

$$\begin{aligned} \sigma(\tau_1 \otimes \tau_2) &= \sigma(\tau(\tau_1 \times \tau_2)) = \sigma(\tau(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2))) = \sigma(\tau(\mathcal{E}_1 \times \mathcal{E}_2)) \\ &= \sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = \sigma(\mathcal{M}_1 \times \mathcal{M}_2) = \mathcal{M}_1 \otimes \mathcal{M}_2. \end{aligned}$$

The proof for n factors works the same way. Indeed,

$$\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n \subset \sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)$$

implies

$$\sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \subset \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)) = \sigma(\pi_1, \dots, \pi_n)$$

and for $E \in \mathcal{E}_i$,

$$\begin{aligned} \pi_i^{-1}(E) &= X_1 \times X_2 \times \cdots \times X_{i-1} \times E \times X_{i+1} \cdots \times X_n \in \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n \\ &\subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n). \end{aligned}$$

This show π_i is $\sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) - \mathcal{M}_i = \sigma(\mathcal{E}_i)$ measurable and therefore,

$$\sigma(\pi_1, \dots, \pi_n) \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \subset \sigma(\pi_1, \dots, \pi_n).$$

If the \mathcal{E}_i are countable, then

$$\begin{aligned} \sigma(\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n) &= \sigma(\tau(\tau_1 \times \tau_2 \times \cdots \times \tau_n)) \\ &= \sigma(\tau(\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2) \times \cdots \times \tau(\mathcal{E}_n))) \\ &= \sigma(\tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n)) \\ &= \sigma(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \\ &= \sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2) \times \cdots \times \sigma(\mathcal{E}_n)) \\ &= \sigma(\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n) \\ &= \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n. \end{aligned}$$

■

Remark 4.52. One can not relax the assumption that $X_i \in \mathcal{E}_i$ in Theorem 4.51. For example, if $X_1 = X_2 = \{1, 2\}$ and $\mathcal{E}_1 = \mathcal{E}_2 = \{\{1\}\}$, then $\sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \{\emptyset, X_1 \times X_2, \{(1, 1)\}\}$ while $\sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = \mathcal{P}(X_1 \times X_2)$.

Proposition 4.53. *If (X_i, d_i) are separable metric spaces for $i = 1, \dots, n$, then*

$$\mathcal{B}_{X_1} \otimes \cdots \otimes \mathcal{B}_{X_n} = \mathcal{B}_{(X_1 \times \cdots \times X_n)}$$

where \mathcal{B}_{X_i} is the Borel σ -algebra on X_i and $\mathcal{B}_{(X_1 \times \cdots \times X_n)}$ is the Borel σ -algebra on $X_1 \times \cdots \times X_n$ equipped with the product topology.

Proof. This follows directly from Proposition 4.22 and Theorem 4.51. ■

Because all norms on finite dimensional spaces are equivalent, the usual Euclidean norm on $\mathbb{R}^m \times \mathbb{R}^n$ is equivalent to the “product” norm defined by

$$\|(x, y)\|_{\mathbb{R}^m \times \mathbb{R}^n} = \|x\|_{\mathbb{R}^m} + \|y\|_{\mathbb{R}^n}.$$

Hence by Lemma 4.49, the Euclidean topology on \mathbb{R}^{m+n} is the same as the product topology on $\mathbb{R}^{m+n} \cong \mathbb{R}^m \times \mathbb{R}^n$. Here we are identifying $\mathbb{R}^m \times \mathbb{R}^n$ with \mathbb{R}^{m+n} by the map

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \rightarrow (x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{R}^{m+n}.$$

Proposition 4.53 and these comments leads to the following corollaries.

