7. L^p-spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space and for $0 < p < \infty$ and a measurable function $f : X \to \mathbb{C}$ let

$$\|f\|_p \equiv (\int |f|^p d\mu)^{1/p}. \tag{7.1}$$

When $p = \infty$, let

$$\|f\|_\infty = \inf \{a \geq 0 : \mu(|f| > a) = 0\} \tag{7.2}$$

For $0 < p \leq \infty$, let

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty\}/\sim$$

where $f \sim g$ iff $f = g$ a.e. Notice that $\|f - g\|_p = 0$ iff $f \sim g$ and if $f \sim g$ then $\|f\|_p = \|g\|_p$. In general we will (by abuse of notation) use f to denote both the function f and the equivalence class containing f.

Remark 7.1. Suppose that $\|f\|_\infty \leq M$, then for all $a > M$, $\mu(|f| > a) = 0$ and therefore $\mu(|f| > M) = \lim_{n\to\infty} \mu(|f| > M + 1/n) = 0$, i.e. $|f(x)| \leq M$ for μ - a.e. x. Conversely, if $|f| \leq M$ a.e. and $a > M$ then $\mu(|f| > a) = 0$ and hence $\|f\|_\infty \leq M$. This leads to the identity:

$$\|f\|_\infty = \inf \{a \geq 0 : |f(x)| \leq a \text{ for } \mu - \text{a.e. } x\}.$$

Theorem 7.2 (Hölder's inequality)**.** *Suppose that $1 \leq p \leq \infty$ and $q := \frac{p}{p-1}$, or equivalently $p^{-1} + q^{-1} = 1$. If f and g are measurable functions then*

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q. \tag{7.3}$$

Assuming $p \in (1, \infty)$ and $\|f\|_p \cdot \|g\|_q < \infty$, equality holds in Eq. (7.3) iff $|f|^p$ and $|g|^q$ are linearly dependent as elements of L^1. If we further assume that $\|f\|_p$ and $\|g\|_q$ are positive then equality holds in Eq. (7.3) iff

$$|g|^q \|f\|_p^p = \|g\|_q^q |f|^p \text{ a.e.} \tag{7.4}$$

Proof. The cases where $\|f\|_q = 0$ or ∞ or $\|g\|_p = 0$ or ∞ are easy to deal with and are left to the reader. So we will now assume that $0 < \|f\|_q, \|g\|_p < \infty$. Let $s = |f|/\|f\|_p$ and $t = |g|/\|g\|_q$ then Lemma 2.27 implies

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f|^p}{\|f\|_p} + \frac{1}{q} \frac{|g|^q}{\|g\|^q} \tag{7.5}$$

with equality iff $|g/\|g\|_q| = |f|^{p-1}/\|f\|_p^{(p-1)} = |f|^{p/q}/\|f\|_p^{p/q}$, i.e. $|g|^q \|f\|_p^p = \|g\|_q^q |f|^p$. Integrating Eq. (7.5) implies

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

with equality iff Eq. (7.4) holds. The proof is finished since it is easily checked that equality holds in Eq. (7.3) when $|f|^p = c\,|g|^q$ of $|g|^q = c\,|f|^p$ for some constant c. ■

The following corollary is an easy extension of Hölder's inequality.

Corollary 7.3. *Suppose that $f_i : X \to \mathbb{C}$ are measurable functions for $i = 1, \ldots, n$ and $p_1, \ldots, p_n$ and r are positive numbers such that $\sum_{i=1}^n p_i^{-1} = r^{-1}$, then*

$$\left\| \prod_{i=1}^n f_i \right\|_r \leq \prod_{i=1}^n \|f_i\|_{p_i} \text{ where } \sum_{i=1}^n p_i^{-1} = r^{-1}.$$

Proof. To prove this inequality, start with $n = 2$, then for any $p \in [1, \infty]$,

$$\|fg\|_r^r = \int f^r g^r d\mu \le \|f^r\|_p \|g^r\|_{p^*}$$

where $p^* = \frac{p}{p-1}$ is the conjugate exponent. Let $p_1 = pr$ and $p_2 = p^* r$ so that $p_1^{-1} + p_2^{-1} = r^{-1}$ as desired. Then the previous equation states that

$$\|fg\|_r \le \|f\|_{p_1} \|g\|_{p_2}$$

as desired. The general case is now proved by induction. Indeed,

$$\left\| \prod_{i=1}^{n+1} f_i \right\|_r = \left\| \prod_{i=1}^{n} f_i \cdot f_{n+1} \right\|_r \le \left\| \prod_{i=1}^{n} f_i \right\|_q \|f_{n+1}\|_{p_{n+1}}$$

where $q^{-1} + p_{n+1}^{-1} = r^{-1}$. Since $\sum_{i=1}^n p_i^{-1} = q^{-1}$, we may now use the induction hypothesis to conclude

$$\left\| \prod_{i=1}^{n} f_i \right\|_q \le \prod_{i=1}^{n} \|f_i\|_{p_i},$$

which combined with the previous displayed equation proves the generalized form of Holder's inequality. ■

Theorem 7.4 (Minkowski's Inequality)**.** *If $1 \le p \le \infty$ and $f, g \in L^p$ then*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \tag{7.6}$$

Moreover if $p < \infty$, then equality holds in this inequality iff

$$\operatorname{sgn}(f) = \operatorname{sgn}(g) \text{ when } p = 1 \text{ and}$$
$$f = cg \text{ or } g = cf \text{ for some } c > 0 \text{ when } p > 1.$$

Proof. When $p = \infty$, $|f| \le \|f\|_\infty$ a.e. and $|g| \le \|g\|_\infty$ a.e. so that $|f + g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ a.e. and therefore

$$\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

When $p < \infty$,

$$|f + g|^p \le (2 \max(|f|, |g|))^p = 2^p \max(|f|^p, |g|^p) \le 2^p (|f|^p + |g|^p),$$

$$\|f + g\|_p^p \le 2^p \left(\|f\|_p^p + \|g\|_p^p \right) < \infty.$$

In case $p = 1$,

$$\|f + g\|_1 = \int_X |f + g| d\mu \le \int_X |f| d\mu + \int_X |g| d\mu$$

with equality iff $|f| + |g| = |f + g|$ a.e. which happens iff $\operatorname{sgn}(f) = \operatorname{sgn}(g)$ a.e.

In case $p \in (1, \infty)$, we may assume $\|f + g\|_p$, $\|f\|_p$ and $\|g\|_p$ are all positive since otherwise the theorem is easily verified. Now

$$|f + g|^p = |f + g||f + g|^{p-1} \le (|f| + |g|)|f + g|^{p-1}$$

with equality iff $\operatorname{sgn}(f) = \operatorname{sgn}(g)$. Integrating this equation and applying Holder's inequality with $q = p/(p-1)$ gives

$$\begin{aligned} \int_X |f + g|^p d\mu &\le \int_X |f| \, |f + g|^{p-1} d\mu + \int_X |g| \, |f + g|^{p-1} d\mu \\ &\le (\|f\|_p + \|g\|_p) \| \, |f + g|^{p-1} \, \|_q \end{aligned} \tag{7.7}$$

with equality iff

$$\operatorname{sgn}(f) = \operatorname{sgn}(g) \text{ and}$$

$$\left(\frac{|f|}{\|f\|_p}\right)^p = \frac{|f+g|^p}{\|f+g\|_p^p} = \left(\frac{|g|}{\|g\|_p}\right)^p \text{ a.e.} \tag{7.8}$$

Therefore

$$\||f+g|^{p-1}\|_q^q = \int_X (|f+g|^{p-1})^q d\mu = \int_X |f+g|^p d\mu. \tag{7.9}$$

Combining Eqs. (7.7) and (7.9) implies

$$\|f+g\|_p^p \le \|f\|_p \|f+g\|_p^{p/q} + \|g\|_p \|f+g\|_p^{p/q} \tag{7.10}$$

with equality iff Eq. (7.8) holds which happens iff $f = cg$ a.e. with $c > 0$.. Solving for $\|f+g\|_p$ in Eq. (7.10) gives Eq. (7.6). ■

The next theorem gives another example of using Hölder's inequality

Theorem 7.5. *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, $p \in [1, \infty]$, $q = p/(p-1)$ and $k : X \times Y \to \mathbb{C}$ be a $\mathcal{M} \otimes \mathcal{N}$ – measurable function. Assume there exist finite constants C_1 and C_2 such that*

$$\int_X |k(x,y)|\, d\mu(x) \le C_1 \text{ for } \nu \text{ a.e. } y \text{ and}$$
$$\int_Y |k(x,y)|\, d\nu(y) \le C_2 \text{ for } \mu \text{ a.e. } x.$$

If $f \in L^p(\nu)$, then

$$\int_Y |k(x,y) f(y)|\, d\nu(y) < \infty \text{ for } \mu - \text{a.e. } x,$$

$x \to Kf(x) := \int k(x,y) f(y) d\nu(y) \in L^p(\mu)$ and

$$\|Kf\|_{L^p(\mu)} \le C_1^{1/p} C_2^{1/q} \|f\|_{L^p(\nu)} \tag{7.11}$$

Proof. Suppose $p \in (1, \infty)$ to begin with and let $q = p/(p-1)$, then by Hölder's inequality,

$$\begin{aligned}
\int_Y |k(x,y) f(y)|\, d\nu(y) &= \int_Y |k(x,y)|^{1/q} |k(x,y)|^{1/p} |f(y)|\, d\nu(y) \\
&\le \left[\int_Y |k(x,y)|\, d\nu(y)\right]^{1/q} \left[\int_X |k(x,y)|\, |f(y)|^p\, d\nu(y)\right]^{1/p} \\
&\le C_2^{1/q} \left[\int_X |k(x,y)|\, |f(y)|^p\, d\nu(y)\right]^{1/p}.
\end{aligned}$$

Therefore, using Tonelli's theorem,

$$\begin{aligned}
\left\|\int_Y |k(\cdot,y) f(y)|\, d\nu(y)\right\|_p^p &\le C_2^{p/q} \int_Y d\mu(x) \int_X d\nu(y)\, |k(x,y)|\, |f(y)|^p \\
&= C_2^{p/q} \int_X d\nu(y)\, |f(y)|^p \int_Y d\mu(x)\, |k(x,y)| \\
&\le C_2^{p/q} C_1 \int_X d\nu(y)\, |f(y)|^p = C_2^{p/q} C_1 \|f\|_p^p.
\end{aligned}$$

From this it follows that $x \to Kf(x) := \int k(x,y)f(y)d\nu(y) \in L^p(\mu)$ and that Eq. (7.11) holds.

