Part III

Topological Spaces

10

Topological Space Basics

Using the metric space results above as motivation we will axiomatize the notion of being an open set to more general settings.

Definition 10.1. *A collection of subsets τ of X is a **topology** if*

1. $\emptyset, X \in \tau$
2. *τ is closed under arbitrary unions, i.e. if $V_\alpha \in \tau$, for $\alpha \in I$ then $\bigcup_{\alpha \in I} V_\alpha \in \tau$.*
3. *τ is closed under finite intersections, i.e. if $V_1, \dots, V_n \in \tau$ then $V_1 \cap \dots \cap V_n \in \tau$.*

   *A pair (X, τ) where τ is a topology on X will be called a **topological space**.*

Notation 10.2 *Let (X, τ) be a topological space.*

1. *The elements, $V \in \tau$, are called **open** sets. We will often write $V \subset_o X$ to indicate V is an open subset of X.*
2. *A subset $F \subset X$ is **closed** if F^c is open and we will write $F \sqsubset X$ if F is a closed subset of X.*
3. *An **open neighborhood** of a point $x \in X$ is an open set $V \subset X$ such that $x \in V$. Let $\tau_x = \{V \in \tau : x \in V\}$ denote the collection of open neighborhoods of x.*
4. *A subset $W \subset X$ is a **neighborhood** of x if there exists $V \in \tau_x$ such that $V \subset W$.*
5. *A collection $\eta \subset \tau_x$ is called a **neighborhood base** at $x \in X$ if for all $V \in \tau_x$ there exists $W \in \eta$ such that $W \subset V$.*

The notation τ_x should not be confused with

$$\tau_{\{x\}} := i_{\{x\}}^{-1}(\tau) = \{\{x\} \cap V : V \in \tau\} = \{\emptyset, \{x\}\}.$$

Example 10.3. 1. Let (X, d) be a metric space, we write τ_d for the collection of d – open sets in X. We have already seen that τ_d is a topology, see Exercise 6.2. The collection of sets $\eta = \{B_x(\varepsilon) : \varepsilon \in \mathbb{D}\}$ where $\mathbb{D}$ is any dense subset of $(0, 1]$ is a neighborhood base at x.

2. Let X be any set, then $\tau = 2^X$ is a topology. In this topology all subsets of X are both open and closed. At the opposite extreme we have the **trivial** topology, $\tau = \{\emptyset, X\}$. In this topology only the empty set and X are open (closed).
3. Let $X = \{1,2,3\}$, then $\tau = \{\emptyset, X, \{2,3\}\}$ is a topology on X which does not come from a metric.
4. Again let $X = \{1,2,3\}$. Then $\tau = \{\{1\},\{2,3\},\emptyset, X\}$. is a topology, and the sets X, $\{1\}$, $\{2,3\}$, $\emptyset$ are open and closed. The sets $\{1,2\}$ and $\{1,3\}$ are neither open nor closed.

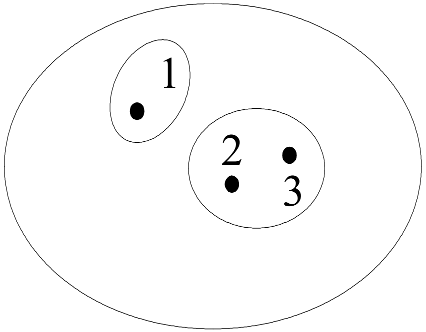

Fig. 10.1. A topology.

Definition 10.4. *Let (X,τ_X) and (Y,τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous** if*

$$f^{-1}(\tau_Y) := \left\{f^{-1}(V) : V \in \tau_Y\right\} \subset \tau_X.$$

We will also say that f is τ_X/τ_Y –continuous or (τ_X,τ_Y) – continuous. Let $C(X,Y)$ denote the set of continuous functions from X to Y.

Exercise 10.1. Show $f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for all closed subsets C of Y.

Definition 10.5. *A map $f : X \to Y$ between topological spaces is called a **homeomorphism** provided that f is bijective, f is continuous and $f^{-1} : Y \to X$ is continuous. If there exists $f : X \to Y$ which is a homeomorphism, we say that X and Y are homeomorphic. (As topological spaces X and Y are essentially the same.)*

10.1 Constructing Topologies and Checking Continuity

Proposition 10.6. *Let $\mathcal{E}$ be any collection of subsets of X. Then there exists a unique smallest topology $\tau(\mathcal{E})$ which contains $\mathcal{E}$.*

Proof. Since 2^X is a topology and $\mathcal{E} \subset 2^X$, $\mathcal{E}$ is always a subset of a topology. It is now easily seen that

$$\tau(\mathcal{E}) := \bigcap\{\tau : \tau \text{ is a topology and } \mathcal{E} \subset \tau\}$$

is a topology which is clearly the smallest possible topology containing $\mathcal{E}$. ∎

The following proposition gives an explicit descriptions of $\tau(\mathcal{E})$.

Proposition 10.7. *Let X be a set and $\mathcal{E} \subset 2^X$. For simplicity of notation, assume that $X, \emptyset \in \mathcal{E}$. (If this is not the case simply replace $\mathcal{E}$ by $\mathcal{E} \cup \{X, \emptyset\}$.) Then*

$$\tau(\mathcal{E}) := \{\text{arbitrary unions of finite intersections of elements from } \mathcal{E}\}. \tag{10.1}$$

Proof. Let τ be given as in the right side of Eq. (10.1). From the definition of a topology any topology containing $\mathcal{E}$ must contain τ and hence $\mathcal{E} \subset \tau \subset \tau(\mathcal{E})$. The proof will be completed by showing τ is a topology. The validation of τ being a topology is routine except for showing that τ is closed under taking finite intersections. Let $V, W \in \tau$ which by definition may be expressed as

$$V = \cup_{\alpha\in A} V_\alpha \text{ and } W = \cup_{\beta\in B} W_\beta,$$

where V_α and W_β are sets which are finite intersection of elements from $\mathcal{E}$. Then

$$V \cap W = (\cup_{\alpha\in A} V_\alpha) \cap (\cup_{\beta\in B} W_\beta) = \bigcup_{(\alpha,\beta)\in A\times B} V_\alpha \cap W_\beta.$$

Since for each $(\alpha,\beta) \in A \times B$, $V_\alpha \cap W_\beta$ is still a finite intersection of elements from $\mathcal{E}$, $V \cap W \in \tau$ showing τ is closed under taking finite intersections. ∎

Definition 10.8. *Let (X,τ) be a topological space. We say that $\mathcal{S} \subset \tau$ is a **sub-base** for the topology τ iff $\tau = \tau(\mathcal{S})$ and $X = \cup\mathcal{S} := \cup_{V\in\mathcal{S}} V$. We say $\mathcal{V} \subset \tau$ is a **base** for the topology τ iff $\mathcal{V}$ is a sub-base with the property that every element $V \in \tau$ may be written as*

$$V = \cup\{B \in \mathcal{V} : B \subset V\}.$$

Exercise 10.2. Suppose that $\mathcal{S}$ is a sub-base for a topology τ on a set X.

1. Show $\mathcal{V} := \mathcal{S}_f$ ($\mathcal{S}_f$ is the collection of finite intersections of elements from $\mathcal{S}$) is a base for τ.
2. Show $\mathcal{S}$ is itself a base for τ iff

$$V_1 \cap V_2 = \cup\{S \in \mathcal{S} : S \subset V_1 \cap V_2\}.$$

for every pair of sets $V_1, V_2 \in \mathcal{S}$.

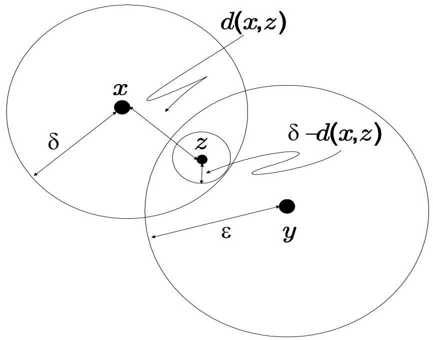

Fig. 10.2. Fitting balls in the intersection.

Remark 10.9. Let (X, d) be a metric space, then $\mathcal{E} = \{B_x(\delta) : x \in X$ and $\delta > 0\}$ is a base for τ_d – the topology associated to the metric d. This is the content of Exercise 6.3.

Let us check directly that $\mathcal{E}$ is a base for a topology. Suppose that $x, y \in X$ and $\varepsilon, \delta > 0$. If $z \in B(x, \delta) \cap B(y, \varepsilon)$, then

$$B(z, \alpha) \subset B(x, \delta) \cap B(y, \varepsilon) \tag{10.2}$$

where $\alpha = \min\{\delta - d(x, z), \varepsilon - d(y, z)\}$, see Figure 10.2. This is a formal consequence of the triangle inequality. For example let us show that $B(z, \alpha) \subset B(x, \delta)$. By the definition of α, we have that $\alpha \le \delta - d(x, z)$ or that $d(x, z) \le \delta - \alpha$. Hence if $w \in B(z, \alpha)$, then

$$d(x, w) \le d(x, z) + d(z, w) \le \delta - \alpha + d(z, w) < \delta - \alpha + \alpha = \delta$$

which shows that $w \in B(x, \delta)$. Similarly we show that $w \in B(y, \varepsilon)$ as well.

Owing to Exercise 10.2, this shows $\mathcal{E}$ is a base for a topology. We do not need to use Exercise 10.2 here since in fact Equation (10.2) may be generalized to finite intersection of balls. Namely if $x_i \in X$, $\delta_i > 0$ and $z \in \cap_{i=1}^n B(x_i, \delta_i)$, then

$$B(z, \alpha) \subset \cap_{i=1}^n B(x_i, \delta_i) \tag{10.3}$$

where now $\alpha := \min\{\delta_i - d(x_i, z) : i = 1, 2, \ldots, n\}$. By Eq. (10.3) it follows that any finite intersection of open balls may be written as a union of open balls.

Exercise 10.3. Suppose $f : X \to Y$ is a function and τ_X and τ_Y are topologies on X and Y respectively. Show

$$f^{-1}\tau_Y := \left\{f^{-1}(V) \subset X : V \in \tau_Y\right\} \text{ and } f_*\tau_X := \left\{V \subset Y : f^{-1}(V) \in \tau_X\right\}$$

(as in Notation 2.7) are also topologies on X and Y respectively.

Remark 10.10. Let $f : X \to Y$ be a function. Given a topology $\tau_Y \subset 2^Y$, the topology $\tau_X := f^{-1}(\tau_Y)$ is the smallest topology on X such that f is (τ_X, τ_Y) - continuous. Similarly, if τ_X is a topology on X then $\tau_Y = f_*\tau_X$ is the largest topology on Y such that f is (τ_X, τ_Y) - continuous.

Definition 10.11. *Let (X, τ) be a topological space and A subset of X. The **relative topology** or **induced topology** on A is the collection of sets*

$$\tau_A = i_A^{-1}(\tau) = \{A \cap V : V \in \tau\},$$

where $i_A : A \to X$ be the inclusion map as in Definition 2.8.

Lemma 10.12. *The relative topology, τ_A, is a topology on A. Moreover a subset $B \subset A$ is τ_A – closed iff there is a τ – closed subset, C, of X such that $B = C \cap A$.*

Proof. The first assertion is a consequence of Exercise 10.3. For the second, $B \subset A$ is τ_A – closed iff $A \setminus B = A \cap V$ for some $V \in \tau$ which is equivalent to $B = A \setminus (A \cap V) = A \cap V^c$ for some $V \in \tau$. ■

Exercise 10.4. Show if (X, d) is a metric space and $\tau = \tau_d$ is the topology coming from d, then $(\tau_d)_A$ is the topology induced by making A into a metric space using the metric $d|_{A \times A}$.

Lemma 10.13. *Suppose that (X, τ_X), (Y, τ_Y) and (Z, τ_Z) are topological spaces. If $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (Y, \tau_Y) \to (Z, \tau_Z)$ are continuous functions then $g \circ f : (X, \tau_X) \to (Z, \tau_Z)$ is continuous as well.*

Proof. This is easy since by assumption $g^{-1}(\tau_Z) \subset \tau_Y$ and $f^{-1}(\tau_Y) \subset \tau_X$ so that

$$(g \circ f)^{-1}(\tau_Z) = f^{-1}\left(g^{-1}(\tau_Z)\right) \subset f^{-1}(\tau_Y) \subset \tau_X.$$

■

The following elementary lemma turns out to be extremely useful because it may be used to greatly simplify the verification that a given function is continuous.

Lemma 10.14. *Suppose that $f : X \to Y$ is a function, $\mathcal{E} \subset 2^Y$ and $A \subset Y$, then*

$$\tau\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\tau(\mathcal{E})) \text{ and} \tag{10.4}$$

$$\tau(\mathcal{E}_A) = (\tau(\mathcal{E}))_A. \tag{10.5}$$

Moreover, if $\tau_Y = \tau(\mathcal{E})$ and τ_X is a topology on X, then f is (τ_X, τ_Y) – continuous iff $f^{-1}(\mathcal{E}) \subset \tau_X$.

Proof. We will give two proof of Eq. (10.4). The first proof is more constructive than the second, but the second proof will work in the context of σ – algebras to be developed later. **First Proof.** There is no harm (as the reader should verify) in replacing $\mathcal{E}$ by $\mathcal{E} \cup \{\emptyset, Y\}$ if necessary so that we may assume that $\emptyset, Y \in \mathcal{E}$. By Proposition 10.7, the general element V of $\tau(\mathcal{E})$ is an arbitrary unions of finite intersections of elements from $\mathcal{E}$. Since f^{-1} preserves all of the set operations, it follows that $f^{-1}\tau(\mathcal{E})$ consists of sets which are arbitrary unions of finite intersections of elements from $f^{-1}\mathcal{E}$, which is precisely $\tau\left(f^{-1}(\mathcal{E})\right)$ by another application of Proposition 10.7. **Second Proof.** By Exercise 10.3, $f^{-1}(\tau(\mathcal{E}))$ is a topology and since $\mathcal{E} \subset \tau(\mathcal{E})$, $f^{-1}(\mathcal{E}) \subset f^{-1}(\tau(\mathcal{E}))$. It now follows that $\tau(f^{-1}(\mathcal{E})) \subset f^{-1}(\tau(\mathcal{E}))$. For the reverse inclusion notice that

$$f_*\tau\left(f^{-1}(\mathcal{E})\right) = \left\{B \subset Y : f^{-1}(B) \in \tau\left(f^{-1}(\mathcal{E})\right)\right\}$$

is a topology which contains $\mathcal{E}$ and thus $\tau(\mathcal{E}) \subset f_*\tau\left(f^{-1}(\mathcal{E})\right)$. Hence if $B \in \tau(\mathcal{E})$ we know that $f^{-1}(B) \in \tau\left(f^{-1}(\mathcal{E})\right)$, i.e. $f^{-1}(\tau(\mathcal{E})) \subset \tau\left(f^{-1}(\mathcal{E})\right)$ and Eq. (10.4) has been proved. Applying Eq. (10.4) with $X = A$ and $f = i_A$ being the inclusion map implies

$$(\tau(\mathcal{E}))_A = i_A^{-1}(\tau(\mathcal{E})) = \tau(i_A^{-1}(\mathcal{E})) = \tau(\mathcal{E}_A).$$

Lastly if $f^{-1}\mathcal{E} \subset \tau_X$, then $f^{-1}\tau(\mathcal{E}) = \tau\left(f^{-1}\mathcal{E}\right) \subset \tau_X$ which shows f is (τ_X, τ_Y) – continuous. ■

Corollary 10.15. *If (X, τ) is a topological space and $f : X \to \mathbb{R}$ is a function then the following are equivalent:*

1. *f is $(\tau, \tau_{\mathbb{R}})$ - continuous,*
2. *$f^{-1}((a, b)) \in \tau$ for all $-\infty < a < b < \infty$,*
3. *$f^{-1}((a, \infty)) \in \tau$ and $f^{-1}((-\infty, b)) \in \tau$ for all $a, b \in \mathbb{Q}$.*

(We are using $\tau_{\mathbb{R}}$ to denote the standard topology on $\mathbb{R}$ induced by the metric $d(x, y) = |x - y|$.)

Proof. Apply Lemma 10.14 with appropriate choices of $\mathcal{E}$. ■

Definition 10.16. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous at a point** $x \in X$ if for every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $U \subset f^{-1}(V)$. See Figure 10.3.*

Exercise 10.5. Show $f : X \to Y$ is continuous (Definition 10.16) iff f is continuous at all points $x \in X$.