Corollary 4.54. *After identifying $\mathbb{R}^m \times \mathbb{R}^n$ with \mathbb{R}^{m+n} as above and letting $\mathcal{B}_{\mathbb{R}^n}$ denote the Borel σ -algebra on \mathbb{R}^n , we have*

$$\mathcal{B}_{\mathbb{R}^{m+n}} = \mathcal{B}_{\mathbb{R}^n} \otimes \mathcal{B}_{\mathbb{R}^m} \text{ and } \mathcal{B}_{\mathbb{R}^n} = \overbrace{\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}}^{n\text{-times}}.$$

Corollary 4.55. *If (X, \mathcal{M}) is a measurable space, then*

$$f = (f_1, f_2, \dots, f_n) : X \rightarrow \mathbb{R}^n$$

is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^n})$ -measurable iff $f_i : X \rightarrow \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable for each i . In particular, a function $f : X \rightarrow \mathbb{C}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ -measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ -measurable.

Corollary 4.56. *Let (X, \mathcal{M}) be a measurable space and $f, g : X \rightarrow \mathbb{C}$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ -measurable functions. Then $f \pm g$ and $f \cdot g$ are also $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ -measurable.*

Proof. Define $F : X \rightarrow \mathbb{C} \times \mathbb{C}$, $A_{\pm} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and $M : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ by $F(x) = (f(x), g(x))$, $A_{\pm}(w, z) = w \pm z$ and $M(w, z) = wz$. Then A_{\pm} and M are continuous and hence $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$ -measurable. Also F is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}}) = (\mathcal{M}, \mathcal{B}_{\mathbb{C}^2})$ -measurable since $\pi_1 \circ F = f$ and $\pi_2 \circ F = g$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ -measurable. Therefore $A_{\pm} \circ F = f \pm g$ and $M \circ F = f \cdot g$, being the composition of measurable functions, are also measurable. ■

Lemma 4.57. *Let $\alpha \in \mathbb{C}$, (X, \mathcal{M}) be a measurable space and $f : X \rightarrow \mathbb{C}$ be a $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ -measurable function. Then*

$$F(x) := \begin{cases} \frac{1}{f(x)} & \text{if } f(x) \neq 0 \\ \alpha & \text{if } f(x) = 0 \end{cases}$$

is measurable.

Proof. Define $i : \mathbb{C} \rightarrow \mathbb{C}$ by

$$i(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq 0 \\ \alpha & \text{if } z = 0. \end{cases}$$

For any open set $V \subset \mathbb{C}$ we have

$$i^{-1}(V) = i^{-1}(V \setminus \{0\}) \cup i^{-1}(V \cap \{0\})$$

Because i is continuous except at $z = 0$, $i^{-1}(V \setminus \{0\})$ is an open set and hence in $\mathcal{B}_{\mathbb{C}}$. Moreover, $i^{-1}(V \cap \{0\}) \in \mathcal{B}_{\mathbb{C}}$ since $i^{-1}(V \cap \{0\})$ is either the empty set or the one point set $\{\alpha\}$. Therefore $i^{-1}(\tau_{\mathbb{C}}) \subset \mathcal{B}_{\mathbb{C}}$ and hence $i^{-1}(\sigma(\tau_{\mathbb{C}})) = \sigma(i^{-1}(\tau_{\mathbb{C}})) \subset \mathcal{B}_{\mathbb{C}}$ which shows that i is Borel measurable. Since $F = i \circ f$ is the composition of measurable functions, F is also measurable. ■

4.6.2. *General Product spaces .*

Definition 4.58. Suppose $(X_\alpha, \mathcal{M}_\alpha)_{\alpha \in A}$ is a collection of measurable spaces and let X be the product space

$$X = \prod_{\alpha \in A} X_\alpha$$

and $\pi_\alpha : X \rightarrow X_\alpha$ be the canonical projection maps. Then the product σ – algebra, $\bigotimes_{\alpha} \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha \equiv \sigma(\pi_\alpha : \alpha \in A) = \sigma \left(\bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{M}_\alpha) \right).$$

Similarly if $(X_\alpha, \mathcal{M}_\alpha)_{\alpha \in A}$ is a collection of topological spaces, the product topology $\bigotimes_{\alpha} \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha \equiv \tau(\pi_\alpha : \alpha \in A) = \tau \left(\bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{M}_\alpha) \right).$$

Remark 4.59. Let (Z, \mathcal{M}) be a measurable (topological) space and

$$\left(X = \prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{M}_\alpha \right)$$

be as in Definition 4.58. By Proposition 4.45, a function $f : Z \rightarrow X$ is measurable (continuous) iff $\pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{M}_\alpha)$ – measurable (continuous) for all $\alpha \in A$.