Similarly, if $p = \infty$,

$$\int_Y |k(x,y)f(y)|\, d\nu(y) \le \|f\|_\infty \int_Y |k(x,y)|\, d\nu(y) \le C_2 \|f\|_\infty \text{ for } \mu - \text{a.e. } x.$$

so that $\|Kf\|_{L^\infty(\mu)} \le C_2 \|f\|_{L^\infty(\nu)}$. If $p = 1$, then

$$\begin{aligned}\int_X d\mu(x) \int_Y d\nu(y)\, |k(x,y)f(y)| &= \int_Y d\nu(y)\, |f(y)| \int_X d\mu(x)\, |k(x,y)| \\ &\le C_1 \int_Y d\nu(y)\, |f(y)|\end{aligned}$$

which shows $\|Kf\|_{L^1(\mu)} \le C_1 \|f\|_{L^1(\nu)}$. ■

7.1. **Jensen's Inequality.**

Definition 7.6. A function $\phi : (a,b) \to \mathbb{R}$ is convex if for all $a < x_0 < x_1 < b$ and $t \in [0,1]$ $\phi(x_t) \le t\phi(x_1) + (1-t)\phi(x_0)$ where $x_t = tx_1 + (1-t)x_0$.

The following Proposition is clearly motivated by Figure 15.

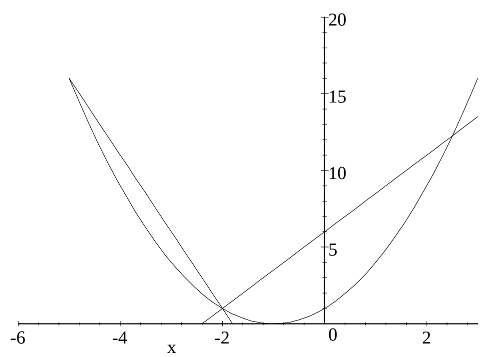

FIGURE 15. A convex function along with two cords corresponding to $x_0 = -2$ and $x_1 = 4$ and $x_0 = -5$ and $x_1 = -2$.

Proposition 7.7. *Suppose $\phi : (a,b) \to \mathbb{R}$ is a convex function, then*

1. *For all $u, v, w, z \in (a,b)$ such that $u < z$, $w \in [u,z)$ and $v \in (u,z]$,*

$$\frac{\phi(v) - \phi(u)}{v - u} \le \frac{\phi(z) - \phi(w)}{z - w}. \tag{7.12}$$

2. *For each $c \in (a,b)$, the right and left sided derivatives $\phi'_\pm(c)$ exists in $\mathbb{R}$ and if $a < u < v < b$, then $\phi'_+(u) \le \phi'_-(v) \le \phi'_+(v)$.*
3. *The function ϕ is continuous.*
4. *For all $t \in (a,b)$ and $\beta \in [\phi'_-(t), \phi'_+(t)]$, $\phi(x) \ge \phi(t) + \beta(x-t)$ for all $x \in (a,b)$. In particular,*

$$\phi(x) \ge \phi(t) + \phi'_-(t)(x-t) \text{ for all } x, t \in (a,b).$$

Proof. 1a) Suppose first that $u < v = w < z$, in which case Eq. (7.12) is equivalent to

$$(\phi(v) - \phi(u))(z - v) \le (\phi(z) - \phi(v))(v - u)$$

which after solving for $\phi(v)$ is equivalent to the following equations holding:

$$\phi(v) \le \phi(z)\frac{v-u}{z-u} + \phi(u)\frac{z-v}{z-u}.$$

But this last equation states that $\phi(v) \le \phi(z)t + \phi(u)(1-t)$ where $t = \frac{v-u}{z-u}$ and $v = tz + (1-t)u$ and hence is valid by the definition of ϕ being convex.

1b) Now assume $u = w < v < z$, in which case Eq. (7.12) is equivalent to

$$(\phi(v) - \phi(u))(z - u) \le (\phi(z) - \phi(u))(v - u)$$

which after solving for $\phi(v)$ is equivalent to

$$\phi(v)(z-u) \le \phi(z)(v-u) + \phi(u)(z-v)$$

which is equivalent to

$$\phi(v) \le \phi(z)\frac{v-u}{z-u} + \phi(u)\frac{z-v}{z-u}.$$

Again this equation is valid by the convexity of ϕ.

1c) $u < w < v = z$, in which case Eq. (7.12) is equivalent to

$$(\phi(z) - \phi(u))(z - w) \le (\phi(z) - \phi(w))(z - u)$$

and this is equivalent to the inequality,

$$\phi(w) \le \phi(z)\frac{w-u}{z-u} + \phi(u)\frac{z-w}{z-u}$$

which again is true by the convexity of ϕ.

1) General case. If $u < w < v < z$, then by 1a-1c)

$$\frac{\phi(z)-\phi(w)}{z-w} \ge \frac{\phi(v)-\phi(w)}{v-w} \ge \frac{\phi(v)-\phi(u)}{v-u}$$

and if $u < v < w < z$

$$\frac{\phi(z)-\phi(w)}{z-w} \ge \frac{\phi(w)-\phi(v)}{w-v} \ge \frac{\phi(w)-\phi(u)}{w-u}.$$

We have now taken care of all possible cases.

2) On the set $a < w < z < b$, Eq. (7.12) shows that $(\phi(z) - \phi(w))/(z-w)$ is a decreasing function in w and an increasing function in z and therefore $\phi'_\pm(x)$ exists for all $x \in (a, b)$. Also from Eq. (7.12) we learn that

$$\phi'_+(u) \le \frac{\phi(z)-\phi(w)}{z-w} \text{ for all } a < u < w < z < b, \tag{7.13}$$

$$\frac{\phi(v)-\phi(u)}{v-u} \le \phi'_-(z) \text{ for all } a < u < v < z < b, \tag{7.14}$$

and letting $w \uparrow z$ in the first equation also implies that

$$\phi'_+(u) \le \phi'_-(z) \text{ for all } a < u < z < b.$$

The inequality, $\phi'_-(z) \le \phi'_+(z)$, is also an easy consequence of Eq. (7.12).

3) Since $\phi(x)$ has both left and right finite derivatives, it follows that ϕ is continuous. (For an alternative proof, see Rudin.)

4) Given t, let $\beta \in [\phi'_-(t), \phi'_+(t)]$, then by Eqs. (7.13) and (7.14),

$$\frac{\phi(t)-\phi(u)}{t-u} \leq \phi'_-(t) \leq \beta \leq \phi'_+(t) \leq \frac{\phi(z)-\phi(t)}{z-t}$$

for all $a < u < t < z < b$. Item 4. now follows. ■

Corollary 7.8. *Suppose $\phi : (a,b) \to \mathbb{R}$ is differential then ϕ is convex iff ϕ' is non decreasing. In particular if $\phi \in C^2(a,b)$ then ϕ is convex iff $\phi'' \geq 0$.*

Proof. By Proposition 7.7, if ϕ is convex then ϕ' is non-decreasing. Conversely if ϕ' is increasing then by the mean value theorem,

$$\frac{\phi(x_1)-\phi(c)}{x_1-c} = \phi'(\xi_1) \text{ for some } \xi_1 \in (c, x_1)$$

and

$$\frac{\phi(c)-\phi(x_0)}{c-x_0} = \phi'(\xi_2) \text{ for some } \xi_2 \in (x_0, c).$$

Hence

$$\frac{\phi(x_1)-\phi(c)}{x_1-c} \geq \frac{\phi(c)-\phi(x_0)}{c-x_0}$$

for all $x_0 < c < x_1$. Solving this inequality for $\phi(c)$ gives

$$\phi(c) \leq \frac{c-x_0}{x_1-x_0}\phi(x_1) + \frac{x_1-c}{x_1-x_0}\phi(x_0)$$

showing ϕ is convex. ■

Example 7.9. The function $\exp(x)$ is convex , x^p is convex iff $p \geq 1$ and $-\log(x)$ is convex.

Theorem 7.10 (Jensen's Inequality)**.** *Suppose that $(X, \mathcal{M}, \mu)$ is a probability space, i.e. μ is a positive measure and $\mu(X) = 1$. Also suppose that $f \in L^1(\mu)$, $f : X \to (a,b)$, and $\phi : (a,b) \to \mathbb{R}$ is a convex function. Then*

$$\phi\left(\int_X f d\mu\right) \leq \int_X \phi(f) d\mu$$

where if $\phi \circ f \notin L^1(\mu)$, then $\phi \circ f$ is integrable in the extended sense and $\int_X \phi(f)d\mu = \infty$.

Proof. Let $t = \int_X f d\mu \in (a,b)$ and let $\beta \in \mathbb{R}$ be such that $\phi(s) - \phi(t) \geq \beta(s-t)$ for all $s \in (a,b)$. Then integrating the inequality, $\phi(f) - \phi(t) \geq \beta(f-t)$, implies that

$$0 \leq \int_X \phi(f)d\mu - \phi(t) = \int_X \phi(f)d\mu - \phi(\int_X f d\mu).$$

Moreover, if $\phi(f)$ is not integrable, then $\phi(f) \geq \phi(t) + \beta(f-t)$ which shows that negative part of $\phi(f)$ is integrable. Therefore, $\int_X \phi(f)d\mu = \infty$ in this case. ■

Example 7.11. The convex function in Example 7.9 lead to the following inequalities,

$$\exp\left(\int_X f d\mu\right) \leq \int_X e^f d\mu,$$
$$\int_X \log(|f|)d\mu \leq \log\left(\int_X |f|\, d\mu\right) \leq \log\left(\int_X f d\mu\right)$$

and for $p \geq 1$,

$$\left|\int_X f d\mu\right|^p \leq \left(\int_X |f|\, d\mu\right)^p \leq \int_X |f|^p\, d\mu.$$

The last equation may also easily be derived using Hölder's inequality. As a special case of the first equation, we get another proof of Lemma 2.27. Indeed, let p and q be conjugate exponents, $s, t > 0$, and $a = \ln s$ and $b = \ln t$, then

$$st = e^{(a+b)} = e^{\left(\frac{1}{q}qa+\frac{1}{p}pa\right)} \leq \frac{1}{q}e^{qa} + \frac{1}{p}e^{pa} = \frac{1}{q}s^q + \frac{1}{p}t^p.$$

Of course the above considerations may also be viewed as just using directly the property that the exponential function is convex.