Definition 10.17. *Given topological spaces (X, τ) and (Y, τ') and a subset $A \subset X$. We say a function $f : A \to Y$ is **continuous** iff f is τ_A/τ' – continuous.*

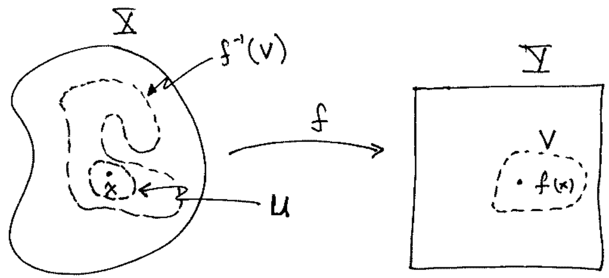

Fig. 10.3. Checking that a function is continuous at $x \in X$.

Definition 10.18. *Let (X, τ) be a topological space and $A \subset X$. A collection of subsets $\mathcal{U} \subset \tau$ is an **open cover** of A if $A \subset \bigcup \mathcal{U} := \bigcup_{U \in \mathcal{U}} U$.*

Proposition 10.19 (Localizing Continuity). *Let (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be a function.*

1. *If f is continuous and $A \subset X$ then $f|_A : A \to Y$ is continuous.*
2. *Suppose there exist an open cover, $\mathcal{U} \subset \tau$, of X such that $f|_A$ is continuous for all $A \in \mathcal{U}$, then f is continuous.*

Proof. 1. If $f : X \to Y$ is a continuous, $f^{-1}(V) \in \tau$ for all $V \in \tau'$ and therefore

$$f|_A^{-1}(V) = A \cap f^{-1}(V) \in \tau_A \text{ for all } V \in \tau'.$$

2. Let $V \in \tau'$, then

$$f^{-1}(V) = \cup_{A \in \mathcal{U}}\left(f^{-1}(V) \cap A\right) = \cup_{A \in \mathcal{U}} f|_A^{-1}(V). \tag{10.6}$$

Since each $A \in \mathcal{U}$ is open, $\tau_A \subset \tau$ and by assumption, $f|_A^{-1}(V) \in \tau_A \subset \tau$. Hence Eq. (10.6) shows $f^{-1}(V)$ is a union of τ – open sets and hence is also τ – open. ■

Exercise 10.6 (A Baby Extension Theorem). Suppose $V \in \tau$ and $f : V \to \mathbb{C}$ is a continuous function. Further assume there is a closed subset C such that $\{x \in V : f(x) \neq 0\} \subset C \subset V$, then $F : X \to \mathbb{C}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \notin V \end{cases}$$

is continuous.

Exercise 10.7 (Building Continuous Functions). Prove the following variant of item 2. of Proposition 10.19. Namely, suppose there exists a **finite** collection $\mathcal{F}$ of closed subsets of X such that $X = \cup_{A \in \mathcal{F}} A$ and $f|_A$ is continuous for all $A \in \mathcal{F}$, then f is continuous. Given an example showing that the assumption that $\mathcal{F}$ is finite can not be eliminated. **Hint:** consider $f^{-1}(C)$ where C is a closed subset of Y.

10.2 Product Spaces I

Definition 10.20. *Let X be a set and suppose there is a collection of topological spaces $\{(Y_\alpha, \tau_\alpha) : \alpha \in A\}$ and functions $f_\alpha : X \to Y_\alpha$ for all $\alpha \in A$. Let $\tau(f_\alpha : \alpha \in A)$ denote the smallest topology on X such that each f_α is continuous, i.e.*

$$\tau(f_\alpha : \alpha \in A) = \tau(\cup_\alpha f_\alpha^{-1}(\tau_\alpha)).$$

Proposition 10.21 (Topologies Generated by Functions). *Assuming the notation in Definition 10.20 and additionally let (Z, τ_Z) be a topological space and $g : Z \to X$ be a function. Then g is $(\tau_Z, \tau(f_\alpha : \alpha \in A))$ – continuous iff $f_\alpha \circ g$ is (τ_Z, τ_α)–continuous for all $\alpha \in A$.*

Proof. $(\Rightarrow)$ If g is $(\tau_Z, \tau(f_\alpha : \alpha \in A))$ – continuous, then the composition $f_\alpha \circ g$ is (τ_Z, τ_α) – continuous by Lemma 10.13. $(\Leftarrow)$ Let

$$\tau_X = \tau(f_\alpha : \alpha \in A) = \tau\left(\cup_{\alpha\in A} f_\alpha^{-1}(\tau_\alpha)\right).$$

If $f_\alpha \circ g$ is (τ_Z, τ_α) – continuous for all α, then

$$g^{-1} f_\alpha^{-1}(\tau_\alpha) \subset \tau_Z \,\forall\, \alpha \in A$$

and therefore

$$g^{-1}\left(\cup_{\alpha\in A} f_\alpha^{-1}(\tau_\alpha)\right) = \cup_{\alpha\in A} g^{-1} f_\alpha^{-1}(\tau_\alpha) \subset \tau_Z$$

Hence

$$g^{-1}(\tau_X) = g^{-1}\left(\tau\left(\cup_{\alpha\in A} f_\alpha^{-1}(\tau_\alpha)\right)\right) = \tau(g^{-1}\left(\cup_{\alpha\in A} f_\alpha^{-1}(\tau_\alpha)\right) \subset \tau_Z$$

which shows that g is (τ_Z, τ_X) – continuous. ■

Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in A}$ be a collection of topological spaces, $X = X_A = \prod_{\alpha\in A} X_\alpha$ and $\pi_\alpha : X_A \to X_\alpha$ be the canonical projection map as in Notation 2.2.

Definition 10.22. *The* ***product topology*** *$\tau = \otimes_{\alpha\in A}\tau_\alpha$ is the smallest topology on X_A such that each projection π_α is continuous. Explicitly, τ is the topology generated by the collection of sets,*

$$\mathcal{E} = \{\pi_\alpha^{-1}(V_\alpha) : \alpha \in A, V_\alpha \in \tau_\alpha\} = \cup_{\alpha\in A} \pi^{-1}\tau_\alpha. \tag{10.7}$$

Applying Proposition 10.21 in this setting implies the following proposition.

Proposition 10.23. *Suppose Y is a topological space and $f : Y \to X_A$ is a map. Then f is continuous iff $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for all $\alpha \in A$. In particular if $A = \{1, 2, \dots, n\}$ so that $X_A = X_1 \times X_2 \times \cdots \times X_n$ and $f(y) = (f_1(y), f_2(y), \dots, f_n(y)) \in X_1 \times X_2 \times \cdots \times X_n$, then $f : Y \to X_A$ is continuous iff $f_i : Y \to X_i$ is continuous for all i.*

Proposition 10.24. *Suppose that (X, τ) is a topological space and $\{f_n\} \subset X^A$ (see Notation 2.2) is a sequence. Then $f_n \to f$ in the product topology of X^A iff $f_n(\alpha) \to f(\alpha)$ for all $\alpha \in A$.*

Proof. Since π_α is continuous, if $f_n \to f$ then $f_n(\alpha) = \pi_\alpha(f_n) \to \pi_\alpha(f) = f(\alpha)$ for all $\alpha \in A$. Conversely, $f_n(\alpha) \to f(\alpha)$ for all $\alpha \in A$ iff $\pi_\alpha(f_n) \to \pi_\alpha(f)$ for all $\alpha \in A$. Therefore if $V = \pi_\alpha^{-1}(V_\alpha) \in \mathcal{E}$ (with $\mathcal{E}$ as in Eq. (10.7)) and $f \in V$, then $\pi_\alpha(f) \in V_\alpha$ and $\pi_\alpha(f_n) \in V_\alpha$ for a.a. n and hence $f_n \in V$ for a.a. n. This shows that $f_n \to f$ as $n \to \infty$. ■

Proposition 10.25. *Suppose that $(X_\alpha, \tau_\alpha)_{\alpha\in A}$ is a collection of topological spaces and $\otimes_{\alpha\in A}\tau_\alpha$ is the product topology on $X := \prod_{\alpha\in A} X_\alpha$.*

1. *If $\mathcal{E}_\alpha \subset \tau_\alpha$ generates τ_α for each $\alpha \in A$, then*

$$\otimes_{\alpha\in A}\tau_\alpha = \tau\left(\cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \tag{10.8}$$

2. *If $\mathcal{B}_\alpha \subset \tau_\alpha$ is a base for τ_α for each α, then the collection of sets, $\mathcal{V}$, of the form*

$$V = \cap_{\alpha\in\Lambda}\pi_\alpha^{-1}V_\alpha = \prod_{\alpha\in\Lambda} V_\alpha \times \prod_{\alpha\notin\Lambda} X_\alpha =: V_\Lambda \times X_{A\setminus\Lambda}, \tag{10.9}$$

where $\Lambda \subset\subset A$ and $V_\alpha \in \mathcal{B}_\alpha$ for all $\alpha \in \Lambda$ is base for $\otimes_{\alpha\in A}\tau_\alpha$.

Proof. 1. Since

$$\begin{aligned}\cup_\alpha\pi_\alpha^{-1}\mathcal{E}_\alpha \subset \cup_\alpha\pi_\alpha^{-1}\tau_\alpha &= \cup_\alpha\pi_\alpha^{-1}(\tau(\mathcal{E}_\alpha)) \\ &= \cup_\alpha\tau(\pi_\alpha^{-1}\mathcal{E}_\alpha) \subset \tau\left(\cup_\alpha\pi_\alpha^{-1}\mathcal{E}_\alpha\right),\end{aligned}$$

it follows that

$$\tau\left(\cup_\alpha\pi_\alpha^{-1}\mathcal{E}_\alpha\right) \subset \otimes_\alpha\tau_\alpha \subset \tau\left(\cup_\alpha\pi_\alpha^{-1}\mathcal{E}_\alpha\right).$$

2. Now let $\mathcal{U} = \left[\cup_\alpha\pi_\alpha^{-1}\tau_\alpha\right]_f$ denote the collection of sets consisting of finite intersections of elements from $\cup_\alpha\pi_\alpha^{-1}\tau_\alpha$. Notice that $\mathcal{U}$ may be described as those sets in Eq. (10.9) where $V_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda$. By Exercise 10.2, $\mathcal{U}$ is a base for the product topology, $\otimes_{\alpha\in A}\tau_\alpha$. Hence for $W \in \otimes_{\alpha\in A}\tau_\alpha$ and $x \in W$, there exists a $V \in \mathcal{U}$ of the form in Eq. (10.9) such that $x \in V \subset W$. Since $\mathcal{B}_\alpha$ is a base for τ_α, there exists $U_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in U_\alpha \subset V_\alpha$ for each $\alpha \in \Lambda$. With this notation, the set $U_\Lambda \times X_{A\setminus\Lambda} \in \mathcal{V}$ and $x \in U_\Lambda \times X_{A\setminus\Lambda} \subset V \subset W$. This shows that every open set in X may be written as a union of elements from $\mathcal{V}$, i.e. $\mathcal{V}$ is a base for the product topology. ■

Notation 10.26 *Let $\mathcal{E}_i \subset 2^{X_i}$ be a collection of subsets of a set X_i for each $i = 1, 2, \dots, n$. We will write, by abuse of notation, $\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ for the collection of subsets of $X_1 \times \cdots \times X_n$ of the form $A_1 \times A_2 \times \cdots \times A_n$ with $A_i \in \mathcal{E}_i$ for all i. That is we are identifying $(A_1, A_2, \dots, A_n)$ with $A_1 \times A_2 \times \cdots \times A_n$.*

Corollary 10.27. *Suppose $A = \{1, 2, \ldots, n\}$ so $X = X_1 \times X_2 \times \cdots \times X_n$.*

1. *If $\mathcal{E}_i \subset 2^{X_i}$, $\tau_i = \tau(\mathcal{E}_i)$ and $X_i \in \mathcal{E}_i$ for each i, then*
$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n = \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \tag{10.10}$$
and in particular
$$\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n = \tau(\tau_1 \times \cdots \times \tau_n). \tag{10.11}$$
2. *Furthermore if $\mathcal{B}_i \subset \tau_i$ is a base for the topology τ_i for each i, then $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is a base for the product topology, $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$.*

Proof. (The proof is a minor variation on the proof of Proposition 10.25.) 1. Let $\left[\cup_{i \in A} \pi_i^{-1}(\mathcal{E}_i)\right]_f$ denotes the collection of sets which are finite intersections from $\cup_{i \in A} \pi_i^{-1}(\mathcal{E}_i)$, then, using $X_i \in \mathcal{E}_i$ for all i,
$$\cup_{i \in A} \pi_i^{-1}(\mathcal{E}_i) \subset \mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n \subset \left[\cup_{i \in A} \pi_i^{-1}(\mathcal{E}_i)\right]_f.$$
Therefore
$$\tau = \tau\left(\cup_{i \in A} \pi_i^{-1}(\mathcal{E}_i)\right) \subset \tau(\mathcal{E}_1 \times \mathcal{E}_2 \times \cdots \times \mathcal{E}_n) \subset \tau\left(\left[\cup_{i \in A} \pi_i^{-1}(\mathcal{E}_i)\right]_f\right) = \tau.$$
2. Observe that $\tau_1 \times \cdots \times \tau_n$ is closed under finite intersections and generates $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$, therefore $\tau_1 \times \cdots \times \tau_n$ is a base for the product topology. The proof that $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is also a base for $\tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ follows the same method used to prove item 2. in Proposition 10.25. ∎

Lemma 10.28. *Let (X_i, d_i) for $i = 1, \ldots, n$ be metric spaces, $X := X_1 \times \cdots \times X_n$ and for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in X let*
$$d(x, y) = \sum_{i=1}^n d_i(x_i, y_i). \tag{10.12}$$
Then the topology, τ_d, associated to the metric d is the product topology on X, i.e.
$$\tau_d = \tau_{d_1} \otimes \tau_{d_2} \otimes \cdots \otimes \tau_{d_n}.$$

Proof. Let $\rho(x, y) = \max\{d_i(x_i, y_i) : i = 1, 2, \ldots, n\}$. Then ρ is equivalent to d and hence $\tau_\rho = \tau_d$. Moreover if $\varepsilon > 0$ and $x = (x_1, x_2, \ldots, x_n) \in X$, then
$$B_x^\rho(\varepsilon) = B_{x_1}^{d_1}(\varepsilon) \times \cdots \times B_{x_n}^{d_n}(\varepsilon).$$
By Remark 10.9,
$$\mathcal{E} := \{B_x^\rho(\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$$
is a base for τ_ρ and by Proposition 10.25 $\mathcal{E}$ is also a base for $\tau_{d_1} \otimes \tau_{d_2} \otimes \cdots \otimes \tau_{d_n}$. Therefore,
$$\tau_{d_1} \otimes \tau_{d_2} \otimes \cdots \otimes \tau_{d_n} = \tau(\mathcal{E}) = \tau_\rho = \tau_d.$$
∎

10.3 Closure operations

Definition 10.29. *Let (X, τ) be a topological space and A be a subset of X.*

1. *The **closure** of A is the smallest closed set $\bar{A}$ containing A, i.e.*
$$\bar{A} := \cap\{F : A \subset F \sqsubset X\}.$$
(Because of Exercise 6.4 this is consistent with Definition 6.10 for the closure of a set in a metric space.)
2. *The **interior** of A is the largest open set A^o contained in A, i.e.*
$$A^o = \cup\{V \in \tau : V \subset A\}.$$
(With this notation the definition of a neighborhood of $x \in X$ may be stated as: $A \subset X$ is a neighborhood of a point $x \in X$ if $x \in A^o$.)
3. *The **accumulation points** of A is the set*
$$\mathrm{acc}(A) = \{x \in X : V \cap A \setminus \{x\} \neq \emptyset \text{ for all } V \in \tau_x\}.$$
4. *The **boundary** of A is the set $\mathrm{bd}(A) := \bar{A} \setminus A^o$.*

Remark 10.30. The relationships between the interior and the closure of a set are:
$$(A^o)^c = \bigcap\{V^c : V \in \tau \text{ and } V \subset A\} = \bigcap\{C : C \text{ is closed } C \supset A^c\} = \overline{A^c}$$
and similarly, $(\bar{A})^c = (A^c)^o$. Hence the boundary of A may be written as
$$\mathrm{bd}(A) := \bar{A} \setminus A^o = \bar{A} \cap (A^o)^c = \bar{A} \cap \overline{A^c}, \tag{10.13}$$
which is to say $\mathrm{bd}(A)$ consists of the points in both the closure of A and A^c.