Proposition 4.60. *Suppose that $(X_\alpha, \mathcal{M}_\alpha)_{\alpha \in A}$ is a collection of measurable (topological) spaces and $\mathcal{E}_\alpha \subset \mathcal{M}_\alpha$ generates \mathcal{M}_α for each $\alpha \in A$, then*

$$(4.19) \quad \bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \sigma \left(\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right) \quad \left(\tau \left(\bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right) \right)$$

Moreover, suppose that A is either finite or countably infinite, $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, and $\mathcal{M}_\alpha = \sigma(\mathcal{E}_\alpha)$ for each $\alpha \in A$. Then the product σ – algebra satisfies

$$(4.20) \quad \bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \sigma \left(\left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A \right\} \right).$$

Similarly if A is finite and $\mathcal{M}_\alpha = \tau(\mathcal{E}_\alpha)$, then the product topology satisfies

$$(4.21) \quad \bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \tau \left(\left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A \right\} \right).$$

Proof. We will prove Eq. (4.19) in the measure theoretic case since a similar proof works in the topological category. Since $\bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{M}_\alpha)$, it follows that

$$\mathcal{F} := \sigma \left(\bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right) \subset \sigma \left(\bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{M}_\alpha) \right) = \bigotimes_{\alpha} \mathcal{M}_\alpha.$$

Conversely,

$$\mathcal{F} \supset \sigma(\pi_\alpha^{-1}(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\sigma(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\mathcal{M}_\alpha)$$

holds for all α implies that

$$\bigcup_{\alpha} \pi_{\alpha}^{-1}(\mathcal{M}_{\alpha}) \subset \mathcal{F}$$

and hence that $\bigotimes_{\alpha} \mathcal{M}_{\alpha} \subset \mathcal{F}$.

We now prove Eq. (4.20). Since we are assuming that $X_{\alpha} \in \mathcal{E}_{\alpha}$ for each $\alpha \in A$, we see that

$$\bigcup_{\alpha} \pi_{\alpha}^{-1}(\mathcal{E}_{\alpha}) \subset \left\{ \prod_{\alpha \in A} E_{\alpha} : E_{\alpha} \in \mathcal{E}_{\alpha} \text{ for all } \alpha \in A \right\}$$

and therefore by Eq. (4.19)

$$\bigotimes_{\alpha \in A} \mathcal{M}_{\alpha} = \sigma \left(\bigcup_{\alpha} \pi_{\alpha}^{-1}(\mathcal{E}_{\alpha}) \right) \subset \sigma \left(\left\{ \prod_{\alpha \in A} E_{\alpha} : E_{\alpha} \in \mathcal{E}_{\alpha} \text{ for all } \alpha \in A \right\} \right).$$

This last statement is true independent as to whether A is countable or not. For the reverse inclusion it suffices to notice that since A is countable,

$$\prod_{\alpha \in A} E_{\alpha} = \bigcap_{\alpha \in A} \pi_{\alpha}^{-1}(E_{\alpha}) \in \bigotimes_{\alpha \in A} \mathcal{M}_{\alpha}$$

and hence

$$\sigma \left(\left\{ \prod_{\alpha \in A} E_{\alpha} : E_{\alpha} \in \mathcal{E}_{\alpha} \text{ for all } \alpha \in A \right\} \right) \subset \bigotimes_{\alpha \in A} \mathcal{M}_{\alpha}.$$

Here is a generalization of Theorem 4.51 to the case of countable number of factors.