7.2. **Modes of Convergence.** As usual let $(X, \mathcal{M}, \mu)$ be a fixed measure space and let $\{f_n\}$ be a sequence of measurable functions on X. Also let $f : X \to \mathbb{C}$ be a measurable function. We have the following notions of convergence and Cauchy sequences.

Definition 7.12.
1. $f_n \to f$ a.e. if there is a set $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $\lim_{n\to\infty} 1_E f_n = 1_E f$.
2. $f_n \to f$ in μ – measure if $\lim_{n\to\infty} \mu(|f_n - f| > \epsilon) = 0$ for all $\epsilon > 0$. We will abbreviate this by saying $f_n \to f$ in L^0 or by $f_n \xrightarrow{\mu} f$.
3. $f_n \to f$ in L^p iff $f \in L^p$ and $f_n \in L^p$ for all n, and $\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0$.

Definition 7.13.
1. $\{f_n\}$ is a.e. Cauchy if there is a set $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and$\{1_E\, f_n\}$ is a pointwise Cauchy sequences.
2. $\{f_n\}$ is Cauchy in μ – measure (or L^0 – Cauchy) if $\lim_{m,n\to\infty} \mu(|f_n - f_m| > \epsilon) = 0$ for all $\epsilon > 0$.
3. $\{f_n\}$ is Cauchy in L^p if $\lim_{m,n\to\infty} \int |f_n - f_m|^p d\mu = 0$.

Lemma 7.14 (Chebyshev's inequality again)**.** *Let $p \in [1, \infty)$ and $f \in L^p$, then*

$$\mu\left(|f| \geq \epsilon\right) \leq \frac{1}{\epsilon^p}\|f\|_p^p \text{ for all } \epsilon > 0.$$

In particular if $\{f_n\} \subset L^p$ is L^p – convergent (Cauchy) then $\{f_n\}$ is also convergent (Cauchy) in measure.

Proof. By Chebyshev's inequality (5.12),

$$\mu\left(|f| \geq \epsilon\right) = \mu\left(|f|^p \geq \epsilon^p\right) \leq \frac{1}{\epsilon^p}\int_X |f|^p\, d\mu = \frac{1}{\epsilon^p}\|f\|_p^p$$

and therefore if $\{f_n\}$ is L^p – Cauchy, then

$$\mu\left(|f_n - f_m| \geq \epsilon\right) \leq \frac{1}{\epsilon^p}\|f_n - f_m\|^p \to 0 \text{ as } m, n \to \infty$$

showing $\{f_n\}$ is L^0 – Cauchy. A similar argument holds for the L^p – convergent case. ■

Lemma 7.15. *Suppose $a_n \in \mathbb{C}$ and $|a_{n+1} - a_n| \leq \epsilon_n$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Then $\lim_{n\to\infty} a_n = a \in \mathbb{C}$ exists and $|a - a_n| \leq \delta_n \equiv \sum_{k=n}^{\infty} \epsilon_k$.*

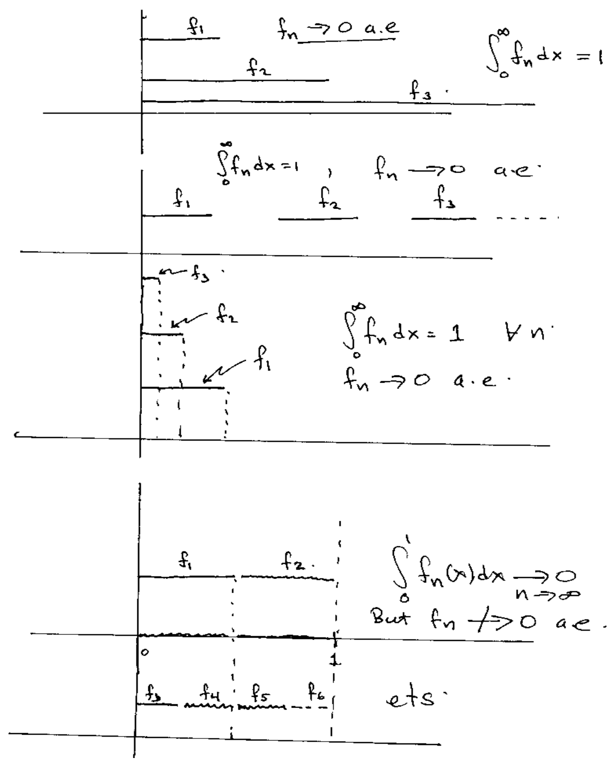

FIGURE 16. Modes of convergence examples. In picture 1. $f_n \to 0$ a.e., $f_n \nrightarrow 0$ in L^1, $f_n \overset{m}{\to} 0$. In picture 2. $f_n \to 0$ a.e., $f_n \nrightarrow 0$ in L^1, $f_n \overset{m}{\nrightarrow} 0$. In picture 3., $f_n \to 0$ a.e., $f_n \overset{m}{\to} 0$ but $f_n \nrightarrow 0$ in L^1. In picture 4., $f_n \to 0$ in L^1, $f_n \nrightarrow 0$ a.e., and $f_n \overset{m}{\to} 0$.

Proof. Let $m > n$ then

$$|a_m - a_n| = \left|\sum_{k=n}^{m-1}(a_{k+1} - a_k)\right| \le \sum_{k=n}^{m-1}|a_{k+1} - a_k| \le \sum_{k=n}^{\infty}\epsilon_k \equiv \delta_n \quad . \tag{7.15}$$

So $|a_m - a_n| \le \delta_{\min(m,n)} \to 0$ as $, m, n \to \infty$, i.e. $\{a_n\}$ is Cauchy. Let $m \to \infty$ in (7.15) to find $|a - a_n| \le \delta_n$. ■

Theorem 7.16. *Suppose $\{f_n\}$ is L^0-Cauchy. Then there exists a subsequence $g_j = f_{n_j}$ of $\{f_n\}$ such that $\lim g_j \equiv f$ exists a.e. and $f_n \overset{\mu}{\longrightarrow} f$ as $n \to \infty$. Moreover if g is a measurable function such that $f_n \overset{\mu}{\longrightarrow} g$ as $n \to \infty$, then $f = g$ a.e.*

Proof. Let $\epsilon_n > 0$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$ ($\epsilon_n = 2^{-n}$ would do) and set $\delta_n = \sum_{k=n}^{\infty} \epsilon_k$. Choose $g_j = f_{n_j}$ such that $\{n_j\}$ is a subsequence of $\mathbb{N}$ and

$$\mu(\{|g_{j+1} - g_j| > \epsilon_j\}) \le \epsilon_j.$$

Let $E_j = \{|g_{j+1} - g_j| > \epsilon_j\}$,

$$F_N = \bigcup_{j=N}^{\infty} E_j = \bigcup_{j=N}^{\infty} \{|g_{j+1} - g_j| > \epsilon_j\}$$

and

$$E \equiv \bigcap_{N=1}^{\infty} F_N = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} E_j = \{|g_{j+1} - g_j| > \epsilon_j \text{ i.o.}\}.$$

Then $\mu(E) = 0$ since

$$\mu(E) \le \sum_{j=N}^{\infty} \mu(E_j) \le \sum_{j=N}^{\infty} \epsilon_j = \delta_N \to 0 \text{ as } N \to \infty.$$

For $x \notin F_N$, $|g_{j+1}(x) - g_j(x)| \le \epsilon_j$ for all $j \ge N$ and by Lemma 7.15, $f(x) = \lim\limits_{j\to\infty} g_j(x)$ exists and $|f(x) - g_j(x)| \le \delta_j$ for all $j \ge N$. Therefore, $\lim\limits_{j\to\infty} g_j(x) = f(x)$ exists for all $x \notin E$. Moreover, $\{x : |f(x) - f_j(x)| > \delta_j\} \subset F_j$ for all $j \ge N$ and hence

$$\mu(|f - g_j| > \delta_j) \le \mu(F_j) \le \delta_j \to 0 \text{ as } j \to \infty.$$

Therefore $g_j \xrightarrow{\mu} f$ as $j \to \infty$.

Since

$$\begin{aligned}\{|f_n - f| > \epsilon\} &= \{|f - g_j + g_j - f_n| > \epsilon\} \\ &\subset \{|f - g_j| > \epsilon/2\} \cup \{|g_j - f_n| > \epsilon/2\},\end{aligned}$$

$$\mu(\{|f_n - f| > \epsilon\}) \le \mu(\{|f - g_j| > \epsilon/2\}) + \mu(|g_j - f_n| > \epsilon/2)$$

and

$$\mu(\{|f_n - f| > \epsilon\}) \le \lim_{j\to\infty} \sup \mu(|g_j - f_n| > \epsilon/2) \to 0 \text{ as } n \to \infty.$$

If also $f_n \xrightarrow{\mu} g$ as $n \to \infty$, then arguing as above

$$\mu(|f - g| > \epsilon) \le \mu(\{|f - f_n| > \epsilon/2\}) + \mu(|g - f_n| > \epsilon/2) \to 0 \text{ as } n \to \infty.$$

Hence

$$\mu(|f - g| > 0) = \mu(\cup_{n=1}^{\infty}\{|f - g| > \frac{1}{n}\}) \le \sum_{n=1}^{\infty} \mu(|f - g| > \frac{1}{n}) = 0,$$

i.e. $f = g$ a.e. ■

Corollary 7.17 (Dominated Convergence Theorem)**.** *Suppose $\{f_n\}$, $\{g_n\}$, and g are in L^1 and $f \in L^0$ are functions such that*

$$|f_n| \le g_n \text{ a.e.}, \; f_n \xrightarrow{\mu} f, \; g_n \xrightarrow{\mu} g, \text{ and } \int g_n \to \int g \text{ as } n \to \infty.$$

Then $f \in L^1$ and $\lim_{n\to\infty} \|f - f_n\|_1 = 0$, i.e. $f_n \to f$ in L^1. In particular $\lim_{n\to\infty} \int f_n = \int f$.