Proposition 10.31. *Let $A \subset X$ and $x \in X$.*

1. *If $V \subset_o X$ and $A \cap V = \emptyset$ then $\bar{A} \cap V = \emptyset$.*
2. *$x \in \bar{A}$ iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$.*
3. *$x \in \mathrm{bd}(A)$ iff $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$ for all $V \in \tau_x$.*
4. *$\bar{A} = A \cup \mathrm{acc}(A)$.*

Proof. 1. Since $A \cap V = \emptyset$, $A \subset V^c$ and since V^c is closed, $\bar{A} \subset V^c$. That is to say $\bar{A} \cap V = \emptyset$. 2. By Remark 10.30[1], $\bar{A} = ((A^c)^o)^c$ so $x \in \bar{A}$ iff $x \notin (A^c)^o$ which happens iff $V \not\subset A^c$ for all $V \in \tau_x$, i.e. iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$. 3. This assertion easily follows from the Item 2. and Eq. (10.13). 4. Item 4. is an easy consequence of the definition of $\mathrm{acc}(A)$ and item 2. ∎

[1] Here is another direct proof of item 2. which goes by showing $x \notin \bar{A}$ iff there exists $V \in \tau_x$ such that $V \cap A = \emptyset$. If $x \notin \bar{A}$ then $V = (\bar{A})^c \in \tau_x$ and $V \cap A \subset V \cap \bar{A} = \emptyset$. Conversely if there exists $V \in \tau_x$ such that $A \cap V = \emptyset$ then by Item 1. $\bar{A} \cap V = \emptyset$.

Lemma 10.32. *Let $A \subset Y \subset X$, $\bar{A}^Y$ denote the closure of A in Y with its relative topology and $\bar{A} = \bar{A}^X$ be the closure of A in X, then $\bar{A}^Y = \bar{A}^X \cap Y$.*

Proof. Using Lemma 10.12,

$$\begin{aligned}\bar{A}^Y &= \cap\{B \sqsubset Y : A \subset B\} = \cap\{C \cap Y : A \subset C \sqsubset X\} \\ &= Y \cap (\cap\{C : A \subset C \sqsubset X\}) = Y \cap \bar{A}^X.\end{aligned}$$

Alternative proof. Let $x \in Y$ then $x \in \bar{A}^Y$ iff $V \cap A \neq \emptyset$ for all $V \in \tau_Y$ such that $x \in V$. This happens iff for all $U \in \tau_x$, $U \cap Y \cap A = U \cap A \neq \emptyset$ which happens iff $x \in \bar{A}^X$. That is to say $\bar{A}^Y = \bar{A}^X \cap Y$. ■

The support of a function may now be defined as in Definition 8.25 above.

Definition 10.33 (Support). *Let $f : X \to Y$ be a function from a topological space (X, τ_X) to a vector space Y. Then we define the support of f by*

$$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}},$$

a closed subset of X.

The next result is included for completeness but will not be used in the sequel so may be omitted.

Lemma 10.34. *Suppose that $f : X \to Y$ is a map between topological spaces. Then the following are equivalent:*

1. *f is continuous.*
2. *$f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$*
3. *$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for all $B \subset X$.*

Proof. If f is continuous, then $f^{-1}\left(\overline{f(A)}\right)$ is closed and since $A \subset f^{-1}(f(A)) \subset f^{-1}\left(\overline{f(A)}\right)$ it follows that $\bar{A} \subset f^{-1}\left(\overline{f(A)}\right)$. From this equation we learn that $f(\bar{A}) \subset \overline{f(A)}$ so that (1) implies (2) Now assume (2), then for $B \subset Y$ (taking $A = f^{-1}(\bar{B})$) we have

$$f(\overline{f^{-1}(B)}) \subset f(\overline{f^{-1}(\bar{B})}) \subset \overline{f(f^{-1}(\bar{B}))} \subset \bar{B}$$

and therefore

$$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}). \tag{10.14}$$

This shows that (2) implies (3) Finally if Eq. (10.14) holds for all B, then when B is closed this shows that

$$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) = f^{-1}(B) \subset \overline{f^{-1}(B)}$$

which shows that

$$f^{-1}(B) = \overline{f^{-1}(B)}.$$

Therefore $f^{-1}(B)$ is closed whenever B is closed which implies that f is continuous. ■

10.4 Countability Axioms

Definition 10.35. *Let (X, τ) be a topological space. A sequence $\{x_n\}_{n=1}^\infty \subset X$ **converges** to a point $x \in X$ if for all $V \in \tau_x$, $x_n \in V$ almost always (abbreviated a.a.), i.e. $\#(\{n : x_n \notin V\}) < \infty$. We will write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$ when x_n converges to x.*

Example 10.36. Let $Y = \{1, 2, 3\}$ and $\tau = \{Y, \emptyset, \{1, 2\}, \{2, 3\}, \{2\}\}$ and $y_n = 2$ for all n. Then $y_n \to y$ for every $y \in Y$. So limits need not be unique!

Definition 10.37 (First Countable). *A topological space, (X, τ), is **first countable** iff every point $x \in X$ has a countable neighborhood base as defined in Notation 10.2*

All metric spaces are first countable and, like for metric spaces,when τ is first countable, we may formulate many topological notions in terms of sequences.

Proposition 10.38. *If $f : X \to Y$ is continuous at $x \in X$ and $\lim_{n\to\infty} x_n = x \in X$, then $\lim_{n\to\infty} f(x_n) = f(x) \in Y$. Moreover, if there exists a countable neighborhood base η of $x \in X$, then f is continuous at x iff $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \to x$ as $n \to \infty$.*

Proof. If $f : X \to Y$ is continuous and $W \in \tau_Y$ is a neighborhood of $f(x) \in Y$, then there exists a neighborhood V of $x \in X$ such that $f(V) \subset W$. Since $x_n \to x$, $x_n \in V$ a.a. and therefore $f(x_n) \in f(V) \subset W$ a.a., i.e. $f(x_n) \to f(x)$ as $n \to \infty$. Conversely suppose that $\eta := \{W_n\}_{n=1}^\infty$ is a countable neighborhood base at x and $\lim\limits_{n\to\infty} f(x_n) = f(x)$ for all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \to x$. By replacing W_n by $W_1 \cap \dots \cap W_n$ if necessary, we may assume that $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of sets. If f were **not** continuous at x then there exists $V \in \tau_{f(x)}$ such that $x \notin \left[f^{-1}(V)\right]^o$. Therefore, W_n is not a subset of $f^{-1}(V)$ for all n. Hence for each n, we may choose $x_n \in W_n \setminus f^{-1}(V)$. This sequence then has the property that $x_n \to x$ as $n \to \infty$ while $f(x_n) \notin V$ for all n and hence $\lim_{n\to\infty} f(x_n) \neq f(x)$. ■

Lemma 10.39. *Suppose there exists $\{x_n\}_{n=1}^\infty \subset A$ such that $x_n \to x$, then $x \in \bar{A}$. Conversely if (X, τ) is a first countable space (like a metric space) then if $x \in \bar{A}$ there exists $\{x_n\}_{n=1}^\infty \subset A$ such that $x_n \to x$.*

Proof. Suppose $\{x_n\}_{n=1}^\infty \subset A$ and $x_n \to x \in X$. Since $\bar{A}^c$ is an open set, if $x \in \bar{A}^c$ then $x_n \in \bar{A}^c \subset A^c$ a.a. contradicting the assumption that $\{x_n\}_{n=1}^\infty \subset A$. Hence $x \in \bar{A}$. For the converse we now assume that (X, τ) is first countable and that $\{V_n\}_{n=1}^\infty$ is a countable neighborhood base at x such that $V_1 \supset V_2 \supset V_3 \supset \dots$. By Proposition 10.31, $x \in \bar{A}$ iff $V \cap A \neq \emptyset$ for all $V \in \tau_x$. Hence $x \in \bar{A}$ implies there exists $x_n \in V_n \cap A$ for all n. It is now easily seen that $x_n \to x$ as $n \to \infty$. ■

Definition 10.40. *A topological space, (X,τ), is* ***second countable*** *if there exists a countable base $\mathcal{V}$ for τ, i.e. $\mathcal{V} \subset \tau$ is a countable set such that for every $W \in \tau$,*

$$W = \cup\{V : V \in \mathcal{V} \ni V \subset W\}.$$

Definition 10.41. *A subset D of a topological space X is* ***dense*** *if $\bar{D} = X$. A topological space is said to be* ***separable*** *if it contains a countable dense subset, D.*

Example 10.42. The following are examples of countable dense sets.

1. The rational number $\mathbb{Q}$ are dense in $\mathbb{R}$ equipped with the usual topology.
2. More generally, $\mathbb{Q}^d$ is a countable dense subset of $\mathbb{R}^d$ for any $d \in \mathbb{N}$.
3. Even more generally, for any function $\mu : \mathbb{N} \to (0,\infty)$, $\ell^p(\mu)$ is separable for all $1 \le p < \infty$. For example, let $\Gamma \subset \mathbb{F}$ be a countable dense set, then
$$D := \{x \in \ell^p(\mu) : x_i \in \lneqq \text{ for all } i \text{ and } \#\{j : x_j \neq 0\} < \infty\}.$$
The set Γ can be taken to be $\mathbb{Q}$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q} + i\mathbb{Q}$ if $\mathbb{F} = \mathbb{C}$.
4. If (X,d) is a metric space which is separable then every subset $Y \subset X$ is also separable in the induced topology.

To prove 4. above, let $A = \{x_n\}_{n=1}^\infty \subset X$ be a countable dense subset of X. Let $d_Y(x) = \inf\{d(x,y) : y \in Y\}$ be the distance from x to Y and recall that $d_Y : X \to [0,\infty)$ is continuous. Let $\varepsilon_n = \max\left\{d_Y(x_n), \frac{1}{n}\right\} \ge 0$ and for each n let $y_n \in B_{x_n}(2\varepsilon_n)$. Then if $y \in Y$ and $\varepsilon > 0$ we may choose $n \in \mathbb{N}$ such that $d(y,x_n) \le \varepsilon_n < \varepsilon/3$. Then $d(y_n,x_n) \le 2\varepsilon_n < 2\varepsilon/3$ and therefore

$$d(y,y_n) \le d(y,x_n) + d(x_n,y_n) < \varepsilon.$$

This shows that $B := \{y_n\}_{n=1}^\infty$ is a countable dense subset of Y.

Exercise 10.8. Show $\ell^\infty(\mathbb{N})$ is not separable.

Exercise 10.9. Show every second countable topological space (X,τ) is separable. Show the converse is not true by by showing $X := \mathbb{R}$ with $\tau = \{\emptyset\} \cup \{V \subset \mathbb{R} : 0 \in V\}$ is a separable, first countable but not second countable topological space.

Exercise 10.10. Every separable metric space, (X,d) is second countable.

Exercise 10.11. Suppose $\mathcal{E} \subset 2^X$ is a countable collection of subsets of X, then $\tau = \tau(\mathcal{E})$ is a second countable topology on X.

10.5 Connectedness

Definition 10.43. *(X,τ) is* ***disconnected*** *if there exists non-empty open sets U and V of X such that $U \cap V = \emptyset$ and $X = U \cup V$. We say $\{U,V\}$ is a* ***disconnection*** *of X. The topological space (X,τ) is called* ***connected*** *if it is not disconnected, i.e. if there is no disconnection of X. If $A \subset X$ we say A is connected iff (A,τ_A) is connected where τ_A is the relative topology on A. Explicitly, A is disconnected in (X,τ) iff there exists $U,V \in \tau$ such that $U \cap A \neq \emptyset$, $U \cap A \neq \emptyset$, $A \cap U \cap V = \emptyset$ and $A \subset U \cup V$.*

The reader should check that the following statement is an equivalent definition of connectivity. A topological space (X,τ) is connected iff the only sets $A \subset X$ which are both open and closed are the sets X and $\emptyset$.

Remark 10.44. Let $A \subset Y \subset X$. Then A is connected in X iff A is connected in Y.

Proof. Since

$$\tau_A := \{V \cap A : V \subset X\} = \{V \cap A \cap Y : V \subset X\} = \{U \cap A : U \subset_o Y\},$$

the relative topology on A inherited from X is the same as the relative topology on A inherited from Y. Since connectivity is a statement about the relative topologies on A, A is connected in X iff A is connected in Y. ■

The following elementary but important lemma is left as an exercise to the reader.

Lemma 10.45. *Suppose that $f : X \to Y$ is a continuous map between topological spaces. Then $f(X) \subset Y$ is connected if X is connected.*

Here is a typical way these connectedness ideas are used.

Example 10.46. Suppose that $f : X \to Y$ is a continuous map between two topological spaces, the space X is connected and the space Y is "T_1," i.e. $\{y\}$ is a closed set for all $y \in Y$ as in Definition 12.36 below. Further assume f is locally constant, i.e. for all $x \in X$ there exists an open neighborhood V of x in X such that $f|_V$ is constant. Then f is constant, i.e. $f(X) = \{y_0\}$ for some $y_0 \in Y$. To prove this, let $y_0 \in f(X)$ and let $W := f^{-1}(\{y_0\})$. Since $\{y_0\} \subset Y$ is a closed set and since f is continuous $W \subset X$ is also closed. Since f is locally constant, W is open as well and since X is connected it follows that $W = X$, i.e. $f(X) = \{y_0\}$.

Theorem 10.47 (Properties of Connected Sets). *Let (X,τ) be a topological space.*

1. *If $B \subset X$ is a connected set and X is the disjoint union of two open sets U and V, then either $B \subset U$ or $B \subset V$.*
2. *If $A \subset X$ is connected,*

a) then $\bar{A}$ is connected.

b) More generally, if A is connected and $B \subset \operatorname{acc}(A)$, then $A \cup B$ is connected as well. (Recall that $\operatorname{acc}(A)$ – the set of accumulation points of A was defined in Definition 10.29 above.)

3. If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected sets such that $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$, then $Y := \bigcup_{\alpha \in A} E_\alpha$ is connected as well.

4. Suppose $A, B \subset X$ are non-empty connected subsets of X such that $\bar{A} \cap B \neq \emptyset$, then $A \cup B$ is connected in X.

5. Every point $x \in X$ is contained in a unique maximal connected subset C_x of X and this subset is closed. The set C_x is called the ***connected component*** *of x.*

Proof.

1. Since B is the disjoint union of the relatively open sets $B \cap U$ and $B \cap V$, we must have $B \cap U = B$ or $B \cap V = B$ for otherwise $\{B \cap U, B \cap V\}$ would be a disconnection of B.
2. a. Let $Y = \bar{A}$ be equipped with the relative topology from X. Suppose that $U, V \subset_o Y$ form a disconnection of $Y = \bar{A}$. Then by 1. either $A \subset U$ or $A \subset V$. Say that $A \subset U$. Since U is both open an closed in Y, it follows that $Y = \bar{A} \subset U$. Therefore $V = \emptyset$ and we have a contradiction to the assumption that $\{U, V\}$ is a disconnection of $Y = \bar{A}$. Hence we must conclude that $Y = \bar{A}$ is connected as well.
   b. Now let $Y = A \cup B$ with $B \subset \operatorname{acc}(A)$, then
   $$\bar{A}^Y = \bar{A} \cap Y = (A \cup \operatorname{acc}(A)) \cap Y = A \cup B.$$
   Because A is connected in Y, by (2a) $Y = A \cup B = \bar{A}^Y$ is also connected.
3. Let $Y := \bigcup_{\alpha \in A} E_\alpha$. By Remark 10.44, we know that E_α is connected in Y for each $\alpha \in A$. If $\{U, V\}$ were a disconnection of Y, by item (1), either $E_\alpha \subset U$ or $E_\alpha \subset V$ for all α. Let $\Lambda = \{\alpha \in A : E_\alpha \subset U\}$ then $U = \cup_{\alpha \in \Lambda} E_\alpha$ and $V = \cup_{\alpha \in A \setminus \Lambda} E_\alpha$. (Notice that neither Λ or $A \setminus \Lambda$ can be empty since U and V are not empty.) Since
   $$\emptyset = U \cap V = \bigcup\nolimits_{\alpha \in \Lambda, \beta \in \Lambda^c} (E_\alpha \cap E_\beta) \supset \bigcap_{\alpha \in A} E_\alpha \neq \emptyset.$$
   we have reached a contradiction and hence no such disconnection exists.
4. (A good example to keep in mind here is $X = \mathbb{R}$, $A = (0,1)$ and $B = [1,2)$.) For sake of contradiction suppose that $\{U, V\}$ were a disconnection of $Y = A \cup B$. By item (1) either $A \subset U$ or $A \subset V$, say $A \subset U$ in which case $B \subset V$. Since $Y = A \cup B$ we must have $A = U$ and $B = V$ and so we may conclude: A and B are disjoint subsets of Y which are both open and closed. This implies
   $$A = \bar{A}^Y = \bar{A} \cap Y = \bar{A} \cap (A \cup B) = A \cup \left(\bar{A} \cap B\right)$$
   and therefore
   $$\emptyset = A \cap B = \left[A \cup \left(\bar{A} \cap B\right)\right] \cap B = \bar{A} \cap B \neq \emptyset$$
   which gives us the desired contradiction.
5. Let $\mathcal{C}$ denote the collection of connected subsets $C \subset X$ such that $x \in C$. Then by item 3., the set $C_x := \cup \mathcal{C}$ is also a connected subset of X which contains x and clearly this is the unique maximal connected set containing x. Since $\bar{C}_x$ is also connected by item (2) and C_x is maximal, $C_x = \bar{C}_x$, i.e. C_x is closed.