■

Proposition 4.61. *Let $\{X_{\alpha}\}_{\alpha \in A}$ be a sequence of sets where A is at most countable. Suppose for each $\alpha \in A$ we are given a countable set $\mathcal{E}_{\alpha} \subset \mathcal{P}(X_{\alpha})$. Let $\tau_{\alpha} = \tau(\mathcal{E}_{\alpha})$ be the topology on X_{α} generated by \mathcal{E}_{α} and X be the product space $\prod_{\alpha \in A} X_{\alpha}$ with equipped with the product topology $\tau := \bigotimes_{\alpha \in A} \tau(\mathcal{E}_{\alpha})$. Then the Borel σ -algebra $\mathcal{B}_X = \sigma(\tau)$ is the same as the product σ -algebra:*

$$\mathcal{B}_X = \bigotimes_{\alpha \in A} \mathcal{B}_{X_{\alpha}},$$

where $\mathcal{B}_{X_{\alpha}} = \sigma(\tau(\mathcal{E}_{\alpha})) = \sigma(\mathcal{E}_{\alpha})$ for all $\alpha \in A$.

Proof. By Proposition 4.60, the topology τ may be described as the smallest topology containing $\mathcal{E} = \bigcup_{\alpha \in A} \pi_{\alpha}^{-1}(\mathcal{E}_{\alpha})$. Now \mathcal{E} is the countable union of countable sets so is still countable. Therefore by Proposition 4.24 and Proposition 4.60 we have

$$\mathcal{B}_X = \sigma(\tau) = \sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E}) = \bigotimes_{\alpha \in A} \sigma(\mathcal{E}_{\alpha}) = \bigotimes_{\alpha \in A} \sigma(\tau_{\alpha}) = \bigotimes_{\alpha \in A} \mathcal{B}_{X_{\alpha}}.$$

■

Lemma 4.62. *Suppose that (Y, \mathcal{F}) is a measurable space and $F : X \rightarrow Y$ is a map. Then to every $(\sigma(F), \mathcal{B}_{\mathbb{R}})$ -measurable function, H from $X \rightarrow \mathbb{R}$, there is a $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable function $h : Y \rightarrow \mathbb{R}$ such that $H = h \circ F$.*

Proof. First suppose that $H = 1_A$ where $A \in \sigma(F) = F^{-1}(\mathcal{B}_{\mathbb{R}})$. Let $J \in \mathcal{B}_{\mathbb{R}}$ such that $A = F^{-1}(J)$ then $1_A = 1_{F^{-1}(J)} = 1_J \circ F$ and hence the Lemma is valid in this case with $h = 1_J$. More generally if $H = \sum a_i 1_{A_i}$ is a simple function, then

there exists $J_i \in \mathcal{B}_{\mathbb{R}}$ such that $1_{A_i} = 1_{J_i} \circ F$ and hence $H = h \circ F$ with $h := \sum a_i 1_{J_i}$ – a simple function on \mathbb{R} .

For general $(\sigma(F), \mathcal{B}_{\mathbb{R}})$ – measurable function, H , from $X \rightarrow \mathbb{R}$, choose simple functions H_n converging to H . Let h_n be simple functions on \mathbb{R} such that $H_n = h_n \circ F$. Then it follows that

$$H = \lim_{n \rightarrow \infty} H_n = \limsup_{n \rightarrow \infty} H_n = \limsup_{n \rightarrow \infty} h_n \circ F = h \circ F$$

where $h := \limsup_{n \rightarrow \infty} h_n$ – a measurable function from Y to \mathbb{R} . ■

The following is an immediate corollary of Proposition 4.45 and Lemma 4.62.

Corollary 4.63. *Let X and A be sets, and suppose for $\alpha \in A$ we are give a measurable space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \rightarrow Y_\alpha$. Let $Y := \prod_{\alpha \in A} Y_\alpha$, $\mathcal{F} := \otimes_{\alpha \in A} \mathcal{F}_\alpha$ be the product σ – algebra on Y and $\mathcal{M} := \sigma(f_\alpha : \alpha \in A)$ be the smallest σ -algebra on X such that each f_α is measurable. Then the function $F : X \rightarrow Y$ defined by $[F(x)]_\alpha := f_\alpha(x)$ for each $\alpha \in A$ is $(\mathcal{M}, \mathcal{F})$ – measurable and a function $H : X \rightarrow \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable iff there exists a $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ – measurable function h from Y to \mathbb{R} such that $H = h \circ F$.*

4.7. Exercises.

Exercise 4.7. Prove Corollary 4.33. **Hint:** See Exercise 4.5.