Proof. First notice that $|f| \le g$ a.e. and hence $f \in L^1$ since $g \in L^1$. To see that $|f| \le g$, use Theorem 7.16 to find subsequences $\{f_{n_k}\}$ and $\{g_{n_k}\}$ of $\{f_n\}$ and $\{g_n\}$ respectively which are almost everywhere convergent. Then

$$|f| = \lim_{k\to\infty} |f_{n_k}| \le \lim_{k\to\infty} g_{n_k} = g \text{ a.e.}$$

If (for sake of contradiction) $\lim_{n\to\infty} \|f - f_n\|_1 \ne 0$ there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\int |f - f_{n_k}| \ge \epsilon \text{ for all } k. \tag{7.16}$$

Using Theorem 7.16 again, we may assume (by passing to a further subsequences if necessary) that $f_{n_k} \to f$ and $g_{n_k} \to g$ almost everywhere. Noting, $|f - f_{n_k}| \le g + g_{n_k} \to 2g$ and $\int (g + g_{n_k}) \to \int 2g$, an application of the dominated convergence Theorem 5.38 implies $\lim_{k\to\infty} \int |f - f_{n_k}| = 0$ which contradicts Eq. (7.16). ■

Exercise 7.1 (Fatou's Lemma)**.** If $f_n \ge 0$ and $f_n \to f$ in measure, then $\int f \le \liminf_{n\to\infty} \int f_n$.

Theorem 7.18 (Egoroff's Theorem)**.** *Suppose $\mu(X) < \infty$ and $f_n \to f$ a.e. Then for all $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. In particular $f_n \xrightarrow{\mu} f$ as $n \to \infty$.*

Proof. Let $f_n \to f$ a.e. Then $\mu(\{|f_n - f| > \frac{1}{k} \text{ i.o. } n\}) = 0$ for all $k > 0$, i.e.

$$\lim_{N\to\infty} \mu\left(\bigcup_{n\ge N} \{|f_n - f| > \frac{1}{k}\}\right) = \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n\ge N} \{|f_n - f| > \frac{1}{k}\}\right) = 0.$$

Let $E_k := \bigcup_{n\ge N_k} \{|f_n - f| > \frac{1}{k}\}$ and choose an increasing sequence $\{N_k\}_{k=1}^\infty$ such that $\mu(E_k) < \epsilon 2^{-k}$ for all k. Setting $E := \cup E_k$, $\mu(E) < \sum_k \epsilon 2^{-k} = \epsilon$ and if $x \notin E$, then $|f_n - f| \le \frac{1}{k}$ for all $n \ge N_k$ and all k. That is $f_n \to f$ uniformly on E^c. ■

Exercise 7.2. Show that Egoroff's Theorem remains valid when the assumption $\mu(X) < \infty$ is replaced by the assumption that $|f_n| \le g \in L^1$ for all n.

7.3. **Completeness of L^p – spaces.**

Theorem 7.19. *Let $\|\cdot\|_\infty$ be as defined in Eq. (7.2), then $(L^\infty(X, \mathcal{M}, \mu), \|\cdot\|_\infty)$ is a Banach space. A sequence $\{f_n\}_{n=1}^\infty \subset L^\infty$ converges to $f \in L^\infty$ iff there exists $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f_n \to f$ uniformly on E^c. Moreover, bounded simple functions are dense in L^∞.*

Proof. By Minkowski's Theorem 7.4, $\|\cdot\|_\infty$ satisfies the triangle inequality. The reader may easily check the remaining conditions that ensure $\|\cdot\|_\infty$ is a norm.

Suppose that $\{f_n\}_{n=1}^\infty \subset L^\infty$ is a sequence such $f_n \to f \in L^\infty$, i.e. $\|f - f_n\|_\infty \to 0$ as $n \to \infty$. Then for all $k \in \mathbb{N}$, there exists $N_k < \infty$ such that

$$\mu\left(|f - f_n| > k^{-1}\right) = 0 \text{ for all } n \ge N_k.$$

Let

$$E = \cup_{k=1}^\infty \cup_{n\ge N_k} \left\{|f - f_n| > k^{-1}\right\}.$$

Then $\mu(E) = 0$ and for $x \in E^c$, $|f(x) - f_n(x)| \leq k^{-1}$ for all $n \geq N_k$. This shows that $f_n \to f$ uniformly on E^c. Conversely, if there exists $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $f_n \to f$ uniformly on E^c, then for any $\epsilon > 0$,

$$\mu(|f - f_n| \geq \epsilon) = \mu(\{|f - f_n| \geq \epsilon\} \cap E^c) = 0$$

for all n sufficiently large. That is to say $\limsup_{n\to\infty} \|f - f_n\|_\infty \leq \epsilon$ for all $\epsilon > 0$. The density of simple functions follows from the approximation Theorem 5.12.

So the last item to prove is the completeness of L^∞ for which we will use Theorem 3.66. Suppose that $\{f_n\}_{n=1}^\infty \subset L^\infty$ is a sequence such that $\sum_{n=1}^\infty \|f_n\|_\infty < \infty$. Let $M_n := \|f_n\|_\infty$, $E_n := \{|f_n| > M_n\}$, and $E := \cup_{n=1}^\infty E_n$ so that $\mu(E) = 0$. Then

$$\sum_{n=1}^\infty \sup_{x\in E^c} |f_n(x)| \leq \sum_{n=1}^\infty M_n < \infty$$

which shows that $S_N(x) = \sum_{n=1}^N f_n(x)$ converges uniformly to $S(x) := \sum_{n=1}^\infty f_n(x)$ on E^c, i.e. $\lim_{n\to\infty} \|S - S_n\|_\infty = 0$.

Alternatively, suppose $\epsilon_{m,n} := \|f_m - f_n\|_\infty \to 0$ as $m, n \to \infty$. Let $E_{m,n} = \{|f_n - f_m| > \epsilon_{m,n}\}$ and $E := \cup E_{m,n}$, then $\mu(E) = 0$ and $\|f_m - f_n\|_{E^c,u} = \epsilon_{m,n} \to 0$ as $m, n \to \infty$. Therefore, $f := \lim_{n\to\infty} f_n$ exists on E^c and the limit is uniform on E^c. Letting $f = \limsup_{n\to\infty} f_n$, it then follows that $\|f_m - f\|_\infty \to 0$ as $m \to \infty$. ■

Theorem 7.20 (Completeness of $L^p(\mu)$). *For $1 \leq p \leq \infty$, $L^p(\mu)$ equipped with the L^p – norm, $\|\cdot\|_p$ (see Eq. (7.1)), is a Banach space.*

Proof. By Minkowski's Theorem 7.4, $\|\cdot\|_p$ satisfies the triangle inequality. As above the reader may easily check the remaining conditions that ensure $\|\cdot\|_p$ is a norm. So we are left to prove the completeness of $L^p(\mu)$ for $1 \leq p < \infty$, the case $p = \infty$ being done in Theorem 7.19. By Chebyshev's inequality (Lemma 7.14), $\{f_n\}$ is L^0-Cauchy (i.e. Cauchy in measure) and by Theorem 7.16 there exists a subsequence $\{g_j\}$ of $\{f_n\}$ such that $g_j \to f$ a.e. By Fatou's Lemma,

$$\begin{aligned} \|g_j - f\|_p^p &= \int \lim_{k\to\infty} \inf |g_j - g_k|^p d\mu \leq \lim_{k\to\infty} \inf \int |g_j - g_k|^p d\mu \\ &= \lim_{k\to\infty} \inf \|g_j - g_k\|_p^p \to 0 \text{ as } j \to \infty. \end{aligned}$$

In particular, $\|f\|_p \leq \|g_j - f\|_p + \|g_j\|_p < \infty$ so the $f \in L^p$ and $g_j \xrightarrow{L^p} f$. The proof is finished because,

$$\|f_n - f\|_p \leq \|f_n - g_j\|_p + \|g_j - f\|_p \to 0 \text{ as } j, n \to \infty.$$

■

The $L^p(\mu)$ – norm controls two types of behaviors of f, namely the "behavior at infinity" and the behavior of local singularities. So in particular, if f is blows up at a point $x_0 \in X$, then locally near x_0 it is harder for f to be in $L^p(\mu)$ as p increases. On the other hand a function $f \in L^p(\mu)$ is allowed to decay at "infinity" slower and slower as p increases. With these insights in mind, we should not in general expect $L^p(\mu) \subset L^q(\mu)$ or $L^q(\mu) \subset L^p(\mu)$. However, there are two notable exceptions. (1) If $\mu(X) < \infty$, then there is no behavior at infinity to worry about and $L^q(\mu) \subset L^p(\mu)$ for all $q \leq p$ as is shown in Corollary 7.21 below. (2) If μ is counting measure, i.e. $\mu(A) = \#(A)$, then all functions in $L^p(\mu)$ for any p can not blow up on a set of positive measure, so there are no local singularities. In this case $L^p(\mu) \subset L^q(\mu)$ for all $q \leq p$, see Corollary 7.25 below.

Corollary 7.21. *If $\mu(X) < \infty$, then $L^p(\mu) \subset L^q(\mu)$ for all $0 < p < q \le \infty$ and the inclusion map is bounded.*

Proof. Choose $a \in [1, \infty]$ such that

$$\frac{1}{p} = \frac{1}{a} + \frac{1}{q}, \text{ i.e. } a = \frac{pq}{q-p}.$$

Then by Corollary 7.3,

$$\|f\|_p = \|f \cdot 1\|_p \le \|f\|_q \cdot \|1\|_a = \mu(X)^{1/a} \|f\|_q = \mu(X)^{(\frac{1}{p} - \frac{1}{q})} \|f\|_q.$$

The reader may easily check this final formula is correct even when $q = \infty$ provided we interpret $1/p - 1/\infty$ to be $1/p$. ■

Proposition 7.22. *Suppose that $0 < p < q < r \le \infty$, then $L^q \subset L^p + L^r$, i.e. every function $f \in L^q$ may be written as $f = g + h$ with $g \in L^p$ and $h \in L^r$. For $1 \le p < r \le \infty$ and $f \in L^p + L^r$ let*

$$\|f\| := \inf \left\{ \|g\|_p + \|h\|_r : f = g + h \right\}.$$

Then $(L^p + L^r, \|\cdot\|)$ is a Banach space and the inclusion map from L^q to $L^p + L^r$ is bounded; in fact $\|f\| \le 2 \|f\|_q$ for all $f \in L^q$.