■

Theorem 10.48 (The Connected Subsets of $\mathbb{R}$). *The connected subsets of $\mathbb{R}$ are intervals.*

Proof. Suppose that $A \subset \mathbb{R}$ is a connected subset and that $a, b \in A$ with $a < b$. If there exists $c \in (a, b)$ such that $c \notin A$, then $U := (-\infty, c) \cap A$ and $V := (c, \infty) \cap A$ would form a disconnection of A. Hence $(a, b) \subset A$. Let $\alpha := \inf(A)$ and $\beta := \sup(A)$ and choose $\alpha_n, \beta_n \in A$ such that $\alpha_n < \beta_n$ and $\alpha_n \downarrow \alpha$ and $\beta_n \uparrow \beta$ as $n \to \infty$. By what we have just shown, $(\alpha_n, \beta_n) \subset A$ for all n and hence $(\alpha, \beta) = \cup_{n=1}^\infty (\alpha_n, \beta_n) \subset A$. From this it follows that $A = (\alpha, \beta)$, $[\alpha, \beta)$, $(\alpha, \beta]$ or $[\alpha, \beta]$, i.e. A is an interval.

Conversely suppose that A is an interval, and for sake of contradiction, suppose that $\{U, V\}$ is a disconnection of A with $a \in U$, $b \in V$. After relabelling U and V if necessary we may assume that $a < b$. Since A is an interval $[a, b] \subset A$. Let $p = \sup([a, b] \cap U)$, then because U and V are open, $a < p < b$. Now p can not be in U for otherwise $\sup([a, b] \cap U) > p$ and p can not be in V for otherwise $p < \sup([a, b] \cap U)$. From this it follows that $p \notin U \cup V$ and hence $A \neq U \cup V$ contradicting the assumption that $\{U, V\}$ is a disconnection. ■

Theorem 10.49 (Intermediate Value Theorem). *Suppose that (X, τ) is a connected topological space and $f : X \to \mathbb{R}$ is a continuous map. Then f satisfies the intermediate value property. Namely, for every pair $x, y \in X$ such that $f(x) < f(y)$ and $c \in (f(x), f(y))$, there exits $z \in X$ such that $f(z) = c$.*

Proof. By Lemma 10.45, $f(X)$ is connected subset of $\mathbb{R}$. So by Theorem 10.48, $f(X)$ is a subinterval of $\mathbb{R}$ and this completes the proof. ■

Definition 10.50. *A topological space X is* ***path connected*** *if to every pair of points $\{x_0, x_1\} \subset X$ there exists a continuous* ***path,*** *$\sigma \in C([0,1], X)$, such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The space X is said to be* ***locally path connected*** *if for each $x \in X$, there is an open neighborhood $V \subset X$ of x which is path connected.*

Proposition 10.51. *Let X be a topological space.*

1. *If X is path connected then X is connected.*
2. *If X is connected and locally path connected, then X is path connected.*
3. *If X is any connected open subset of $\mathbb{R}^n$, then X is path connected.*

Proof. The reader is asked to prove this proposition in Exercises 10.18 – 10.20 below. ■

Proposition 10.52 (Stability of Connectedness Under Products). *Let (X_α, τ_α) be connected topological spaces. Then the product space $X_A = \prod_{\alpha \in A} X_\alpha$ equipped with the product topology is connected.*

Proof. Let us begin with the case of two factors, namely assume that X and Y are connected topological spaces, then we will show that $X \times Y$ is connected as well. To do this let $p = (x_0, y_0) \in X \times Y$ and E denote the connected component of p. Since $\{x_0\} \times Y$ is homeomorphic to Y, $\{x_0\} \times Y$ is connected in $X \times Y$ and therefore $\{x_0\} \times Y \subset E$, i.e. $(x_0, y) \in E$ for all $y \in Y$. A similar argument now shows that $X \times \{y\} \subset E$ for any $y \in Y$, that is to $X \times Y = E$, see Figure 10.4. By induction the theorem holds whenever A is a finite set, i.e. for products of a finite number of connected spaces.

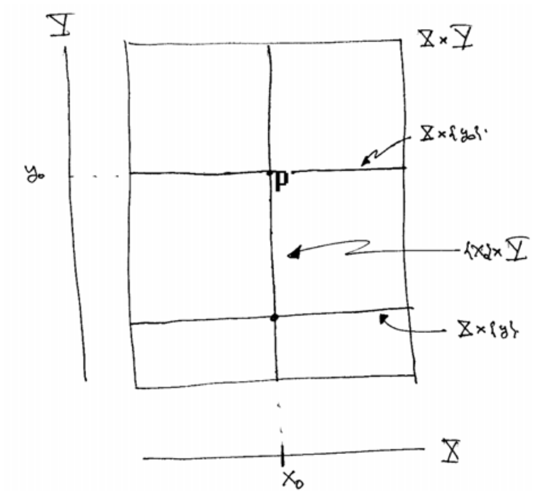

Fig. 10.4. This picture illustrates why the connected component of p in $X \times Y$ must contain all points of $X \times Y$.

For the general case, again choose a point $p \in X_A = X^A$ and let $C = C_p$ be the connected component of p in X_A. Recall that C_p is closed and therefore if C_p is a proper subset of X_A, then $X_A \setminus C_p$ is a non-empty open set. By the definition of the product topology, this would imply that $X_A \setminus C_p$ contains an open set of the form

$$V := \cap_{\alpha \in \Lambda} \pi_\alpha^{-1}(V_\alpha) = V_\Lambda \times X_{A \setminus \Lambda}$$

where $\Lambda \subset\subset A$ and $V_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda$. We will now show that no such V can exist and hence $X_A = C_p$, i.e. X_A is connected. Define $\phi : X_\Lambda \to X_A$ by $\phi(y) = x$ where

$$x_\alpha = \begin{cases} y_\alpha & \text{if } \alpha \in \Lambda \\ p_\alpha & \text{if } \alpha \notin \Lambda. \end{cases}$$

If $\alpha \in \Lambda$, $\pi_\alpha \circ \phi(y) = y_\alpha = \pi_\alpha(y)$ and if $\alpha \in A \setminus \Lambda$ then $\pi_\alpha \circ \phi(y) = p_\alpha$ so that in every case $\pi_\alpha \circ \phi : X_\Lambda \to X_\alpha$ is continuous and therefore ϕ is continuous. Since X_Λ is a product of a finite number of connected spaces it is connected by step 1. above. Hence so is the continuous image, $\phi(X_\Lambda) = X_\Lambda \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda}$, of X_Λ. Now $p \in \phi(X_\Lambda)$ and $\phi(X_\Lambda)$ is connected implies that $\phi(X_\Lambda) \subset C$. On the other hand one easily sees that

$$\emptyset \neq V \cap \phi(X_\Lambda) \subset V \cap C$$

contradicting the assumption that $V \subset C^c$. ■

10.6 Exercises

10.6.1 General Topological Space Problems

Exercise 10.12. Let V be an open subset of $\mathbb{R}$. Show V may be written as a disjoint union of open intervals $J_n = (a_n, b_n)$, where $a_n, b_n \in \mathbb{R} \cup \{\pm\infty\}$ for $n = 1, 2, \dots < N$ with $N = \infty$ possible.

Exercise 10.13. Let (X, τ) and (Y, τ') be a topological spaces, $f : X \to Y$ be a function, $\mathcal{U}$ be an open cover of X and $\{F_j\}_{j=1}^n$ be a finite cover of X by closed sets.

1. If $A \subset X$ is any set and $f : X \to Y$ is (τ, τ') – continuous then $f|_A : A \to Y$ is (τ_A, τ') – continuous.
2. Show $f : X \to Y$ is (τ, τ') – continuous iff $f|_U : U \to Y$ is (τ_U, τ') – continuous for all $U \in \mathcal{U}$.
3. Show $f : X \to Y$ is (τ, τ') – continuous iff $f|_{F_j} : F_j \to Y$ is (τ_{F_j}, τ') – continuous for all $j = 1, 2, \dots, n$.

Exercise 10.14. Suppose that X is a set, $\{(Y_\alpha, \tau_\alpha) : \alpha \in A\}$ is a family of topological spaces and $f_\alpha : X \to Y_\alpha$ is a given function for all $\alpha \in A$. Assuming that $\mathcal{S}_\alpha \subset \tau_\alpha$ is a sub-base for the topology τ_α for each $\alpha \in A$, show $\mathcal{S} := \cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{S}_\alpha)$ is a sub-base for the topology $\tau := \tau(f_\alpha : \alpha \in A)$.

10.6.2 Connectedness Problems

Exercise 10.15. Show any non-trivial interval in $\mathbb{Q}$ is disconnected.

Exercise 10.16. Suppose $a < b$ and $f : (a, b) \to \mathbb{R}$ is a non-decreasing function. Show if f satisfies the intermediate value property (see Theorem 10.49), then f is continuous.

Exercise 10.17. Suppose $-\infty < a < b \le \infty$ and $f : [a, b) \to \mathbb{R}$ is a strictly increasing continuous function. By Lemma 10.45, $f([a, b))$ is an interval and since f is strictly increasing it must of the form $[c, d)$ for some $c \in \mathbb{R}$ and $d \in \bar{\mathbb{R}}$ with $c < d$. Show the inverse function $f^{-1} : [c, d) \to [a, b)$ is continuous and is strictly increasing. In particular if $n \in \mathbb{N}$, apply this result to $f(x) = x^n$ for $x \in [0, \infty)$ to construct the positive n^{th} – root of a real number. Compare with Exercise 3.8

Exercise 10.18. Prove item 1. of Proposition 10.51. **Hint:** show X is not connected implies X is not path connected.

Exercise 10.19. Prove item 2. of Proposition 10.51. **Hint:** fix $x_0 \in X$ and let W denote the set of $x \in X$ such that there exists $\sigma \in C([0, 1], X)$ satisfying $\sigma(0) = x_0$ and $\sigma(1) = x$. Then show W is both open and closed.

Exercise 10.20. Prove item 3. of Proposition 10.51.

Exercise 10.21. Let

$$X := \left\{(x, y) \in \mathbb{R}^2 : y = \sin(x^{-1})\right\} \cup \{(0, 0)\}$$

equipped with the relative topology induced from the standard topology on $\mathbb{R}^2$. Show X is connected but not path connected.

10.6.3 Metric Spaces as Topological Spaces

Definition 10.53. *Two metrics d and ρ on a set X are said to be **equivalent** if there exists a constant $c \in (0, \infty)$ such that $c^{-1}\rho \le d \le c\rho$.*

Exercise 10.22. Suppose that d and ρ are two metrics on X.

1. Show $\tau_d = \tau_\rho$ if d and ρ are equivalent.
2. Show by example that it is possible for $\tau_d = \tau_\rho$ even thought d and ρ are inequivalent.

Exercise 10.23. Let (X_i, d_i) for $i = 1, \dots, n$ be a finite collection of metric spaces and for $1 \le p \le \infty$ and $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $X := \prod_{i=1}^n X_i$, let

$$\rho_p(x, y) = \begin{cases} \left(\sum_{i=1}^n [d_i(x_i, y_i)]^p\right)^{1/p} & \text{if } p \ne \infty \\ \max_i d_i(x_i, y_i) & \text{if } p = \infty \end{cases}.$$

1. Show (X, ρ_p) is a metric space for $p \in [1, \infty]$. **Hint**: Minkowski's inequality.
2. Show for any $p, q \in [1, \infty]$, the metrics ρ_p and ρ_q are equivalent. **Hint:** This can be done with explicit estimates or you could use Theorem 11.12 below.

Notation 10.54 *Let X be a set and $\mathbf{p} := \{p_n\}_{n=0}^\infty$ be a family of semi-metrics on X, i.e. $p_n : X \times X \to [0, \infty)$ are functions satisfying the assumptions of metric except for the assertion that $p_n(x, y) = 0$ implies $x = y$. Further assume that $p_n(x, y) \le p_{n+1}(x, y)$ for all n and if $p_n(x, y) = 0$ for all $n \in \mathbb{N}$ then $x = y$. Given $n \in \mathbb{N}$ and $x \in X$ let*

$$B_n(x, \varepsilon) := \{y \in X : p_n(x, y) < \varepsilon\}.$$

We will write $\tau(\mathbf{p})$ form the smallest topology on X such that $p_n(x, \cdot) : X \to [0, \infty)$ is continuous for all $n \in \mathbb{N}$ and $x \in X$, i.e. $\tau(\mathbf{p}) := \tau(p_n(x\cdot) : n \in \mathbb{N}$ and $x \in X)$.

Exercise 10.24. Using Notation 10.54, show that collection of balls,

$$\mathcal{B} := \{B_n(x, \varepsilon) : n \in \mathbb{N},\ x \in X \text{ and } \varepsilon > 0\},$$

forms a base for the topology $\tau(\mathbf{p})$. **Hint:** Use Exercise 10.14 to show $\mathcal{B}$ is a sub-base for the topology $\tau(\mathbf{p})$ and then use Exercise 10.2 to show $\mathcal{B}$ is in fact a base for the topology $\tau(\mathbf{p})$.

Exercise 10.25 (A minor variant of Exercise 6.12). Let p_n be as in Notation 10.54 and

$$d(x, y) := \sum_{n=0}^\infty 2^{-n} \frac{p_n(x, y)}{1 + p_n(x, y)}.$$

Show d is a metric on X and $\tau_d = \tau(\mathbf{p})$. Conclude that a sequence $\{x_k\}_{k=1}^\infty \subset X$ converges to $x \in X$ iff

$$\lim_{k\to\infty} p_n(x_k, x) = 0 \text{ for all } n \in \mathbb{N}.$$

Exercise 10.26. Let $\{(X_n, d_n)\}_{n=1}^\infty$ be a sequence of metric spaces, $X := \prod_{n=1}^\infty X_n$, and for $x = (x(n))_{n=1}^\infty$ and $y = (y(n))_{n=1}^\infty$ in X let

$$d(x, y) = \sum_{n=1}^\infty 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.$$

(See Exercise 6.12.) Moreover, let $\pi_n : X \to X_n$ be the projection maps, show

$$\tau_d = \otimes_{n=1}^\infty \tau_{d_n} := \tau(\{\pi_n : n \in \mathbb{N}\}).$$

That is show the d – metric topology is the same as the product topology on X. **Suggestions:** 1) show π_n is τ_d continuous for each n and 2) show for each $x \in X$ that $d(x, \cdot)$ is $\otimes_{n=1}^\infty \tau_{d_n}$ – continuous. For the second assertion notice that $d(x, \cdot) = \sum_{n=1}^\infty f_n$ where $f_n = 2^{-n}\left(\frac{d_n(x(n), \cdot)}{1 + d_n(x(n), \cdot)}\right) \circ \pi_n$.

11

Compactness

Definition 11.1. *The subset A of a topological space $(X\ \tau)$ is said to be* ***compact*** *if every open cover (Definition 10.18) of A has finite a sub-cover, i.e. if $\mathcal{U}$ is an open cover of A there exists $\mathcal{U}_0 \subset\subset \mathcal{U}$ such that $\mathcal{U}_0$ is a cover of A. (We will write $A \sqsubset\sqsubset X$ to denote that $A \subset X$ and A is compact.) A subset $A \subset X$ is* ***precompact*** *if $\bar{A}$ is compact.*

Proposition 11.2. *Suppose that $K \subset X$ is a compact set and $F \subset K$ is a closed subset. Then F is compact. If $\{K_i\}_{i=1}^n$ is a finite collections of compact subsets of X then $K = \cup_{i=1}^n K_i$ is also a compact subset of X.*

Proof. Let $\mathcal{U} \subset \tau$ be an open cover of F, then $\mathcal{U} \cup \{F^c\}$ is an open cover of K. The cover $\mathcal{U} \cup \{F^c\}$ of K has a finite subcover which we denote by $\mathcal{U}_0 \cup \{F^c\}$ where $\mathcal{U}_0 \subset\subset \mathcal{U}$. Since $F \cap F^c = \emptyset$, it follows that $\mathcal{U}_0$ is the desired subcover of F. For the second assertion suppose $\mathcal{U} \subset \tau$ is an open cover of K. Then $\mathcal{U}$ covers each compact set K_i and therefore there exists a finite subset $\mathcal{U}_i \subset\subset \mathcal{U}$ for each i such that $K_i \subset \cup \mathcal{U}_i$. Then $\mathcal{U}_0 := \cup_{i=1}^n \mathcal{U}_i$ is a finite cover of K. ■

Exercise 11.1. Suppose $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is a compact subset of Y. Give an example of continuous map, $f : X \to Y$, and a compact subset K of Y such that $f^{-1}(K)$ is not compact.