Exercise 4.8. Folland, Problem 1.5 on p.24. If \mathcal{M} is the σ – algebra generated by $\mathcal{E} \subset \mathcal{P}(X)$, then \mathcal{M} is the union of the σ – algebras generated by countable subsets $\mathcal{F} \subset \mathcal{E}$.

Exercise 4.9. Let (X, \mathcal{M}) be a measure space and $f_n : X \rightarrow \mathbb{F}$ be a sequence of measurable functions on X . Show that $\{x : \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\} \in \mathcal{M}$.

Exercise 4.10. Show that every monotone function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$ – measurable.

Exercise 4.11. Folland problem 2.6 on p. 48.

Exercise 4.12. Suppose that X is a set, $\{(Y_\alpha, \tau_\alpha) : \alpha \in A\}$ is a family of topological spaces and $f_\alpha : X \rightarrow Y_\alpha$ is a given function for all $\alpha \in A$. Assuming that $\mathcal{S}_\alpha \subset \tau_\alpha$ is a sub-basis for the topology τ_α for each $\alpha \in A$, show $\mathcal{S} := \cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{S}_\alpha)$ is a sub-basis for the topology $\tau := \tau(f_\alpha : \alpha \in A)$.

Notation 4.64. Let X be a set and $\mathbf{p} := \{p_n\}_{n=0}^\infty$ be a family of semi-metrics on X , i.e. $p_n : X \times X \rightarrow [0, \infty)$ are functions satisfying the assumptions of metric except for the assertion that $p_n(x, y) = 0$ implies $x = y$. Further assume that $p_n(x, y) \leq p_{n+1}(x, y)$ for all n and if $p_n(x, y) = 0$ for all $n \in \mathbb{N}$ then $x = y$. Given $n \in \mathbb{N}$ and $x \in X$ let

$$B_n(x, \epsilon) := \{y \in X : p_n(x, y) < \epsilon\}.$$

We will write $\tau(\mathbf{p})$ form the smallest topology on X such that $p_n(x, \cdot) : X \rightarrow [0, \infty)$ is continuous for all $n \in \mathbb{N}$ and $x \in X$, i.e. $\tau(\mathbf{p}) := \tau(p_n(x \cdot) : n \in \mathbb{N} \text{ and } x \in X)$.

Exercise 4.13. Using Notation 4.64, show that collection of balls,

$$\mathcal{B} := \{B_n(x, \epsilon) : n \in \mathbb{N}, x \in X \text{ and } \epsilon > 0\},$$

forms a basis for the topology $\tau(\mathbf{p})$. **Hint:** Use Exercise 4.12 to show \mathcal{B} is a sub-basis for the topology $\tau(\mathbf{p})$ and then use Exercise 4.3 to show \mathcal{B} is in fact a basis for the topology $\tau(\mathbf{p})$.

Exercise 4.14. Using the notation in 4.64, let

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x, y)}{1 + p_n(x, y)}.$$

Show d is a metric on X and $\tau_d = \tau(\mathbf{p})$. Conclude that a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converges to $x \in X$ iff

$$\lim_{k \rightarrow \infty} p_n(x_k, x) = 0 \text{ for all } n \in \mathbb{N}.$$

Exercise 4.15. Let $\{(X_n, d_n)\}_{n=1}^{\infty}$ be a sequence of metric spaces, $X := \prod_{n=1}^{\infty} X_n$, and for $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ in X let

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.$$

(See Exercise 3.26.) Moreover, let $\pi_i : X \rightarrow X_i$ be the projection maps, show

$$\tau_d = \otimes_{n=1}^{\infty} \tau_{d_n} := \tau(\{\pi_i : i \in \mathbb{N}\}).$$

That is show the d -metric topology is the same as the product topology on X .