Proof. Let $M > 0$, then the local singularities of f are contained in the set $E := \{|f| > M\}$ and the behavior of f at "infinity" is solely determined by f on E^c. Hence let $g = f 1_E$ and $h = f 1_{E^c}$ so that $f = g + h$. By our earlier discussion we expect that $g \in L^p$ and $h \in L^r$ and this is the case since,

$$\begin{aligned} \|g\|_p^p &= \left\| f 1_{|f|>M} \right\|_p^p = \int |f|^p \, 1_{|f|>M} = M^p \int \left| \frac{f}{M} \right|^p 1_{|f|>M} \\ &\le M^p \int \left| \frac{f}{M} \right|^q 1_{|f|>M} \le M^{p-q} \|f\|_q^q \; < \infty \end{aligned}$$

and

$$\begin{aligned} \|h\|_r^r &= \left\| f 1_{|f|\le M} \right\|_r^r = \int |f|^r \, 1_{|f|\le M} = M^r \int \left| \frac{f}{M} \right|^r 1_{|f|\le M} \\ &\le M^r \int \left| \frac{f}{M} \right|^q 1_{|f|\le M} \le M^{r-q} \|f\|_q^q < \infty. \end{aligned}$$

Moreover this shows

$$\|f\| \le M^{1-q/p} \|f\|_q^{q/p} + M^{1-q/r} \|f\|_q^{q/r}.$$

Taking $M = \lambda \|f\|_q$ then gives

$$\|f\| \le \left(\lambda^{1-q/p} + \lambda^{1-q/r} \right) \|f\|_q$$

and then taking $\lambda = 1$ shows $\|f\| \le 2 \|f\|_q$. The the proof that $(L^p + L^r, \|\cdot\|)$ is a Banach space is left as Exercise 7.7 to the reader. ■

Corollary 7.23. *Suppose that $0 < p < q < r \le \infty$, then $L^p \cap L^r \subset L^q$ and*

$$\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda} \tag{7.17}$$

where $\lambda \in (0, 1)$ is determined so that

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r} \text{ with } \lambda = p/q \text{ if } r = \infty.$$

Further assume $1 \le p < q < r \le \infty$, *and for* $f \in L^p \cap L^r$ *let*

$$\|f\| := \|f\|_p + \|f\|_r .$$

Then $(L^p \cap L^r, \|\cdot\|)$ *is a Banach space and the inclusion map of* $L^p \cap L^r$ *into* L^q *is bounded, in fact*

$$\|f\|_q \le \max\left(\lambda^{-1}, (1-\lambda)^{-1}\right)\left(\|f\|_p + \|f\|_r\right), \tag{7.18}$$

where

$$\lambda = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r}} = \frac{p(r-q)}{q(r-p)}.$$

The heuristic explanation of this corollary is that if $f \in L^p \cap L^r$, then f has local singularities no worse than an L^r function and behavior at infinity no worse than an L^p function. Hence $f \in L^q$ for any q between p and r.

Proof. Let λ be determined as above, $a = p/\lambda$ and $b = r/(1-\lambda)$, then by Corollary 7.3,

$$\|f\|_q = \left\| |f|^\lambda |f|^{1-\lambda} \right\|_q \le \left\| |f|^\lambda \right\|_a \left\| |f|^{1-\lambda} \right\|_b = \|f\|_p^\lambda \|f\|_r^{1-\lambda} .$$

It is easily checked that $\|\cdot\|$ is a norm on $L^p \cap L^r$. To show this space is complete, suppose that $\{f_n\} \subset L^p \cap L^r$ is a $\|\cdot\|$ – Cauchy sequence. Then $\{f_n\}$ is both L^p and L^r – Cauchy. Hence there exist $f \in L^p$ and $g \in L^r$ such that $\lim_{n\to\infty} \|f - f_n\|_p = 0$ and $\lim_{n\to\infty} \|g - f_n\|_q = 0$. By Chebyshev's inequality (Lemma 7.14) $f_n \to f$ and $f_n \to g$ in measure and therefore by Theorem 7.16, $f = g$ a.e. It now is clear that $\lim_{n\to\infty} \|f - f_n\| = 0$. The estimate in Eq. (7.18) is left as Exercise 7.6 to the reader. ■

Remark 7.24. Let $p = p_1$, $r = p_0$ and for $\lambda \in (0,1)$ let p_λ be defined by

$$\frac{1}{p_\lambda} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}. \tag{7.19}$$

Combining Proposition 7.22 and Corollary 7.23 gives

$$L^{p_0} \cap L^{p_1} \subset L^{p_\lambda} \subset L^{p_0} + L^{p_1}$$

and Eq. (7.17) becomes

$$\|f\|_{p_\lambda} \le \|f\|_{p_0}^{1-\lambda} \|f\|_{p_1}^{\lambda} .$$

Corollary 7.25. *Suppose now that* μ *is counting measure on* X. *Then* $L^p(\mu) \subset L^q(\mu)$ *for all* $0 < p < q \le \infty$ *and* $\|f\|_q \le \|f\|_p$.

Proof. Suppose that $0 < p < q = \infty$, then

$$\|f\|_\infty^p = \sup\{|f(x)|^p : x \in X\} \le \sum_{x\in X} |f(x)|^p = \|f\|_p^p ,$$

i.e. $\|f\|_\infty \le \|f\|_p$ for all $0 < p < \infty$. For $0 < p \le q \le \infty$, apply Corollary 7.23 with $r = \infty$ to find

$$\|f\|_q \le \|f\|_p^{p/q} \|f\|_\infty^{1-p/q} \le \|f\|_p^{p/q} \|f\|_p^{1-p/q} = \|f\|_p .$$

■

7.3.1. *Summary:*

1. Since $\mu(|f| > \epsilon) \le \epsilon^{-p} \|f\|_p^p$ it follows that L^p – convergence implies L^0 – convergence.
2. L^0 – convergence implies almost everywhere convergence for some subsequence.
3. If $\mu(X) < \infty$, then $L^q \subset L^p$ for all $p \le q$ in fact
$$\|f\|_p \le [\mu(X)]^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_q ,$$
i.e. L^q – convergence implies L^p – convergence.
4. $L^{p_0} \cap L^{p_1} \subset L^{p_\lambda} \subset L^{p_0} + L^{p_1}$ where
$$\frac{1}{p_\lambda} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}.$$
5. $\ell^p \subset \ell^q$ if $p \le q$. In fact $\|f\|_q \le \|f\|_p$ in this case. To prove this write
$$\frac{1}{q} = \frac{\lambda}{p} + \frac{(1-\lambda)}{\infty},$$
then using $\|f\|_\infty \le \|f\|_p$ for all p,
$$\|f\|_q \le \|f\|_p^\lambda \|f\|_\infty^{1-\lambda} \le \|f\|_p^\lambda \|f\|_p^{1-\lambda} = \|f\|_p .$$
6. If $\mu(X) < \infty$ then almost everywhere convergence implies L^0 – convergence.

7.4. **Converse of Hölder's Inequality.** Throughout this section we assume $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, $q \in [1, \infty]$ and $p \in [1, \infty]$ are conjugate exponents, i.e. $p^{-1} + q^{-1} = 1$. For $g \in L^q$, let $\phi_g \in (L^p)^*$ be given by

$$\phi_g(f) = \int gf \, d\mu. \tag{7.20}$$

By Hölder's inequality

$$|\phi_g(f)| \le \int |gf| d\mu \le \|g\|_q \|f\|_p \tag{7.21}$$

which implies that

$$\|\phi_g\|_{(L^p)^*} := \sup\{|\phi_g(f)| : \|f\|_p = 1\} \le \|g\|_q. \tag{7.22}$$

Proposition 7.26 (Converse of Hölder's Inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $1 \le p \le \infty$ as above. For all $g \in L^q$,*

$$\|g\|_q = \|\phi_g\|_{(L^p)^*} := \sup\left\{|\phi_g(f)| : \|f\|_p = 1\right\} \tag{7.23}$$

and for any measurable function $g : X \to \mathbb{C}$,

$$\|g\|_q = \sup\left\{\int_X |g| \, f d\mu : \|f\|_p = 1 \text{ and } f \ge 0\right\}. \tag{7.24}$$

Proof. We begin by proving Eq. (7.23). Assume first that $q < \infty$ so $p > 1$. Then

$$|\phi_g(f)| = \left|\int gf \, d\mu\right| \le \int |gf| \, d\mu \le \|g\|_q \|f\|_p$$

and equality occurs in the first inequality when $\operatorname{sgn}(gf)$ is constant a.e. while equality in the second occurs, by Theorem 7.2, when $|f|^p = c|g|^q$ for some constant

$c > 0$. So let $f := \overline{\operatorname{sgn}(g)}|g|^{q/p}$ which for $p = \infty$ is to be interpreted as $f = \overline{\operatorname{sgn}(g)}$, i.e. $|g|^{q/\infty} \equiv 1$.

When $p = \infty$,

$$|\phi_g(f)| = \int_X g\,\overline{\operatorname{sgn}(g)}d\mu = \|g\|_{L^1(\mu)} = \|g\|_1 \|f\|_\infty$$

which shows that $\|\phi_g\|_{(L^\infty)^*} \geq \|g\|_1$. If $p < \infty$, then

$$\|f\|_p^p = \int |f|^p = \int |g|^q = \|g\|_q^q$$

while

$$\phi_g(f) = \int gf d\mu = \int |g||g|^{q/p} d\mu = \int |g|^q d\mu = \|g\|_q^q.$$

Hence

$$\frac{|\phi_g(f)|}{\|f\|_p} = \frac{\|g\|_q^q}{\|g\|_q^{q/p}} = \|g\|_q^{q(1-\frac{1}{p})} = \|g\|_q.$$

This shows that $||\phi_g|| \geq \|g\|_q$ which combined with Eq. (7.22) implies Eq. (7.23).