Exercise 11.2 (Dini's Theorem). Let X be a compact topological space and $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x \in X} f_n(x) \downarrow 0$ as $n \to \infty$. **Hint:** Given $\varepsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \varepsilon\}$.

Definition 11.3. *A collection $\mathcal{F}$ of closed subsets of a topological space (X, τ) has the* ***finite intersection property*** *if $\cap \mathcal{F}_0 \neq \emptyset$ for all $\mathcal{F}_0 \subset\subset \mathcal{F}$.*

The notion of compactness may be expressed in terms of closed sets as follows.

Proposition 11.4. *A topological space X is compact iff every family of closed sets $\mathcal{F} \subset 2^X$ having the **finite intersection property** satisfies $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. ($\Rightarrow$) Suppose that X is compact and $\mathcal{F} \subset 2^X$ is a collection of closed sets such that $\bigcap \mathcal{F} = \emptyset$. Let

$$\mathcal{U} = \mathcal{F}^c := \{C^c : C \in \mathcal{F}\} \subset \tau,$$

then $\mathcal{U}$ is a cover of X and hence has a finite subcover, $\mathcal{U}_0$. Let $\mathcal{F}_0 = \mathcal{U}_0^c \subset\subset \mathcal{F}$, then $\cap \mathcal{F}_0 = \emptyset$ so that $\mathcal{F}$ does not have the finite intersection property. ($\Leftarrow$) If X is not compact, there exists an open cover $\mathcal{U}$ of X with no finite subcover. Let

$$\mathcal{F} = \mathcal{U}^c := \{U^c : U \in \mathcal{U}\},$$

then $\mathcal{F}$ is a collection of closed sets with the finite intersection property while $\bigcap \mathcal{F} = \emptyset$. ■

Exercise 11.3. Let (X, τ) be a topological space. Show that $A \subset X$ is compact iff (A, τ_A) is a compact topological space.

11.1 Metric Space Compactness Criteria

Let (X, d) be a metric space and for $x \in X$ and $\varepsilon > 0$ let $B'_x(\varepsilon) = B_x(\varepsilon) \setminus \{x\}$ – the deleted ball centered at x of radius $\varepsilon > 0$. Recall from Definition 10.29 that a point $x \in X$ is an accumulation point of a subset $E \subset X$ if $\emptyset \neq E \cap V \setminus \{x\}$ for all open neighborhoods, V, of x. The proof of the following elementary lemma is left to the reader.

Lemma 11.5. *Let $E \subset X$ be a subset of a metric space (X, d). Then the following are equivalent:*

1. *$x \in X$ is an accumulation point of E.*
2. *$B'_x(\varepsilon) \cap E \neq \emptyset$ for all $\varepsilon > 0$.*
3. *$B_x(\varepsilon) \cap E$ is an infinite set for all $\varepsilon > 0$.*
4. *There exists $\{x_n\}_{n=1}^\infty \subset E \setminus \{x\}$ with $\lim_{n\to\infty} x_n = x$.*

Definition 11.6. *A metric space (X, d) is ε – **bounded** ($\varepsilon > 0$) if there exists a finite cover of X by balls of radius ε and it is **totally bounded** if it is ε – bounded for all $\varepsilon > 0$.*

Theorem 11.7. *Let (X, d) be a metric space. The following are equivalent.*

(a) X is compact.
(b) Every infinite subset of X has an accumulation point.
(c) Every sequence $\{x_n\}_{n=1}^\infty \subset X$ has a convergent subsequence.
(d) X is totally bounded and complete.

Proof. The proof will consist of showing that $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$. ($a \Rightarrow b$) We will show that **not** $b \Rightarrow$ **not** a. Suppose there exists an infinite subset $E \subset X$ which has no accumulation points. Then for all $x \in X$ there exists $\delta_x > 0$ such that $V_x := B_x(\delta_x)$ satisfies $(V_x \setminus \{x\}) \cap E = \emptyset$. Clearly $\mathcal{V} = \{V_x\}_{x\in X}$ is a cover of X, yet $\mathcal{V}$ has no finite sub cover. Indeed, for each $x \in X$, $V_x \cap E \subset \{x\}$ and hence if $\Lambda \subset\subset X$, $\cup_{x\in\Lambda} V_x$ can only contain a finite number of points from E (namely $\Lambda \cap E$). Thus for any $\Lambda \subset\subset X$, $E \nsubseteq \cup_{x\in\Lambda} V_x$ and in particular $X \neq \cup_{x\in\Lambda} V_x$. (See Figure 11.1.) ($b \Rightarrow c$) Let $\{x_n\}_{n=1}^\infty \subset X$

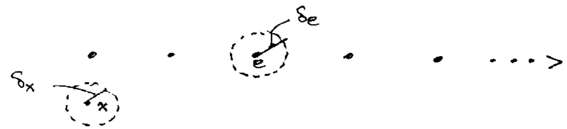

Fig. 11.1. The construction of an open cover with no finite sub-cover.

be a sequence and $E := \{x_n : n \in \mathbb{N}\}$. If $\#(E) < \infty$, then $\{x_n\}_{n=1}^\infty$ has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which is constant and hence convergent. On the other hand if $\#(E) = \infty$ then by assumption E has an accumulation point and hence by Lemma 11.5, $\{x_n\}_{n=1}^\infty$ has a convergent subsequence. ($c \Rightarrow d$) Suppose $\{x_n\}_{n=1}^\infty \subset X$ is a Cauchy sequence. By assumption there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which is convergent to some point $x \in X$. Since $\{x_n\}_{n=1}^\infty$ is Cauchy it follows that $x_n \to x$ as $n \to \infty$ showing X is complete. We now show that X is totally bounded. Let $\varepsilon > 0$ be given and choose an arbitrary point $x_1 \in X$. If possible choose $x_2 \in X$ such that $d(x_2, x_1) \geq \varepsilon$, then if possible choose $x_3 \in X$ such that $d_{\{x_1,x_2\}}(x_3) \geq \varepsilon$ and continue inductively choosing points $\{x_j\}_{j=1}^n \subset X$ such that $d_{\{x_1,\dots,x_{n-1}\}}(x_n) \geq \varepsilon$. (See Figure 11.2.) This process must terminate, for otherwise we would produce a sequence $\{x_n\}_{n=1}^\infty \subset X$ which can have no convergent subsequences. Indeed, the x_n have been chosen so that $d(x_n, x_m) \geq \varepsilon > 0$ for every $m \neq n$ and hence no subsequence of $\{x_n\}_{n=1}^\infty$ can be Cauchy. ($d \Rightarrow a$) For sake of contradiction, assume there exists an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ of X with no finite subcover. Since X is totally bounded for each $n \in \mathbb{N}$ there exists $\Lambda_n \subset\subset X$ such that

$$X = \bigcup_{x\in\Lambda_n} B_x(1/n) \subset \bigcup_{x\in\Lambda_n} C_x(1/n).$$

Choose $x_1 \in \Lambda_1$ such that no finite subset of $\mathcal{V}$ covers $K_1 := C_{x_1}(1)$. Since $K_1 = \cup_{x\in\Lambda_2} K_1 \cap C_x(1/2)$, there exists $x_2 \in \Lambda_2$ such that $K_2 := K_1 \cap C_{x_2}(1/2)$ can not be covered by a finite subset of $\mathcal{V}$, see Figure 11.3. Continuing this way inductively, we construct sets $K_n = K_{n-1} \cap C_{x_n}(1/n)$ with $x_n \in \Lambda_n$ such no K_n can be covered by a finite subset of $\mathcal{V}$. Now choose $y_n \in K_n$ for each n. Since $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of closed sets such that $\operatorname{diam}(K_n) \leq 2/n$, it follows that $\{y_n\}$ is a Cauchy and hence convergent with

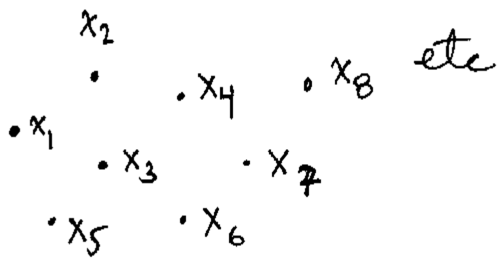

Fig. 11.2. Constructing a set with out an accumulation point.

$$y = \lim_{n\to\infty} y_n \in \cap_{m=1}^{\infty} K_m.$$

Since $\mathcal{V}$ is a cover of X, there exists $V \in \mathcal{V}$ such that $x \in V$. Since $K_n \downarrow \{y\}$ and $\operatorname{diam}(K_n) \to 0$, it now follows that $K_n \subset V$ for some n large. But this violates the assertion that K_n can not be covered by a finite subset of $\mathcal{V}$.

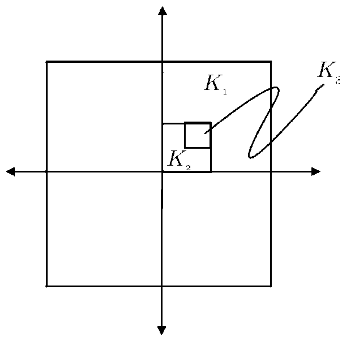

Fig. 11.3. Nested Sequence of cubes.

∎

Corollary 11.8. *Any compact metric space (X, d) is second countable and hence also separable by Exercise 10.9. (See Example 12.25 below for an example of a compact topological space which is not separable.)*

Proof. To each integer n, there exists $\Lambda_n \subset\subset X$ such that $X = \cup_{x\in\Lambda_n} B(x, 1/n)$. The collection of open balls,

$$\mathcal{V} := \cup_{n\in\mathbb{N}} \cup_{x\in\Lambda_n} \{B(x, 1/n)\}$$

forms a countable basis for the metric topology on X. To check this, suppose that $x_0 \in X$ and $\varepsilon > 0$ are given and choose $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$ and $x \in \Lambda_n$ such that $d(x_0, x) < 1/n$. Then $B(x, 1/n) \subset B(x_0, \varepsilon)$ because for $y \in B(x, 1/n)$,

$$d(y, x_0) \le d(y, x) + d(x, x_0) < 2/n < \varepsilon.$$

∎

Corollary 11.9. *The compact subsets of $\mathbb{R}^n$ are the closed and bounded sets.*

Proof. This is a consequence of Theorem 8.2 and Theorem 11.7. Here is another proof. If K is closed and bounded then K is complete (being the closed subset of a complete space) and K is contained in $[-M, M]^n$ for some positive integer M. For $\delta > 0$, let

$$\Lambda_\delta = \delta\mathbb{Z}^n \cap [-M, M]^n := \{\delta x : x \in \mathbb{Z}^n \text{ and } \delta|x_i| \le M \text{ for } i = 1, 2, \dots, n\}.$$

We will show, by choosing $\delta > 0$ sufficiently small, that

$$K \subset [-M, M]^n \subset \cup_{x\in\Lambda_\delta} B(x, \varepsilon) \tag{11.1}$$

which shows that K is totally bounded. Hence by Theorem 11.7, K is compact. Suppose that $y \in [-M, M]^n$, then there exists $x \in \Lambda_\delta$ such that $|y_i - x_i| \le \delta$ for $i = 1, 2, \dots, n$. Hence

$$d^2(x, y) = \sum_{i=1}^{n} (y_i - x_i)^2 \le n\delta^2$$

which shows that $d(x, y) \le \sqrt{n}\delta$. Hence if choose $\delta < \varepsilon/\sqrt{n}$ we have shows that $d(x, y) < \varepsilon$, i.e. Eq. (11.1) holds. ∎

Example 11.10. Let $X = \ell^p(\mathbb{N})$ with $p \in [1, \infty)$ and $\mu \in \ell^p(\mathbb{N})$ such that $\mu(k) \ge 0$ for all $k \in \mathbb{N}$. The set

$$K := \{x \in X : |x(k)| \le \mu(k) \text{ for all } k \in \mathbb{N}\}$$

is compact. To prove this, let $\{x_n\}_{n=1}^\infty \subset K$ be a sequence. By compactness of closed bounded sets in $\mathbb{C}$, for each $k \in \mathbb{N}$ there is a subsequence of $\{x_n(k)\}_{n=1}^\infty \subset \mathbb{C}$ which is convergent. By Cantor's diagonalization trick, we may choose a subsequence $\{y_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $y(k) := \lim_{n\to\infty} y_n(k)$ exists for all $k \in \mathbb{N}$.[1] Since $|y_n(k)| \le \mu(k)$ for all n it follows that $|y(k)| \le \mu(k)$, i.e. $y \in K$. Finally

[1] The argument is as follows. Let $\{n_j^1\}_{j=1}^\infty$ be a subsequence of $\mathbb{N} = \{n\}_{n=1}^\infty$ such that $\lim_{j\to\infty} x_{n_j^1}(1)$ exists. Now choose a subsequence $\{n_j^2\}_{j=1}^\infty$ of $\{n_j^1\}_{j=1}^\infty$ such that $\lim_{j\to\infty} x_{n_j^2}(2)$ exists and similarly $\{n_j^3\}_{j=1}^\infty$ of $\{n_j^2\}_{j=1}^\infty$ such that $\lim_{j\to\infty} x_{n_j^3}(3)$ exists. Continue on this way inductively to get

$$\{n\}_{n=1}^\infty \supset \{n_j^1\}_{j=1}^\infty \supset \{n_j^2\}_{j=1}^\infty \supset \{n_j^3\}_{j=1}^\infty \supset \dots$$

such that $\lim_{j\to\infty} x_{n_j^k}(k)$ exists for all $k \in \mathbb{N}$. Let $m_j := n_j^j$ so that eventually $\{m_j\}_{j=1}^\infty$ is a subsequence of $\{n_j^k\}_{j=1}^\infty$ for all k. Therefore, we may take $y_j := x_{m_j}$.

$$\lim_{n\to\infty}\|y-y_n\|_p^p=\lim_{n\to\infty}\sum_{k=1}^{\infty}|y(k)-y_n(k)|^p=\sum_{k=1}^{\infty}\lim_{n\to\infty}|y(k)-y_n(k)|^p=0$$

wherein we have used the Dominated convergence theorem. (Note $|y(k)-y_n(k)|^p$ $2^p\mu^p(k)$ and μ^p is summable.) Therefore $y_n\to y$ and we are done.

Alternatively, we can prove K is compact by showing that K is closed and totally bounded. It is simple to show K is closed, for if $\{x_n\}_{n=1}^\infty\subset K$ is a convergent sequence in X, $x:=\lim_{n\to\infty}x_n$, then

$$|x(k)|\le\lim_{n\to\infty}|x_n(k)|\le\mu(k)\ \forall\ k\in\mathbb{N}.$$

This shows that $x\in K$ and hence K is closed. To see that K is totally bounded, let $\varepsilon>0$ and choose N such that $\left(\sum_{k=N+1}^\infty|\mu(k)|^p\right)^{1/p}<\varepsilon$. Since $\prod_{k=1}^N C_{\mu(k)}(0)\subset\mathbb{C}^N$ is closed and bounded, it is compact. Therefore there exists a finite subset $\Lambda\subset\prod_{k=1}^N C_{\mu(k)}(0)$ such that

$$\prod_{k=1}^N C_{\mu(k)}(0)\subset\cup_{z\in\Lambda}B_z^N(\varepsilon)$$

where $B_z^N(\varepsilon)$ is the open ball centered at $z\in\mathbb{C}^N$ relative to the $\ell^p(\{1,2,3,\dots,N\})$ – norm. For each $z\in\Lambda$, let $\tilde z\in X$ be defined by $\tilde z(k)=z(k)$ if $k\le N$ and $\tilde z(k)=0$ for $k\ge N+1$. I now claim that

$$K\subset\cup_{z\in\Lambda}B_{\tilde z}(2\varepsilon) \tag{11.2}$$

which, when verified, shows K is totally bounded. To verify Eq. (11.2), let $x\in K$ and write $x=u+v$ where $u(k)=x(k)$ for $k\le N$ and $u(k)=0$ for $k<N$. Then by construction $u\in B_{\tilde z}(\varepsilon)$ for some $\tilde z\in\Lambda$ and

$$\|v\|_p\le\left(\sum_{k=N+1}^\infty|\mu(k)|^p\right)^{1/p}<\varepsilon.$$

So we have

$$\|x-\tilde z\|_p=\|u+v-\tilde z\|_p\le\|u-\tilde z\|_p+\|v\|_p<2\varepsilon.$$

Exercise 11.4 (Extreme value theorem). Let (X,τ) be a compact topological space and $f:X\to\mathbb{R}$ be a continuous function. Show $-\infty<\inf f\le\sup f<\infty$ and there exists $a,b\in X$ such that $f(a)=\inf f$ and $f(b)=\sup f$[2]. **Hint:** use Exercise 11.1 and Corollary 11.9.

[2] Here is a proof if X is a metric space. Let $\{x_n\}_{n=1}^\infty\subset X$ be a sequence such that $f(x_n)\uparrow\sup f$. By compactness of X we may assume, by passing to a subsequence if necessary that $x_n\to b\in X$ as $n\to\infty$. By continuity of f, $f(b)=\sup f$.

Exercise 11.5 (Uniform Continuity). Let (X,d) be a compact metric space, (Y,ρ) be a metric space and $f:X\to Y$ be a continuous function. Show that f is uniformly continuous, i.e. if $\varepsilon>0$ there exists $\delta>0$ such that $\rho(f(y),f(x))<\varepsilon$ if $x,y\in X$ with $d(x,y)<\delta$. **Hint:** you could follow the argument in the proof of Theorem 8.2.

Definition 11.11. *Let L be a vector space. We say that two norms, $|\cdot|$ and $\|\cdot\|$, on L are equivalent if there exists constants $\alpha,\beta\in(0,\infty)$ such that*

$$\|f\|\le\alpha|f|\ \text{ and }\ |f|\le\beta\|f\|\ \text{ for all } f\in L.$$

Theorem 11.12. *Let L be a finite dimensional vector space. Then any two norms $|\cdot|$ and $\|\cdot\|$ on L are equivalent. (This is typically not true for norms on infinite dimensional spaces, see for example Exercise 7.7.)*

Proof. Let $\{f_i\}_{i=1}^n$ be a basis for L and define a new norm on L by

$$\left\|\sum_{i=1}^n a_if_i\right\|_2:=\sqrt{\sum_{i=1}^n|a_i|^2}\ \text{ for } a_i\in\mathbb{F}.$$

By the triangle inequality for the norm $|\cdot|$, we find

$$\left|\sum_{i=1}^n a_if_i\right|\le\sum_{i=1}^n|a_i|\,|f_i|\le\sqrt{\sum_{i=1}^n|f_i|^2}\sqrt{\sum_{i=1}^n|a_i|^2}\le M\left\|\sum_{i=1}^n a_if_i\right\|_2$$

where $M=\sqrt{\sum_{i=1}^n|f_i|^2}$. Thus we have

$$|f|\le M\|f\|_2$$

for all $f\in L$ and this inequality shows that $|\cdot|$ is continuous relative to $\|\cdot\|_2$. Since the normed space $(L,\|\cdot\|_2)$ is homeomorphic and isomorphic to $\mathbb{F}^n$ with the standard euclidean norm, the closed bounded set, $S:=\{f\in L:\|f\|_2=1\}\subset L$, is a compact subset of L relative to $\|\cdot\|_2$. Therefore by Exercise 11.4 there exists $f_0\in S$ such that

$$m=\inf\{|f|:f\in S\}=|f_0|>0.$$

Hence given $0\neq f\in L$, then $\frac{f}{\|f\|_2}\in S$ so that

$$m\le\left|\frac{f}{\|f\|_2}\right|=|f|\frac{1}{\|f\|_2}$$

or equivalently

$$\|f\|_2\le\frac{1}{m}|f|.$$

This shows that $|\cdot|$ and $\|\cdot\|_2$ are equivalent norms. Similarly one shows that $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent and hence so are $|\cdot|$ and $\|\cdot\|$. ■

Corollary 11.13. *If $(L, \|\cdot\|)$ is a finite dimensional normed space, then $A \subset L$ is compact iff A is closed and bounded relative to the given norm, $\|\cdot\|$.*

Corollary 11.14. *Every finite dimensional normed vector space $(L, \|\cdot\|)$ is complete. In particular any finite dimensional subspace of a normed vector space is automatically closed.*

Proof. If $\{f_n\}_{n=1}^\infty \subset L$ is a Cauchy sequence, then $\{f_n\}_{n=1}^\infty$ is bounded and hence has a convergent subsequence, $g_k = f_{n_k}$, by Corollary 11.13. It is now routine to show $\lim_{n\to\infty} f_n = f := \lim_{k\to\infty} g_k$. ∎

Theorem 11.15. *Suppose that $(X, \|\cdot\|)$ is a normed vector in which the unit ball, $V := B_0(1)$, is precompact. Then* $\dim X < \infty$.

Proof. Since $\bar{V}$ is compact, we may choose $\Lambda \subset\subset X$ such that

$$\bar{V} \subset \cup_{x\in\Lambda}\left(x + \frac{1}{2}V\right)$$

where, for any $\delta > 0$,

$$\delta V := \{\delta x : x \in V\} = B_0(\delta).$$

Let $Y := \text{span}(\Lambda)$, then the previous equation implies that

$$V \subset \bar{V} \subset Y + \frac{1}{2}V.$$

Multiplying this equation by $\frac{1}{2}$ then shows

$$\frac{1}{2}V \subset \frac{1}{2}Y + \frac{1}{4}V = Y + \frac{1}{4}V$$

and hence

$$V \subset Y + \frac{1}{2}V \subset Y + Y + \frac{1}{4}V = Y + \frac{1}{4}V.$$

Continuing this way inductively then shows that

$$V \subset Y + \frac{1}{2^n}V \text{ for all } n \in \mathbb{N}.$$

Hence if $x \in V$, there exists $y_n \in Y$ and $z_n \in B_0(2^{-n})$ such that $y_n + z_n \to x$. Since $\lim_{n\to\infty} z_n = 0$, it follows that $x = \lim_{n\to\infty} y_n \in \bar{Y}$. Since $\dim Y \le \#(\Lambda) < \infty$, Corollary 11.14 implies $Y = \bar{Y}$ and so we have shown that $V \subset Y$. Since for any $x \in X$, $\frac{1}{2\|x\|}x \in V \subset Y$, we have $x \in Y$ for all $x \in X$, i.e. $X = Y$. ∎

Exercise 11.6. Suppose $(Y, \|\cdot\|_Y)$ is a normed space and $(X, \|\cdot\|_X)$ is a finite dimensional normed space. Show **every** linear transformation $T : X \to Y$ is necessarily bounded.

11.2 Compact Operators

Definition 11.16. *Let $A : X \to Y$ be a bounded operator between two (separable) Banach spaces. Then A is* ***compact*** *if $A[B_X(0,1)]$ is precompact in Y or equivalently for any $\{x_n\}_{n=1}^\infty \subset X$ such that $\|x_n\| \le 1$ for all n the sequence $y_n := Ax_n \in Y$ has a convergent subsequence.*

Example 11.17. Let $X = \ell^2 = Y$ and $\lambda_n \in \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = 0$, then $A : X \to Y$ defined by $(Ax)(n) = \lambda_n x(n)$ is compact.

Proof. Suppose $\{x_j\}_{j=1}^\infty \subset \ell^2$ such that $\|x_j\|^2 = \sum |x_j(n)|^2 \le 1$ for all j. By Cantor's Diagonalization argument, there exists $\{j_k\} \subset \{j\}$ such that, for each n, $\tilde{x}_k(n) = x_{j_k}(n)$ converges to some $\tilde{x}(n) \in \mathbb{C}$ as $k \to \infty$. Since for any $M < \infty$,

$$\sum_{n=1}^M |\tilde{x}(n)|^2 = \lim_{k\to\infty} \sum_{n=1}^M |\tilde{x}_k(n)|^2 \le 1$$

we may conclude that $\sum_{n=1}^\infty |\tilde{x}(n)|^2 \le 1$, i.e. $\tilde{x} \in \ell^2$. Let $y_k := A\tilde{x}_k$ and $y := A\tilde{x}$. We will finish the verification of this example by showing $y_k \to y$ in ℓ^2 as $k \to \infty$. Indeed if $\lambda_M^* = \max_{n\ge M} |\lambda_n|$, then

$$\begin{aligned}
\|A\tilde{x}_k - A\tilde{x}\|^2 &= \sum_{n=1}^\infty |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 \\
&= \sum_{n=1}^M |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 + |\lambda_M^*|^2 \sum_{M+1}^\infty |\tilde{x}_k(n) - \tilde{x}(n)|^2 \\
&\le \sum_{n=1}^M |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 + |\lambda_M^*|^2 \|\tilde{x}_k - \tilde{x}\|^2 \\
&\le \sum_{n=1}^M |\lambda_n|^2 |\tilde{x}_k(n) - \tilde{x}(n)|^2 + 4|\lambda_M^*|^2.
\end{aligned}$$

Passing to the limit in this inequality then implies

$$\limsup_{k\to\infty} \|A\tilde{x}_k - A\tilde{x}\|^2 \le 4|\lambda_M^*|^2 \to 0 \text{ as } M \to \infty.$$

∎

Lemma 11.18. *If $X \xrightarrow{A} Y \xrightarrow{B} Z$ are bounded operators such the either A or B is compact then the composition $BA : X \to Z$ is also compact.*

Proof. Let $B_X(0,1)$ be the open unit ball in X. If A is compact and B is bounded, then $BA(B_X(0,1)) \subset B(\overline{AB_X(0,1)})$ which is compact since the image of compact sets under continuous maps are compact. Hence we conclude that $\overline{BA(B_X(0,1))}$ is compact, being the closed subset of the compact set $B(\overline{AB_X(0,1)})$. If A is continuous and B is compact, then $A(B_X(0,1))$ is a bounded set and so by the compactness of B, $BA(B_X(0,1))$ is a precompact subset of Z, i.e. BA is compact. ■

11.3 Local and σ – Compactness

Notation 11.19 *If X is a topological spaces and Y is a normed space, let*

$$BC(X,Y) := \{f \in C(X,Y) : \sup_{x\in X} \|f(x)\|_Y < \infty\}$$

and

$$C_c(X,Y) := \{f \in C(X,Y) : \operatorname{supp}(f) \text{ is compact}\}.$$

If $Y = \mathbb{R}$ or $\mathbb{C}$ we will simply write $C(X)$, $BC(X)$ and $C_c(X)$ for $C(X,Y)$, $BC(X,Y)$ and $C_c(X,Y)$ respectively.

Remark 11.20. Let X be a topological space and Y be a Banach space. By combining Exercise 11.1 and Theorem 11.7 it follows that $C_c(X,Y) \subset BC(X,Y)$.

Definition 11.21 (Local and σ – compactness). *Let (X,τ) be a topological space.*

1. *(X,τ) is **locally compact** if for all $x \in X$ there exists an open neighborhood $V \subset X$ of x such that $\bar{V}$ is compact. (Alternatively, in light of Definition 10.29 (also see Definition 6.5), this is equivalent to requiring that to each $x \in X$ there exists a compact neighborhood N_x of x.)*
2. *(X,τ) is σ – **compact** if there exists compact sets $K_n \subset X$ such that $X = \cup_{n=1}^\infty K_n$. (Notice that we may assume, by replacing K_n by $K_1 \cup K_2 \cup \dots \cup K_n$ if necessary, that $K_n \uparrow X$.)*

Example 11.22. Any open subset of $U \subset \mathbb{R}^n$ is a locally compact and σ – compact metric space. The proof of local compactness is easy and is left to the reader. To see that U is σ – compact, for $k \in \mathbb{N}$, let

$$K_k := \{x \in U : |x| \le k \text{ and } d_{U^c}(x) \ge 1/k\}.$$

Then K_k is a closed and bounded subset of $\mathbb{R}^n$ and hence compact. Moreover $K_k^o \uparrow U$ as $k \to \infty$ since[3]

$$K_k^o \supset \{x \in U : |x| < k \text{ and } d_{U^c}(x) > 1/k\} \uparrow U \text{ as } k \to \infty.$$

Exercise 11.7. If (X,τ) is locally compact and second countable, then there is a countable basis $\mathcal{B}_0$ for the topology consisting of precompact open sets. Use this to show (X,τ) is σ - compact.

[3] In fact this is an equality, but we will not need this here.

Exercise 11.8. Every separable locally compact metric space is σ – compact.

Exercise 11.9. Every σ – compact metric space is second countable (or equivalently separable), see Corollary 11.8.

Exercise 11.10. Suppose that (X,d) is a metric space and $U \subset X$ is an open subset.

1. If X is locally compact then (U,d) is locally compact.
2. If X is σ – compact then (U,d) is σ – compact. **Hint:** Mimic Example 11.22, replacing $\{x \in \mathbb{R}^n : |x| \le k\}$ by compact sets $X_k \sqsubset\sqsubset X$ such that $X_k \uparrow X$.

Lemma 11.23. *Let (X,τ) be locally and σ – compact. Then there exists compact sets $K_n \uparrow X$ such that $K_n \subset K_{n+1}^o \subset K_{n+1}$ for all n.*

Proof. Suppose that $C \subset X$ is a compact set. For each $x \in C$ let $V_x \subset_o X$ be an open neighborhood of x such that $\bar{V}_x$ is compact. Then $C \subset \cup_{x\in C} V_x$ so there exists $\Lambda \subset\subset C$ such that

$$C \subset \cup_{x\in\Lambda} V_x \subset \cup_{x\in\Lambda} \bar{V}_x =: K.$$

Then K is a compact set, being a finite union of compact subsets of X, and $C \subset \cup_{x\in\Lambda} V_x \subset K^o$. Now let $C_n \subset X$ be compact sets such that $C_n \uparrow X$ as $n \to \infty$. Let $K_1 = C_1$ and then choose a compact set K_2 such that $C_2 \subset K_2^o$. Similarly, choose a compact set K_3 such that $K_2 \cup C_3 \subset K_3^o$ and continue inductively to find compact sets K_n such that $K_n \cup C_{n+1} \subset K_{n+1}^o$ for all n. Then $\{K_n\}_{n=1}^\infty$ is the desired sequence. ■

Remark 11.24. Lemma 11.23 may also be stated as saying there exists precompact open sets $\{G_n\}_{n=1}^\infty$ such that $G_n \subset \bar{G}_n \subset G_{n+1}$ for all n and $G_n \uparrow X$ as $n \to \infty$. Indeed if $\{G_n\}_{n=1}^\infty$ are as above, let $K_n := \bar{G}_n$ and if $\{K_n\}_{n=1}^\infty$ are as in Lemma 11.23, let $G_n := K_n^o$.

Proposition 11.25. *Suppose X is a locally compact metric space and $U \subset_o X$ and $K \sqsubset\sqsubset U$. Then there exists $V \subset_o X$ such that $K \subset V \subset \overline{V} \subset U \subset X$ and $\bar{V}$ is compact.*

Proof. (This is done more generally in Proposition 12.7 below.) By local compactness or X, for each $x \in K$ there exists $\varepsilon_x > 0$ such that $\overline{B_x(\varepsilon_x)}$ is compact and by shrinking ε_x if necessary we may assume,

$$\overline{B_x(\varepsilon_x)} \subset C_x(\varepsilon_x) \subset B_x(2\varepsilon_x) \subset U$$

for each $x \in K$. By compactness of K, there exists $\Lambda \subset\subset K$ such that $K \subset \cup_{x\in\Lambda} B_x(\varepsilon_x) =: V$. Notice that $\bar{V} \subset \cup_{x\in\Lambda} \overline{B_x(\varepsilon_x)} \subset U$ and $\bar{V}$ is a closed subset of the compact set $\cup_{x\in\Lambda} \overline{B_x(\varepsilon_x)}$ and hence compact as well. ■

Definition 11.26. *Let U be an open subset of a topological space (X,τ). We will write $f \prec U$ to mean a function $f \in C_c(X,[0,1])$ such that* $\operatorname{supp}(f) := \overline{\{f \neq 0\}} \subset U$.