The last case to consider is $p = 1$ and $q = \infty$. Let $M := \|g\|_\infty$ and choose $X_n \in \mathcal{M}$ such that $X_n \uparrow X$ as $n \to \infty$ and $\mu(X_n) < \infty$ for all n. For any $\epsilon > 0$, $\mu(|g| \geq M - \epsilon) > 0$ and $X_n \cap \{|g| \geq M - \epsilon\} \uparrow \{|g| \geq M - \epsilon\}$. Therefore, $\mu(X_n \cap \{|g| \geq M - \epsilon\}) > 0$ for n sufficiently large. Let

$$f = \overline{\operatorname{sgn}(g)} 1_{X_n \cap \{|g| \geq M-\epsilon\}},$$

then

$$\|f\|_1 = \mu(X_n \cap \{|g| \geq M - \epsilon\}) \in (0, \infty)$$

and

$$\begin{aligned} |\phi_g(f)| &= \int_{X_n \cap \{|g| \geq M-\epsilon\}} \overline{\operatorname{sgn}(g)} g d\mu = \int_{X_n \cap \{|g| \geq M-\epsilon\}} |g| d\mu \\ &\geq (M - \epsilon)\mu(X_n \cap \{|g| \geq M - \epsilon\}) = (M - \epsilon)\|f\|_1. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows from this equation that $\|\phi_g\|_{(L^1)^*} \geq M = \|g\|_\infty$.

We now will prove Eq. (7.24). The key new point is that we no longer are assuming the $g \in L^q$. Let $M(g)$ denote the right member in Eq. (7.24) and set $g_n := 1_{X_n \cap \{|g| \leq n\}} g$. Then $|g_n| \uparrow |g|$ as $n \to \infty$ and it is clear that $M(g_n)$ is increasing in n. Therefore using Lemma 2.10 and the monotone convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} M(g_n) &= \sup_n M(g_n) = \sup_n \sup \left\{ \int_X |g_n|\, f d\mu : \|f\|_p = 1 \text{ and } f \geq 0 \right\} \\ &= \sup \left\{ \sup_n \int_X |g_n|\, f d\mu : \|f\|_p = 1 \text{ and } f \geq 0 \right\} \\ &= \sup \left\{ \lim_{n\to\infty} \int_X |g_n|\, f d\mu : \|f\|_p = 1 \text{ and } f \geq 0 \right\} \\ &= \sup \left\{ \int_X |g|\, f d\mu : \|f\|_p = 1 \text{ and } f \geq 0 \right\} = M(g). \end{aligned}$$

Since $g_n \in L^q$ for all n and $M(g_n) = \|\phi_{g_n}\|_{(L^p)^*}$ (as you should verify), it follows from Eq. (7.23) that $M(g_n) = \|g_n\|_q$. When $q < \infty$, by the monotone convergence theorem, and when $q = \infty$, directly from the definitions, one learns that $\lim_{n\to\infty} \|g_n\|_q = \|g\|_q$. Combining this fact with $\lim_{n\to\infty} M(g_n) = M(g)$ just proved shows $M(g) = \|g\|_q$. ■

As an application we can derive a sweeping generalization of Minkowski's inequality. (See Reed and Simon, Vol II. Appendix IX.4 for a more thorough discussion of complex interpolation theory.)

Theorem 7.27 (Minkowski's Inequality for Integrals)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and $1 \le p \le \infty$. If f is a $\mathcal{M} \otimes \mathcal{N}$ measurable function, then $y \to \|f(\cdot, y)\|_{L^p(\mu)}$ is measurable and*

1. *if f is a positive $\mathcal{M} \otimes \mathcal{N}$ measurable function, then*

$$\| \int_Y f(\cdot, y) d\nu(y)\|_{L^p(\mu)} \le \int_Y \|f(\cdot, y)\|_{L^p(\mu)} d\nu(y). \tag{7.25}$$

2. *If $f : X \times Y \to \mathbb{C}$ is a $\mathcal{M} \otimes \mathcal{N}$ measurable function and $\int_Y \|f(\cdot, y)\|_{L^p(\mu)} d\nu(y) < \infty$ then*
   (a) *for μ – a.e. x, $f(x, \cdot) \in L^1(\nu)$,*
   (b) *the μ –a.e. defined function, $x \to \int_Y f(x, y) d\nu(y)$, is in $L^p(\mu)$ and*
   (c) *the bound in Eq. (7.25) holds.*

Proof. For $p \in [1, \infty]$, let $F_p(y) := \|f(\cdot, y)\|_{L^p(\mu)}$. If $p \in [1, \infty)$

$$F_p(y) = \|f(\cdot, y)\|_{L^p(\mu)} = \left(\int_X |f(x, y)|^p \, d\mu(x)\right)^{1/p}$$

is a measurable function on Y by Fubini's theorem. To see that F_∞ is measurable, let $X_n \in \mathcal{M}$ such that $X_n \uparrow X$ and $\mu(X_n) < \infty$ for all n. Then by Exercise 7.5,

$$F_\infty(y) = \lim_{n\to\infty} \lim_{p\to\infty} \|f(\cdot, y) 1_{X_n}\|_{L^p(\mu)}$$

which shows that F_∞ is $(Y, \mathcal{N})$ – measurable as well. This shows that integral on the right side of Eq. (7.25) is well defined.

Now suppose that $f \ge 0$, $q = p/(p-1)$and $g \in L^q(\mu)$ such that $g \ge 0$ and $\|g\|_{L^q(\mu)} = 1$. Then by Tonelli's theorem and Hölder's inequality,

$$\begin{aligned} \int_X \left[\int_Y f(x, y) d\nu(y)\right] g(x) d\mu(x) &= \int_Y d\nu(y) \int_X d\mu(x) f(x, y) g(x) \\ &\le \|g\|_{L^q(\mu)} \int_Y \|f(\cdot, y)\|_{L^p(\mu)} d\nu(y) \\ &= \int_Y \|f(\cdot, y)\|_{L^p(\mu)} d\nu(y). \end{aligned}$$

Therefore by Proposition 7.26,

$$\begin{aligned} \| \int_Y f(\cdot, y) d\nu(y)\|_{L^p(\mu)} &= \sup\left\{\int_X \left[\int_Y f(x, y) d\nu(y)\right] g(x) d\mu(x) : \|g\|_{L^q(\mu)} = 1 \text{ and } g \ge 0\right\} \\ &\le \int_Y \|f(\cdot, y)\|_{L^p(\mu)} d\nu(y) \end{aligned}$$

proving Eq. (7.25) in this case.

Now let $f : X \times Y \to \mathbb{C}$ be as in item 2) of the theorem. Applying the first part of the theorem to $|f|$ shows

$$\int_Y |f(x, y)| \, d\nu(y) < \infty \text{ for } \mu\text{– a.e. } x,$$

i.e. $f(x,\cdot) \in L^1(\nu)$ for the μ –a.e. x. Since $\left|\int_Y f(x,y)d\nu(y)\right| \le \int_Y |f(x,y)|\, d\nu(y)$ it follows by item 1) that

$$\|\int_Y f(\cdot,y)d\nu(y)\|_{L^p(\mu)} \le \|\int_Y |f(\cdot,y)|\, d\nu(y)\|_{L^p(\mu)} \le \int_Y \|f(\cdot,y)\|_{L^p(\mu)} d\nu(y).$$

Hence the function, $x \in X \to \int_Y f(x,y)d\nu(y)$, is in $L^p(\mu)$ and the bound in Eq. (7.25) holds. ■

Here is an application of Minkowski's inequality for integrals.

Theorem 7.28 (Theorem 6.20 in Folland)**.** *Suppose that $k : (0,\infty)\times(0,\infty) \to \mathbb{C}$ is a measurable function such that k is homogenous of degree -1, i.e. $k(\lambda x, \lambda y) = \lambda^{-1}k(x,y)$ for all $\lambda > 0$. If*

$$C_p := \int_0^\infty |k(x,1)|\, x^{-1/p} dx < \infty$$

for some $p \in [1,\infty]$, then for $f \in L^p((0,\infty),m)$, $k(x,\cdot)f(\cdot) \in L^p((0,\infty),m)$ for m – a.e. x Moreover, the m – a.e. defined function

$$(Kf)(x) = \int_0^\infty k(x,y)f(y)dy \tag{7.26}$$

is in $L^p((0,\infty),m)$ and

$$\|Kf\|_{L^p((0,\infty),m)} \le C_p \|f\|_{L^p((0,\infty),m)}.$$

Proof. By the homogeneity of k, $k(x,y) = y^{-1}k(\frac{x}{y},1)$. Hence

$$\begin{aligned}\int_0^\infty |k(x,y)f(y)|\, dy &= \int_0^\infty x^{-1}\, |k(1,y/x)f(y)|\, dy \\ &= \int_0^\infty x^{-1}\, |k(1,z)f(xz)|\, xdz = \int_0^\infty |k(1,z)f(xz)|\, dz.\end{aligned}$$

Since

$$\|f(\cdot\ z)\|^p_{L^p((0,\infty),m)} = \int_0^\infty |f(yz)|^p dy = \int_0^\infty |f(x)|^p\ \frac{dx}{z},$$

$$\|f(\cdot\ z)\|_{L^p((0,\infty),m)} = z^{-1/p}\|f\|_{L^p((0,\infty),m)}.$$

Using Minkowski's inequality for integrals then shows

$$\begin{aligned}\left\|\int_0^\infty |k(\cdot,y)f(y)|\, dy\right\|_{L^p((0,\infty),m)} &\le \int_0^\infty |k(1,z)|\, \|f(\cdot z)\|_{L^p((0,\infty),m)}\, dz \\ &= \|f\|_{L^p((0,\infty),m)} \int_0^\infty |k(1,z)|\, z^{-1/p}dz \\ &= C_p\|f\|_{L^p((0,\infty),m)} < \infty.\end{aligned}$$

This shows that Kf in Eq. (7.26) is well defined from m – a.e. x. The proof is finished by observing

$$\|Kf\|_{L^p((0,\infty),m)} \le \left\|\int_0^\infty |k(\cdot,y)f(y)|\, dy\right\|_{L^p((0,\infty),m)} \le C_p\|f\|_{L^p((0,\infty),m)}$$

for all $f \in L^p((0,\infty),m)$. ■

7.5. **Uniform Integrability.** This section will address the question as to what extra conditions are needed in order that an L^0 – convergent sequence is L^p – convergent.