Lemma 11.27 (Urysohn's Lemma for Metric Spaces). *Let X be a locally compact metric space and $K \sqsubset\sqsubset U \subset_o X$. Then there exists $f \prec U$ such that $f = 1$ on K. In particular, if K is compact and C is closed in X such that $K \cap C = \emptyset$, there exists $f \in C_c(X,[0,1])$ such that $f = 1$ on K and $f = 0$ on C.*

Proof. Let V be as in Proposition 11.25 and then use Lemma 6.15 to find a function $f \in C(X,[0,1])$ such that $f = 1$ on K and $f = 0$ on V^c. Then $\operatorname{supp}(f) \subset \bar{V} \subset U$ and hence $f \prec U$. ■

11.4 Function Space Compactness Criteria

In this section, let (X,τ) be a topological space.

Definition 11.28. *Let $\mathcal{F} \subset C(X)$.*

1. $\mathcal{F}$ is **equicontinuous at** $x \in X$ iff for all $\varepsilon > 0$ there exists $U \in \tau_x$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U$ and $f \in \mathcal{F}$.
2. $\mathcal{F}$ is **equicontinuous** if $\mathcal{F}$ is equicontinuous at all points $x \in X$.
3. $\mathcal{F}$ is **pointwise bounded** if $\sup\{|f(x)| : |f \in \mathcal{F}\} < \infty$ for all $x \in X$.

Theorem 11.29 (Ascoli-Arzela Theorem). *Let (X,τ) be a compact topological space and $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is precompact in $C(X)$ iff $\mathcal{F}$ is equicontinuous and point-wise bounded.*

Proof. ($\Leftarrow$) Since $C(X) \subset \ell^\infty(X)$ is a complete metric space, we must show $\mathcal{F}$ is totally bounded. Let $\varepsilon > 0$ be given. By equicontinuity, for all $x \in X$, there exists $V_x \in \tau_x$ such that $|f(y) - f(x)| < \varepsilon/2$ if $y \in V_x$ and $f \in \mathcal{F}$. Since X is compact we may choose $\Lambda \subset\subset X$ such that $X = \cup_{x\in\Lambda} V_x$. We have now decomposed X into "blocks" $\{V_x\}_{x\in\Lambda}$ such that each $f \in \mathcal{F}$ is constant to within ε on V_x. Since $\sup\{|f(x)| : x \in \Lambda \text{ and } f \in \mathcal{F}\} < \infty$, it is now evident that

$$\begin{aligned} M &= \sup\{|f(x)| : x \in X \text{ and } f \in \mathcal{F}\} \\ &\leq \sup\{|f(x)| : x \in \Lambda \text{ and } f \in \mathcal{F}\} + \varepsilon < \infty. \end{aligned}$$

Let $\mathbb{D} := \{k\varepsilon/2 : k \in \mathbb{Z}\} \cap [-M, M]$. If $f \in \mathcal{F}$ and $\phi \in \mathbb{D}^\Lambda$ (i.e. $\phi : \Lambda \to \mathbb{D}$ is a function) is chosen so that $|\phi(x) - f(x)| \leq \varepsilon/2$ for all $x \in \Lambda$, then

$$|f(y) - \phi(x)| \leq |f(y) - f(x)| + |f(x) - \phi(x)| < \varepsilon \;\forall\, x \in \Lambda \text{ and } y \in V_x.$$

From this it follows that $\mathcal{F} = \bigcup\{\mathcal{F}_\phi : \phi \in \mathbb{D}^\Lambda\}$ where, for $\phi \in \mathbb{D}^\Lambda$,

$$\mathcal{F}_\phi := \{f \in \mathcal{F} : |f(y) - \phi(x)| < \varepsilon \text{ for } y \in V_x \text{ and } x \in \Lambda\}.$$

Let $\Gamma := \{\phi \in \mathbb{D}^\Lambda : \mathcal{F}_\phi \neq \emptyset\}$ and for each $\phi \in \Gamma$ choose $f_\phi \in \mathcal{F}_\phi \cap \mathcal{F}$. For $f \in \mathcal{F}_\phi$, $x \in \Lambda$ and $y \in V_x$ we have

$$|f(y) - f_\phi(y)| \leq |f(y) - \phi(x))| + |\phi(x) - f_\phi(y)| < 2\varepsilon.$$

So $\|f - f_\phi\|_\infty < 2\varepsilon$ for all $f \in \mathcal{F}_\phi$ showing that $\mathcal{F}_\phi \subset B_{f_\phi}(2\varepsilon)$. Therefore,

$$\mathcal{F} = \cup_{\phi\in\Gamma} \mathcal{F}_\phi \subset \cup_{\phi\in\Gamma} B_{f_\phi}(2\varepsilon)$$

and because $\varepsilon > 0$ was arbitrary we have shown that $\mathcal{F}$ is totally bounded.

($\Rightarrow$) (*The rest of this proof may safely be skipped.) Since $\|\cdot\|_\infty : C(X) \to [0,\infty)$ is a continuous function on $C(X)$ it is bounded on any compact subset $\mathcal{F} \subset C(X)$. This shows that $\sup\{\|f\|_\infty : f \in \mathcal{F}\} < \infty$ which clearly implies that $\mathcal{F}$ is pointwise bounded.[4] Suppose $\mathcal{F}$ were **not** equicontinuous at some point $x \in X$ that is to say there exists $\varepsilon > 0$ such that for all $V \in \tau_x$, $\sup\limits_{y\in V} \sup\limits_{f\in\mathcal{F}} |f(y) - f(x)| > \varepsilon$.[5] Equivalently said, to each $V \in \tau_x$ we may choose

$$f_V \in \mathcal{F} \text{ and } x_V \in V \ni |f_V(x) - f_V(x_V)| \geq \varepsilon. \tag{11.3}$$

Set $\mathcal{C}_V = \overline{\{f_W : W \in \tau_x \text{ and } W \subset V\}}^{\|\cdot\|_\infty} \subset \mathcal{F}$ and notice for any $\mathcal{V} \subset\subset \tau_x$ that

$$\cap_{V\in\mathcal{V}} \mathcal{C}_V \supseteq \mathcal{C}_{\cap\mathcal{V}} \neq \emptyset,$$

so that $\{\mathcal{C}_V\}_V \in \tau_x \subset \mathcal{F}$ has the finite intersection property.[6] Since $\mathcal{F}$ is compact, it follows that there exists some

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[4] One could also prove that $\mathcal{F}$ is pointwise bounded by considering the continuous evaluation maps $e_x : C(X) \to \mathbb{R}$ given by $e_x(f) = f(x)$ for all $x \in X$.

[5] If X is first countable we could finish the proof with the following argument. Let $\{V_n\}_{n=1}^\infty$ be a neighborhood base at x such that $V_1 \supset V_2 \supset V_3 \supset \dots$. By the assumption that $\mathcal{F}$ is not equicontinuous at x, there exist $f_n \in \mathcal{F}$ and $x_n \in V_n$ such that $|f_n(x) - f_n(x_n)| \geq \epsilon \;\forall\, n$. Since $\mathcal{F}$ is a compact metric space by passing to a subsequence if necessary we may assume that f_n converges uniformly to some $f \in \mathcal{F}$. Because $x_n \to x$ as $n \to \infty$ we learn that

$$\begin{aligned} \epsilon &\leq |f_n(x) - f_n(x_n)| \leq |f_n(x) - f(x)| + |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| \\ &\leq 2\|f_n - f\| + |f(x) - f(x_n)| \to 0 \text{ as } n \to \infty \end{aligned}$$

which is a contradiction.

[6] If we are willing to use Net's described in Appendix C below we could finish the proof as follows. Since $\mathcal{F}$ is compact, the net $\{f_V\}_{V\in\tau_x} \subset \mathcal{F}$ has a cluster point $f \in \mathcal{F} \subset C(X)$. Choose a subnet $\{g_\alpha\}_{\alpha\in A}$ of $\{f_V\}_{V\in\tau_X}$ such that $g_\alpha \to f$ uniformly. Then, since $x_V \to x$ implies $x_{V_\alpha} \to x$, we may conclude from Eq. (11.3) that

$$\epsilon \leq |g_\alpha(x) - g_\alpha(x_{V_\alpha})| \to |g(x) - g(x)| = 0$$

which is a contradiction.

$$f \in \bigcap_{V \in \tau_x} \mathcal{C}_V \neq \emptyset.$$

Since f is continuous, there exists $V \in \tau_x$ such that $|f(x) - f(y)| < \varepsilon/3$ for all $y \in V$. Because $f \in \mathcal{C}_V$, there exists $W \subset V$ such that $\|f - f_W\| < \varepsilon/3$. We now arrive at a contradiction;

$$\begin{aligned}\varepsilon &\le |f_W(x) - f_W(x_W)| \\ &\le |f_W(x) - f(x)| + |f(x) - f(x_W)| + |f(x_W) - f_W(x_W)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

■

The following result is a corollary of Lemma 11.23 and Theorem 11.29.

Corollary 11.30 (Locally Compact Ascoli-Arzela Theorem). *Let (X, τ) be a locally compact and σ – compact topological space and $\{f_m\} \subset C(X)$ be a pointwise bounded sequence of functions such that $\{f_m|_K\}$ is equicontinuous for any compact subset $K \subset X$. Then there exists a subsequence $\{m_n\} \subset \{m\}$ such that $\{g_n := f_{m_n}\}_{n=1}^\infty \subset C(X)$ is a sequence which is uniformly convergent on compact subsets of X.*

Proof. Let $\{K_n\}_{n=1}^\infty$ be the compact subsets of X constructed in Lemma 11.23. We may now apply Theorem 11.29 repeatedly to find a nested family of subsequences

$$\{f_m\} \supset \{g_m^1\} \supset \{g_m^2\} \supset \{g_m^3\} \supset \dots$$

such that the sequence $\{g_m^n\}_{m=1}^\infty \subset C(X)$ is uniformly convergent on K_n. Using Cantor's trick, define the subsequence $\{h_n\}$ of $\{f_m\}$ by $h_n := g_n^n$. Then $\{h_n\}$ is uniformly convergent on K_l for each $l \in \mathbb{N}$. Now if $K \subset X$ is an arbitrary compact set, there exists $l < \infty$ such that $K \subset K_l^o \subset K_l$ and therefore $\{h_n\}$ is uniformly convergent on K as well. ■

Proposition 11.31. *Let $\Omega \subset_o \mathbb{R}^d$ such that $\bar{\Omega}$ is compact and $0 \le \alpha < \beta \le 1$. Then the inclusion map $i : C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ is a compact operator. See Chapter 9 and Lemma 9.9 for the notation being used here.*

Let $\{u_n\}_{n=1}^\infty \subset C^\beta(\overline{\Omega})$ such that $\|u_n\|_{C^\beta} \le 1$, i.e. $\|u_n\|_\infty \le 1$ and

$$|u_n(x) - u_n(y)| \le |x - y|^\beta \text{ for all } x, y \in \overline{\Omega}.$$

By the Arzela-Ascoli Theorem 11.29, there exists a subsequence of $\{\tilde{u}_n\}_{n=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $u \in C^o(\bar{\Omega})$ such that $\tilde{u}_n \to u$ in C^0. Since

$$|u(x) - u(y)| = \lim_{n\to\infty} |\tilde{u}_n(x) - \tilde{u}_n(y)| \le |x - y|^\beta,$$

$u \in C^\beta$ as well. Define $g_n := u - \tilde{u}_n \in C^\beta$, then

$$[g_n]_\beta + \|g_n\|_{C^0} = \|g_n\|_{C^\beta} \le 2$$

and $g_n \to 0$ in C^0. To finish the proof we must show that $g_n \to 0$ in C^α. Given $\delta > 0$,

$$[g_n]_\alpha = \sup_{x \neq y} \frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} \le A_n + B_n$$

where

$$\begin{aligned}A_n &= \sup\left\{\frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} : x \neq y \text{ and } |x - y| \le \delta\right\} \\ &= \sup\left\{\frac{|g_n(x) - g_n(y)|}{|x - y|^\beta} \cdot |x - y|^{\beta - \alpha} : x \neq y \text{ and } |x - y| \le \delta\right\} \\ &\le \delta^{\beta - \alpha} \cdot [g_n]_\beta \le 2\delta^{\beta - \alpha}\end{aligned}$$

and

$$B_n = \sup\left\{\frac{|g_n(x) - g_n(y)|}{|x - y|^\alpha} : |x - y| > \delta\right\} \le 2\delta^{-\alpha} \|g_n\|_{C^0} \to 0 \text{ as } n \to \infty.$$

Therefore,

$$\limsup_{n\to\infty} [g_n]_\alpha \le \limsup_{n\to\infty} A_n + \limsup_{n\to\infty} B_n \le 2\delta^{\beta - \alpha} + 0 \to 0 \text{ as } \delta \downarrow 0.$$

This proposition generalizes to the following theorem which the reader is asked to prove in Exercise 11.18 below.

Theorem 11.32. *Let Ω be a precompact open subset of $\mathbb{R}^d$, $\alpha, \beta \in [0, 1]$ and $k, j \in \mathbb{N}_0$. If $j + \beta > k + \alpha$, then $C^{j,\beta}(\bar{\Omega})$ is compactly contained in $C^{k,\alpha}(\bar{\Omega})$.*

11.5 Tychonoff's Theorem

The goal of this section is to show that arbitrary products of compact spaces is still compact. Before going to the general case of an arbitrary number of factors let us start with only two factors.

Proposition 11.33. *Suppose that X and Y are non-empty compact topological spaces, then $X \times Y$ is compact in the product topology.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. Then for each $(x, y) \in X \times Y$ there exist $U \in \mathcal{U}$ such that $(x, y) \in U$. By definition of the product topology, there also exist $V_x \in \tau_x^X$ and $W_y \in \tau_y^Y$ such that $V_x \times W_y \subset U$. Therefore $\mathcal{V} := \{V_x \times W_y : (x, y) \in X \times Y\}$ is also an open cover of $X \times Y$. We will now show that $\mathcal{V}$ has a finite sub-cover, say $\mathcal{V}_0 \subset\subset \mathcal{V}$. Assuming this is proved for the moment, this implies that $\mathcal{U}$ also has a finite subcover because each $V \in \mathcal{V}_0$ is contained in some $U_V \in \mathcal{U}$. So to complete the proof it suffices to show every cover $\mathcal{V}$ of the form $\mathcal{V} = \{V_\alpha \times W_\alpha : \alpha \in A\}$ where $V_\alpha \subset_o X$ and $W_\alpha \subset_o Y$ has a finite subcover. Given $x \in X$, let $f_x : Y \to X \times Y$ be the map $f_x(y) = (x, y)$

and notice that f_x is continuous since $\pi_X \circ f_x(y) = x$ and $\pi_Y \circ f_x(y) = y$ are continuous maps. From this we conclude that $\{x\} \times Y = f_x(Y)$ is compact. Similarly, it follows that $X \times \{y\}$ is compact for all $y \in Y$. Since $\mathcal{V}$ is a cover of $\{x\} \times Y$, there exist $\Gamma_x \subset\subset A$ such that $\{x\} \times Y \subset \bigcup_{\alpha \in \Gamma_x} (V_\alpha \times W_\alpha)$ without loss of generality we may assume that Γ_x is chosen so that $x \in V_\alpha$ for all $\alpha \in \Gamma_x$. Let $U_x := \bigcap_{\alpha \in \Gamma_x} V_\alpha \subset_o X$, and notice that

$$\bigcup_{\alpha \in \Gamma_x} (V_\alpha \times W_\alpha) \supset \bigcup_{\alpha \in \Gamma_x} (U_x \times W_\alpha) = U_x \times Y, \tag{11.4}$$

see Figure 11.4 below. Since $\{U_x\}_{x \in X}$ is now an open cover of X and X is

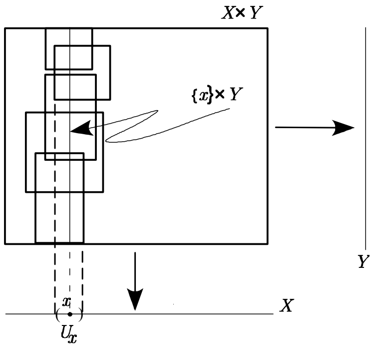

Fig. 11.4. Constructing the open set U_x.

compact, there exists $\Lambda \subset\subset X$ such that $X = \cup_{x \in \Lambda} U_x$. The finite subcollection, $\mathcal{V}_0 := \{V_\alpha \times W_\alpha : \alpha \in \cup_{x \in \Lambda} \Gamma_x\}$, of $\mathcal{V}$ is the desired finite subcover. Indeed using Eq. (11.4),

$$\cup \mathcal{V}_0 = \cup_{x \in \Lambda} \cup_{\alpha \in \Gamma_x} (V_\alpha \times W_\alpha) \supset \cup_{x \in \Lambda} (U_x \times Y) = X \times Y.$$

■

The results of Exercises 11.19 and 10.26 prove Tychonoff's Theorem for a countable product of compact metric spaces. We now state the general version of the theorem.