Notation 7.29. For $f \in L^1(\mu)$ and $E \in \mathcal{M}$, let

$$\mu(f : E) := \int_E f d\mu.$$

and more generally if $A, B \in \mathcal{M}$ let

$$\mu(f : A, B) := \int_{A\cap B} f d\mu.$$

Lemma 7.30. *Suppose $g \in L^1(\mu)$, then for any $\epsilon > 0$ there exist a $\delta > 0$ such that $\mu(|g| : E) < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. If the Lemma is false, there would exist $\epsilon > 0$ and sets E_n such that $\mu(E_n) \to 0$ while $\mu(|g| : E_n) \geq \epsilon$ for all n. Since $|1_{E_n} g| \leq |g| \in L^1$ and for any $\delta \in (0, 1)$, $\mu(1_{E_n} |g| > \delta) \leq \mu(E_n) \to 0$ as $n \to \infty$, the dominated convergence theorem of Corollary 7.17 implies $\lim_{n\to\infty} \mu(|g| : E_n) = 0$. This contradicts $\mu(|g| : E_n) \geq \epsilon$ for all n and the proof is complete. ■

Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions which converge in $L^1(\mu)$ to a function f. Then for $E \in \mathcal{M}$ and $n \in \mathbb{N}$,

$$|\mu(f_n : E)| \leq |\mu(f - f_n : E)| + |\mu(f : E)| \leq \|f - f_n\|_1 + |\mu(f : E)|.$$

Let $\epsilon_N := \sup_{n>N} \|f - f_n\|_1$, then $\epsilon_N \downarrow 0$ as $N \uparrow \infty$ and

$$\sup_n |\mu(f_n : E)| \leq \sup_{n\leq N} |\mu(f_n : E)| \vee (\epsilon_N + |\mu(f : E)|) \leq \epsilon_N + \mu(g_N : E), \tag{7.27}$$

where $g_N = |f| + \sum_{n=1}^N |f_n| \in L^1$. From Lemma 7.30 and Eq. (7.27) one easily concludes,

$$\forall\ \epsilon > 0\ \exists\ \delta > 0\ \ni\ \sup_n |\mu(f_n : E)| < \epsilon \text{ when } \mu(E) < \delta. \tag{7.28}$$

Definition 7.31. Functions $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ satisfying Eq. (7.28) are said to be *uniformly integrable.*

Remark 7.32. Let $\{f_n\}$ be real functions satisfying Eq. (7.28), E be a set where $\mu(E) < \delta$ and $E_n = E \cap \{f_n \geq 0\}$. Then $\mu(E_n) < \delta$ so that $\mu(f_n^+ : E) = \mu(f_n : E_n) < \epsilon$ and similarly $\mu(f_n^- : E) < \epsilon$. Therefore if Eq. (7.28) holds then

$$\sup_n \mu(|f_n| : E) < 2\epsilon \text{ when } \mu(E) < \delta. \tag{7.29}$$

Similar arguments work for the complex case by looking at the real and imaginary parts of f_n. Therefore $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ is uniformly integrable iff

$$\forall\ \epsilon > 0\ \exists\ \delta > 0\ \ni\ \sup_n \mu(|f_n| : E) < \epsilon \text{ when } \mu(E) < \delta. \tag{7.30}$$

Lemma 7.33. *Assume that $\mu(X) < \infty$ and $\{f_n\}$ is uniformly bounded sequence in $L^1(\mu)$ (i.e. $K = \sup_n \|f_n\|_1 < \infty$), then $\{f_n\}$ is uniformly integrable iff*

$$\lim_{M\to\infty} \sup_n \mu(|f_n| : |f_n| \geq M) = 0. \tag{7.31}$$

Proof. Suppose that (7.30) holds, then

$$\mu(|f_n| \geq M) \leq K/M < \delta$$

for M sufficiently large. This shows that

$$\sup_n \mu(|f_n| : |f_n| \geq M) \leq \epsilon.$$

Since ϵ is arbitrary, we concluded that Eq. (7.31) must hold.

Conversely, suppose that Eq. (7.31) holds, then automatically $K = \sup_n \mu(|f_n|) < \infty$ because

$$\begin{aligned}\mu(|f_n|) &= \mu(|f_n| : |f_n| \geq M) + \mu(|f_n| : |f_n| < M) \\ &\leq \sup_n \mu(|f_n| : |f_n| \geq M) + M\mu(X) < \infty.\end{aligned}$$

Moreover,

$$\begin{aligned}\mu(|f_n| : E) &= \mu(|f_n| : |f_n| \geq M, E) + \mu(|f_n| : |f_n| < M, E) \\ &\leq \sup_n \mu(|f_n| : |f_n| \geq M) + M\mu(E).\end{aligned}$$

So given $\epsilon > 0$ choose M so large that $\sup_n \mu(|f_n| : |f_n| \geq M) < \epsilon/2$ and then take $\delta = \epsilon/(2M)$. ∎

Remark 7.34. It is not in general true that if $\{f_n\} \subset L^1(\mu)$ is uniformly integrable then $\sup_n \mu(|f_n|) < \infty$. For example take $X = \{*\}$ and $\mu(\{*\}) = 1$. Let $f_n(*) = n$. Since for $\delta < 1$ a set $E \subset X$ such that $\mu(E) < \delta$ is in fact the empty set, we see that Eq. (7.29) holds in this example. However, for finite measure spaces with out "atoms", for every $\delta > 0$ we may find a finite partition of X by sets $\{E_\ell\}_{\ell=1}^k$ with $\mu(E_\ell) < \delta$. Then if Eq. (7.29) holds with $2\epsilon = 1$, then

$$\mu(|f_n|) = \sum_{\ell=1}^k \mu(|f_n| : E_\ell) \leq k$$

showing that $\mu(|f_n|) \leq k$ for all n.

The following Lemma gives a concrete necessary condition for verifying a sequence of functions is uniformly integrable.

Lemma 7.35. *Suppose that $\mu(X) < \infty$, $\phi(x) \geq 0$ is a strictly monotonically increasing function on $\mathbb{R}_+$ such that $\lim_{x\to\infty} \phi(x) = \infty$. Suppose that $\{f_n\}$ is a sequence of measurable functions such that*

$$\sup_n \mu(|f_n|\phi(|f_n|)) = K < \infty.$$

Then $\{f_n\}_{n=1}^\infty$ is uniformly integrable, and in fact

$$\sup_n \mu(|f_n| : |f_n| \geq M) \leq K/\phi(M)$$

which implies Eq. (7.31).

Proof. Let $M \in (0, \infty)$, then

$$\begin{aligned}\mu(|f_n| : |f_n| \geq M) &= \mu(|f_n| : \{\phi(|f_n|) \geq \phi(M)\}) \\ &\leq \mu(|f_n|\frac{\phi(|f_n|)}{\phi(M)}) \leq K/\phi(M).\end{aligned}$$

From this inequality it is clear that $\{f_n\}$ is uniformly integrable. ∎

Theorem 7.36 (Vitali Convergence Theorem). *(Folland 6.15) Suppose that $1 \leq p < \infty$. A sequence $\{f_n\} \subset L^p$ is Cauchy iff*

1. *$\{f_n\}$ is L^0 – Cauchy,*
2. *$\{|f_n|^p\}$ – is uniformly integrable.*
3. *For all $\epsilon > 0$, there exits a set $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n|^p \, d\mu < \epsilon$ for all n. (This condition is vacuous when $\mu(X) < \infty$.)*

Proof. ($\Longrightarrow$) Suppose $\{f_n\} \subset L^p$ is Cauchy. Then (1) $\{f_n\}$ is L^0 – Cauchy by Lemma 7.14. (2) By completeness of L^p, there exists $f \in L^p$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. By the mean value theorem,

$$||f|^p - |f_n|^p| \leq p(\max(|f|, |f_n|))^{p-1} \, ||f| - |f_n|| \leq p(|f| + |f_n|)^{p-1} \, ||f| - |f_n||$$

and therefore by Hölder's inequality,

$$\begin{aligned}\int ||f|^p - |f_n|^p| \, d\mu &\leq p \int (|f| + |f_n|)^{p-1} \, ||f| - |f_n|| \, d\mu \leq p \int (|f| + |f_n|)^{p-1} |f - f_n| d\mu \\ &\leq p\|f - f_n\|_p \|(|f| + |f_n|)^{p-1}\|_q = p\||f| + |f_n|\|_p^{p/q} \|f - f_n\|_p \\ &\leq p(\|f\|_p + \|f_n\|_p)^{p/q} \|f - f_n\|_p\end{aligned}$$

where $q := p/(p-1)$. This shows that $\int ||f|^p - |f_n|^p| \, d\mu \to 0$ as $n \to \infty$.[15] By the remarks prior to Definition 7.31, $\{|f_n|^p\}$ is uniformly integrable.

To verify (3), for $M > 0$ and $n \in \mathbb{N}$ let $E_M = \{|f| \geq M\}$ and $E_M(n) = \{|f_n| \geq M\}$. Then $\mu(E_M) \leq \frac{1}{M^p} \|f\|_p^p < \infty$ and by the dominated convergence theorem,

$$\int_{E_M^c} |f|^p d\mu = \int |f|^p 1_{|f| < M} d\mu \to 0 \text{ as } M \to 0.$$

Moreover,

$$\left\|f_n 1_{E_M^c}\right\|_p \leq \left\|f 1_{E_M^c}\right\|_p + \left\|(f_n - f) 1_{E_M^c}\right\|_p \leq \left\|f 1_{E_M^c}\right\|_p + \|f_n - f\|_p . \tag{7.32}$$

So given $\epsilon > 0$, choose N sufficiently large such that for all $n \geq N$, $\|f - f_n\|_p^p < \epsilon$. Then choose M sufficiently small such that $\int_{E_M^c} |f|^p \, d\mu < \epsilon$ and $\int_{E_M^c(n)} |f|^p \, d\mu < \epsilon$ for all $n = 1, 2, \dots, N-1$. Letting $E \equiv E_M \cup E_M(1) \cup \dots \cup E_M(N-1)$, we have

$$\mu(E) < \infty, \ \int_{E^c} |f_n|^p \, d\mu < \epsilon \text{ for } n \leq N-1$$

and by Eq. (7.32)

$$\int_{E^c} |f_n|^p \, d\mu < (\epsilon^{1/p} + \epsilon^{1/p})^p \leq 2^p \epsilon \text{ for } n \geq N.$$

Therefore we have found $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and

$$\sup_n \int_{E^c} |f_n|^p \, d\mu \leq 2^p \epsilon$$

which verifies (3) since $\epsilon > 0$ was arbitrary.

($\Longleftarrow$) Now suppose $\{f_n\} \subset L^p$ satisfies conditions (1) - (3). Let $\epsilon > 0$, E be as in (3) and

$$A_{mn} \equiv \{x \in E | f_m(x) - f_n(x)| \geq \epsilon\}.$$

[15] Here is an alternative proof. Let $h_n \equiv ||f_n|^p - |f|^p| \leq |f_n|^p + |f|^p =: g_n \in L^1$ and $g \equiv 2|f|^p$. Then $g_n \xrightarrow{\mu} g$, $h_n \xrightarrow{\mu} 0$ and $\int g_n \to \int g$. Therefore by the dominated convergence theorem in Corollary 7.17, $\lim_{n\to\infty} \int h_n \, d\mu = 0$.

Then

$$\|(f_n - f_m)\ 1_{E^c}\|_p \le \|f_n 1_{E^c}\|_p + \|f_m\, 1_{E^c}\|_p < 2\epsilon^{1/p}$$

and

$$\begin{aligned} \|f_n - f_m\|_p &= \|(f_n - f_m)1_{E^c}\|_p + \|(f_n - f_m)1_{E\setminus A_{mn}}\|_p \\ &\quad + \|(f_n - f_m)1_{A_{mn}}\|_p \\ &\le \|(f_n - f_m)1_{E\setminus A_{mn}}\|_p + \|(f_n - f_m)1_{A_{mn}}\|_p + 2\epsilon^{1/p}. \end{aligned} \tag{7.33}$$

Using properties (1) and (3) and $1_{E\cap\{|f_m - f_n|<\epsilon\}}|f_m - f_n|^p \le \epsilon^p 1_E \in L^1$, the dominated convergence theorem in Corollary 7.17 implies

$$\|(f_n - f_m)\ 1_{E\setminus A_{mn}}\|_p^p = \int 1_{E\cap\{|f_m - f_n|<\epsilon\}}\ |f_m - f_n|^p \underset{m,n\to\infty}{\longrightarrow} 0.$$

which combined with Eq. (7.33) implies

$$\limsup_{m,n\to\infty} \|f_n - f_m\|_p \le \limsup_{m,n\to\infty} \|(f_n - f_m)1_{A_{mn}}\|_p + 2\epsilon^{1/p}.$$

Finally

$$\|(f_n - f_m)1_{A_{mn}}\|_p \le \|f_n 1_{A_{mn}}\|_p + \|f_m\ 1_{A_{mn}}\|_p \le 2\delta(\epsilon)$$

where

$$\delta(\epsilon) \equiv \sup_n\ \sup\{\ \|f_n\ 1_E\|_p : E \in \mathcal{M}\ \ni \mu(E) \le \epsilon\}$$

By property (2), $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore

$$\limsup_{m,n\to\infty} \|f_n - f_m\|_p \le 2\epsilon^{1/p} + 0 + 2\delta(\epsilon) \to 0 \text{ as } \epsilon \downarrow 0$$

and therefore $\{f_n\}$ is L^p-Cauchy. ■

Here is another version of Vitali's Convergence Theorem.

Theorem 7.37 (Vitali Convergence Theorem)**.** *(This is problem 9 on p. 133 in Rudin.) Assume that $\mu(X) < \infty$, $\{f_n\}$ is uniformly integrable, $f_n \to f$ a.e. and $|f| < \infty$ a.e., then $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$.*

Proof. Let $\epsilon > 0$ be given and choose $\delta > 0$ as in the Eq. (7.29). Now use Egoroff's Theorem 7.18 to choose a set E^c where $\{f_n\}$ converges uniformly on E^c and $\mu(E) < \delta$. By uniform convergence on E^c, there is an integer $N < \infty$ such that $|f_n - f_m| \le 1$ on E^c for all $m, n \ge N$. Letting $m \to \infty$, we learn that

$$|f_N - f| \le 1 \text{ on } E^c.$$

Therefore $|f| \le |f_N| + 1$ on E^c and hence

$$\begin{aligned} \mu(|f|) &= \mu(|f| : E^c) + \mu(|f| : E) \\ &\le \mu(|f_N|) + \mu(X) + \mu(|f| : E). \end{aligned}$$

Now by Fatou's lemma,

$$\mu(|f| : E) \le \lim\inf_{n\to\infty} \mu(|f_n| : E) \le 2\epsilon < \infty$$

by Eq. (7.29). This shows that $f \in L^1$. Finally

$$\begin{aligned} \mu(|f - f_n|) &= \mu(|f - f_n| : E^c) + \mu(|f - f_n| : E) \\ &\le \mu(|f - f_n| : E^c) + \mu(|f| + |f_n| : E) \\ &\le \mu(|f - f_n| : E^c) + 4\epsilon \end{aligned}$$

and so by the Dominated convergence theorem we learn that

$$\limsup_{n\to\infty} \mu(|f - f_n|) \le 4\epsilon.$$

Since $\epsilon > 0$ was arbitrary this completes the proof. ■

Theorem 7.38 (Vitali again)**.** *Suppose that $f_n \to f$ in μ measure and Eq. (7.31) holds, then $f_n \to f$ in L^1.*

Proof. This could of course be proved using 7.37 after passing to subsequences to get $\{f_n\}$ to converge a.s. However I wish to give another proof. By Fatou's lemma $f \in L^1(\mu)$. Now let

$$\phi_K(x) = x1_{|x|\le K} + K1_{|x|>K}.$$

then $\phi_K(f_n) \xrightarrow{\mu} \phi_K(f)$ because $|\phi_K(f) - \phi_K(f_n)| \le |f - f_n|$ and since

$$|f - f_n| \le |f - \phi_K(f)| + |\phi_K(f) - \phi_K(f_n)| + |\phi_K(f_n) - f_n|$$

we have that

$$\begin{aligned}\mu|f - f_n| &\le \mu\,|f - \phi_K(f)| + \mu|\phi_K(f) - \phi_K(f_n)| + \mu\,|\phi_K(f_n) - f_n| \\ &= \mu(|f| : |f| \ge K) + \mu|\phi_K(f) - \phi_K(f_n)| + \mu(|f_n| : |f_n| \ge K).\end{aligned}$$

Therefore by the dominated convergence theorem

$$\limsup_{n\to\infty} \mu|f - f_n| \le \mu(|f| : |f| \ge K) + \limsup_{n\to\infty} \mu(|f_n| : |f_n| \ge K).$$

This last expression goes to zero as $K \to \infty$ by uniform integrability. ■

7.6. **Exercises.**

Definition 7.39. The **essential range** of f, essran(f), consists of those $\lambda \in \mathbb{C}$ such that $\mu(|f - \lambda| < \epsilon) > 0$ for all $\epsilon > 0$.

Definition 7.40. Let (X, τ) be a topological space and ν be a measure on $\mathcal{B}_X = \sigma(\tau)$. The **support** of ν, supp(ν), consists of those $x \in X$ such that $\nu(V) > 0$ for all open neighborhoods, V, of x.

Exercise 7.3. Let (X, d) be a separable metric space (see Definition 3.55) and ν be a measure on $\mathcal{B}_X$ – the Borel σ – algebra on X. Show

1. supp(ν) is a closed set. (This is true on all topological spaces.)
2. $\nu(X \setminus \text{supp}(\nu)) = 0$ and use this to conclude that $W := X \setminus \text{supp}(\nu)$ is the largest open set in X such that $\nu(W) = 0$. **Hint:** Let D be a countable dense subset of X and
$$\mathcal{V} := \{B_x(1/n) : x \in D \text{ and } n \in \mathbb{N}\}.$$
Show that W may be written as a union of elements from $V \in \mathcal{V}$ with the property that $\mu(V) = 0$.

Exercise 7.4. Prove the following facts about essran(f).

1. Let $\nu = f_*\mu := \mu \circ f^{-1}$ – a Borel measure on $\mathbb{C}$. Show essran$(f) = \text{supp}(\nu)$.
2. essran(f) is a closed set and $f(x) \in \text{essran}(f)$ for almost every x, i.e. $\mu(f \notin \text{essran}(f)) = 0$.
3. If $F \subset \mathbb{C}$ is a closed set such that $f(x) \in F$ for almost every x then essran$(f) \subset F$. So essran(f) is the smallest closed set F such that $f(x) \in F$ for almost every x.

4. $\|f\|_\infty = \sup\{|\lambda| : \lambda \in \text{essran}(f)\}$.

Exercise 7.5. Let $f \in L^p \cap L^\infty$ for some $p < \infty$. Show $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$. If we further assume $\mu(X) < \infty$, show $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$ for all measurable functions $f : X \to \mathbb{C}$. In particular, $f \in L^\infty$ iff $\lim_{q\to\infty} \|f\|_q < \infty$.

Exercise 7.6. Prove Eq. (7.18) in Corollary 7.23. (Part of Folland 6.3 on p. 186.) **Hint:** Use Lemma 2.27 applied to the right side of Eq. (7.17).

Exercise 7.7. Complete the proof of Proposition 7.22 by showing $(L^p + L^r, \|\cdot\|)$ is a Banach space. (Part of Folland 6.4 on p. 186.)

Exercise 7.8. Folland 6.5 on p. 186.

Exercise 7.9. Folland 6.6 on p. 186.

Exercise 7.10. Folland 6.9 on p. 186.

Exercise 7.11. Folland 6.10 on p. 186. Use the strong form of Theorem 5.38.

Exercise 7.12. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, $f \in L^2(\nu)$ and $k \in L^2(\mu \otimes \nu)$. Show

$$\int |k(x,y)f(y)|\, d\nu(y) < \infty \text{ for } \mu-\text{a.e. } x.$$

Let $Kf(x) := \int k(x,y)f(y)d\nu(y)$ when the integral is defined. Show $Kf \in L^2(\mu)$ and $K : L^2(\nu) \to L^2(\mu)$ is a bounded operator with $\|K\|_{op} \le \|k\|_{L^2(\mu\otimes\nu)}$.

Exercise 7.13. Folland 6.27 on p. 196.

Exercise 7.14. Folland 2.32 on p. 63.

Exercise 7.15. Folland 2.38 on p. 63.