Theorem 11.34 (Tychonoff's Theorem). *Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of non-empty compact spaces. Then $X := X_A = \prod_{\alpha \in A} X_\alpha$ is compact in the product space topology. (Compare with Exercise 11.19 which covers the special case of a countable product of compact metric spaces.)*

Proof. (The proof is taken from Loomis [13] which followed Bourbaki. Remark 11.35 below should help the reader understand the strategy of the proof to follow.) The proof requires a form of "induction" known as Zorn's lemma which is equivalent to the axiom of choice, see Theorem B.7 of Appendix B below. For $\alpha \in A$ let π_α denote the projection map from X to X_α. Suppose that $\mathcal{F}$ is a family of closed subsets of X which has the finite intersection property, see Definition 11.3. By Proposition 11.4 the proof will be complete if we can show $\cap \mathcal{F} \neq \emptyset$. The first step is to apply Zorn's lemma to construct a maximal collection $\mathcal{F}_0$ of (not necessarily closed) subsets of X with the finite intersection property. To do this, let $\Gamma := \left\{\mathcal{G} \subset 2^X : \mathcal{F} \subset \mathcal{G}\right\}$ equipped with the partial order, $\mathcal{G}_1 < \mathcal{G}_2$ if $\mathcal{G}_1 \subset \mathcal{G}_2$. If Φ is a linearly ordered subset of Γ, then $\mathcal{G} := \cup \Phi$ is an upper bound for Γ which still has the finite intersection property as the reader should check. So by Zorn's lemma, Γ has a maximal element $\mathcal{F}_0$. The maximal $\mathcal{F}_0$ has the following properties.

1. $\mathcal{F}_0$ is closed under finite intersections. Indeed, if we let $(\mathcal{F}_0)_f$ denote the collection of all finite intersections of elements from $\mathcal{F}_0$, then $(\mathcal{F}_0)_f$ has the finite intersection property and contains $\mathcal{F}_0$. Since $\mathcal{F}_0$ is maximal, this implies $(\mathcal{F}_0)_f = \mathcal{F}_0$.
2. If $A \subset X$ and $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}_0$ then $A \in \mathcal{F}_0$. For if not $\mathcal{F}_0 \cup \{A\}$ would still satisfy the finite intersection property and would properly contain $\mathcal{F}_0$. But this would violate the maximallity of $\mathcal{F}_0$.
3. For each $\alpha \in A$,
$$\pi_a(\mathcal{F}_0) := \{\pi_\alpha(F) \subset X_\alpha : F \in \mathcal{F}_0\}$$
has the finite intersection property. Indeed, if $\{F_i\}_{i=1}^n \subset \mathcal{F}_0$, then $\cap_{i=1}^n \pi_\alpha(F_i) \supset \pi_\alpha \left(\cap_{i=1}^n F_i\right) \neq \emptyset$.

Since X_α is compact, property 3. above along with Proposition 11.4 implies $\cap_{F \in \mathcal{F}_0} \overline{\pi_\alpha(F)} \neq \emptyset$. Since this true for each $\alpha \in A$, using the axiom of choice, there exists $p \in X$ such that $p_\alpha = \pi_\alpha(p) \in \cap_{F \in \mathcal{F}_0} \overline{\pi_\alpha(F)}$ for all $\alpha \in A$. The proof will be completed by showing $p \in \cap \mathcal{F}$ which will prove $\cap \mathcal{F}$ is not empty as desired.

Since $C := \cap \left\{\bar{F} : F \in \mathcal{F}_0\right\} \subset \cap \mathcal{F}$, it suffices to show $p \in C$. Let U be an open neighborhood of p in X. By the definition of the product topology (or item 2. of Proposition 10.25), there exists $\Lambda \subset\subset A$ and open sets $U_\alpha \subset X_\alpha$ for all $\alpha \in \Lambda$ such that $p \in \cap_{\alpha \in \Lambda} \pi_\alpha^{-1}(U_\alpha) \subset U$. Since $p_\alpha \in \cap_{F \in \mathcal{F}_0} \overline{\pi_\alpha(F)}$ and $p_\alpha \in U_\alpha$ for all $\alpha \in \Lambda$, it follows that $U_\alpha \cap \pi_\alpha(F) \neq \emptyset$ for all $F \in \mathcal{F}_0$ and all $\alpha \in \Lambda$. This then implies $\pi_\alpha^{-1}(U_\alpha) \cap F \neq \emptyset$ for all $F \in \mathcal{F}_0$ and all $\alpha \in \Lambda$. By property 2.[7] above we concluded that $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}_0$ for all $\alpha \in \Lambda$ and then by property 1. that $\cap_{\alpha \in \Lambda} \pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}_0$. In particular

$$\emptyset \neq F \cap \left(\cap_{\alpha \in \Lambda} \pi_\alpha^{-1}(U_\alpha)\right) \subset F \cap U \text{ for all } F \in \mathcal{F}_0$$

[7] Here is where we use that $\mathcal{F}_0$ is maximal among the collection of all, not just closed, sets having the finite intersection property and containing $\mathcal{F}$.

which shows $p \in \bar{F}$ for each $F \in \mathcal{F}_0$, i.e. $p \in C$. ■

Remark 11.35. Consider the following simple example where $X = [-1,1] \times [-1,1]$ and $\mathcal{F} = \{F_1, F_2\}$ as in Figure 11.5. Notice that $\pi_i(F_1) \cap \pi_i(F_2) = [-1,1]$ for each i and so gives no help in trying to find the i^{th} – coordinate of one of the two points in $F_1 \cap F_2$. This is why it is necessary to introduce the collection $\mathcal{F}_0$ in the proof of Theorem 11.34. In this case one might take $\mathcal{F}_0$ to be the collection of all subsets $F \subset X$ such that $p \in F$. We then have $\cap_{F \in \mathcal{F}_0} \pi_i(F) = \{p_i\}$, so the i^{th} – coordinate of p may now be determined by observing the sets, $\{\pi_i(F) : F \in \mathcal{F}_0\}$.

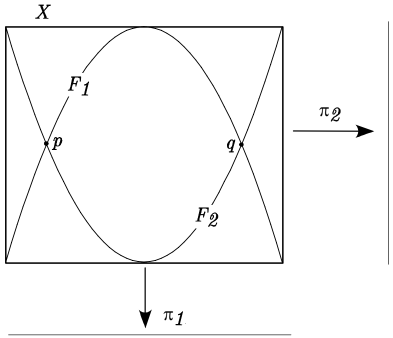

Fig. 11.5. Here $\mathcal{F} = \{F_1, F_2\}$ where F_1 and F_2 are the two parabolic arcs and $F_1 \cap F_2 = \{p, q\}$.

11.6 Exercises

Exercise 11.11. Prove Lemma 11.5.

Exercise 11.12. Let C be a closed proper subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus C$. Show there exists a $y \in C$ such that $d(x,y) = d_C(x)$.

Exercise 11.13. Let $\mathbb{F} = \mathbb{R}$ in this problem and $A \subset \ell^2(\mathbb{N})$ be defined by

$$\begin{aligned} A &= \{x \in \ell^2(\mathbb{N}) : x(n) \geq 1 + 1/n \text{ for some } n \in \mathbb{N}\} \\ &= \cup_{n=1}^{\infty} \{x \in \ell^2(\mathbb{N}) : x(n) \geq 1 + 1/n\}. \end{aligned}$$

Show A is a closed subset of $\ell^2(\mathbb{N})$ with the property that $d_A(0) = 1$ while there is no $y \in A$ such that $d_A(y) = 1$. (Remember that in general an infinite union of closed sets need not be closed.)

Exercise 11.14. Let $p \in [1, \infty]$ and X be an infinite set. Show directly, without using Theorem 11.15, the closed unit ball in $\ell^p(X)$ is not compact.

11.6.1 Ascoli-Arzela Theorem Problems

Exercise 11.15. Let $T \in (0, \infty)$ and $\mathcal{F} \subset C([0,T])$ be a family of functions such that:

1. $\dot{f}(t)$ exists for all $t \in (0,T)$ and $f \in \mathcal{F}$.
2. $\sup_{f \in \mathcal{F}} |f(0)| < \infty$ and
3. $M := \sup_{f \in \mathcal{F}} \sup_{t \in (0,T)} \left|\dot{f}(t)\right| < \infty$.

Show $\mathcal{F}$ is precompact in the Banach space $C([0,T])$ equipped with the norm $\|f\|_\infty = \sup_{t \in [0,T]} |f(t)|$.

Exercise 11.16 (Peano's Existence Theorem). Suppose $Z : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function. Then for each $T < \infty$[8] there exists a solution to the differential equation

$$\dot{x}(t) = Z(t, x(t)) \text{ for } 0 \leq t \leq T \text{ with } x(0) = x_0. \tag{11.5}$$

Do this by filling in the following outline for the proof.

1. Given $\varepsilon > 0$, show there exists a unique function $x_\varepsilon \in C([-\varepsilon, \infty) \to \mathbb{R}^d)$ such that $x_\varepsilon(t) := x_0$ for $-\varepsilon \leq t \leq 0$ and
$$x_\varepsilon(t) = x_0 + \int_0^t Z(\tau, x_\varepsilon(\tau - \varepsilon)) d\tau \text{ for all } t \geq 0. \tag{11.6}$$
Here
$$\int_0^t Z(\tau, x_\varepsilon(\tau - \varepsilon)) d\tau = \left(\int_0^t Z_1(\tau, x_\varepsilon(\tau - \varepsilon)) d\tau, \dots, \int_0^t Z_d(\tau, x_\varepsilon(\tau - \varepsilon)) d\tau \right)$$
where $Z = (Z_1, \dots, Z_d)$ and the integrals are either the Lebesgue or the Riemann integral since they are equal on continuous functions. **Hint:** For $t \in [0, \varepsilon]$, it follows from Eq. (11.6) that
$$x_\varepsilon(t) = x_0 + \int_0^t Z(\tau, x_0) d\tau.$$
Now that $x_\varepsilon(t)$ is known for $t \in [-\varepsilon, \varepsilon]$ it can be found by integration for $t \in [-\varepsilon, 2\varepsilon]$. The process can be repeated.
2. Then use Exercise 11.15 to show there exists $\{\varepsilon_k\}_{k=1}^\infty \subset (0, \infty)$ such that $\lim_{k \to \infty} \varepsilon_k = 0$ and x_{ε_k} converges to some $x \in C([0,T])$ with respect to the sup-norm: $\|x\|_\infty = \sup_{t \in [0,T]} |x(t)|$). Also show for this sequence that
$$\lim_{k \to \infty} \sup_{\varepsilon_k \leq \tau \leq T} |x_{\varepsilon_k}(\tau - \varepsilon_k) - x(\tau)| = 0.$$

[8] Using Corollary 11.30 below, we may in fact allow $T = \infty$.

3. Pass to the limit (**with justification**) in Eq. (11.6) with ε replaced by ε_k to show x satisfies

$$x(t) = x_0 + \int_0^t Z(\tau, x(\tau))d\tau \;\forall\, t \in [0,T].$$

4. Conclude from this that $\dot{x}(t)$ exists for $t \in (0,T)$ and that x solves Eq. (11.5).
5. Apply what you have just prove to the ODE,

$$\dot{y}(t) = -Z(-t, y(t)) \text{ for } 0 \le t \le T \text{ with } x(0) = x_0.$$

Then extend $x(t)$ above to $[-T,T]$ by setting $x(t) = y(-t)$ if $t \in [-T,0]$. Show x so defined solves Eq. (11.5) for $t \in (-T,T)$.

Exercise 11.17. Folland Problem 4.63. (Compactness of integral operators.)

Exercise 11.18. Prove Theorem 11.32. **Hint:** First prove $C^{j,\beta}\left(\bar{\Omega}\right) \sqsubset\sqsubset C^{j,\alpha}\left(\bar{\Omega}\right)$ is compact if $0 \le \alpha < \beta \le 1$. Then use Lemma 11.18 repeatedly to handle all of the other cases.

11.6.2 Tychonoff's Theorem Problem

Exercise 11.19 (Tychonoff's Theorem for Compact Metric Spaces). Let us continue the Notation used in Exercise 6.12. Further assume that the spaces X_n are compact for all n. Show, without using Theorem 11.34, (X,d) is compact. **Hint:** Either use Cantor's method to show every sequence $\{x_m\}_{m=1}^\infty \subset X$ has a convergent subsequence or alternatively show (X,d) is complete and totally bounded. (Compare with Tychonoff's Theorem 11.34 above which covers the general case.)

12

Locally Compact Hausdorff Spaces

In this section X will always be a topological space with topology τ. We are now interested in restrictions on τ in order to insure there are "plenty" of continuous functions. One such restriction is to assume $\tau = \tau_d$ – is the topology induced from a metric on X. For example the results in Lemma 6.15 and Theorem 7.4 above shows that metric spaces have lots of continuous functions.

The main thrust of this section is to study locally compact (and σ – compact) "Hausdorff" spaces as defined in Definitions 12.2 and 11.21. We will see again that this class of topological spaces have an ample supply of continuous functions. We will start out with the notion of a Hausdorff topology. The following example shows a pathology which occurs when there are not enough open sets in a topology.

Example 12.1. Let $X = \{1,2,3\}$ and $\tau = \{X, \emptyset, \{1,2\}, \{2,3\}, \{2\}\}$ and $x_n = 2$ for all n. Then $x_n \to x$ for every $x \in X$!

Definition 12.2 (Hausdorff Topology). *A topological space, (X,τ), is* ***Hausdorff*** *if for each pair of distinct points, $x, y \in X$, there exists disjoint open neighborhoods, U and V of x and y respectively. (Metric spaces are typical examples of Hausdorff spaces.)*

Remark 12.3. When τ is Hausdorff the "pathologies" appearing in Example 12.1 do not occur. Indeed if $x_n \to x \in X$ and $y \in X \setminus \{x\}$ we may choose $V \in \tau_x$ and $W \in \tau_y$ such that $V \cap W = \emptyset$. Then $x_n \in V$ a.a. implies $x_n \notin W$ for all but a finite number of n and hence $x_n \nrightarrow y$, so limits are unique.

Proposition 12.4. *Let (X_α, τ_α) be Hausdorff topological spaces. Then the product space $X_A = \prod_{\alpha\in A} X_\alpha$ equipped with the product topology is Hausdorff.*

Proof. Let $x, y \in X_A$ be distinct points. Then there exists $\alpha \in A$ such that $\pi_\alpha(x) = x_\alpha \ne y_\alpha = \pi_\alpha(y)$. Since X_α is Hausdorff, there exists disjoint open sets $U, V \subset X_\alpha$ such $\pi_\alpha(x) \in U$ and $\pi_\alpha(y) \in V$. Then $\pi_\alpha^{-1}(U)$ and $\pi_\alpha^{-1}(V)$ are disjoint open sets in X_A containing x and y respectively. ■

Proposition 12.5. *Suppose that (X, τ) is a Hausdorff space, $K \sqsubset\sqsubset X$ and $x \in K^c$. Then there exists $U, V \in \tau$ such that $U \cap V = \emptyset$, $x \in U$ and $K \subset V$. In particular K is closed. (So compact subsets of Hausdorff topological spaces are closed.) More generally if K and F are two disjoint compact subsets of X, there exist disjoint open sets $U, V \in \tau$ such that $K \subset V$ and $F \subset U$.*

Proof. Because X is Hausdorff, for all $y \in K$ there exists $V_y \in \tau_y$ and $U_y \in \tau_x$ such that $V_y \cap U_y = \emptyset$. The cover $\{V_y\}_{y \in K}$ of K has a finite subcover, $\{V_y\}_{y \in \Lambda}$ for some $\Lambda \subset\subset K$. Let $V = \cup_{y \in \Lambda} V_y$ and $U = \cap_{y \in \Lambda} U_y$, then $U, V \in \tau$ satisfy $x \in U$, $K \subset V$ and $U \cap V = \emptyset$. This shows that K^c is open and hence that K is closed. Suppose that K and F are two disjoint compact subsets of X. For each $x \in F$ there exists disjoint open sets U_x and V_x such that $K \subset V_x$ and $x \in U_x$. Since $\{U_x\}_{x \in F}$ is an open cover of F, there exists a finite subset Λ of F such that $F \subset U := \cup_{x \in \Lambda} U_x$. The proof is completed by defining $V := \cap_{x \in \Lambda} V_x$. ■

Exercise 12.1. Show any finite set X admits exactly one Hausdorff topology τ.

Exercise 12.2. Let (X, τ) and (Y, τ_Y) be topological spaces.

1. Show τ is Hausdorff iff $\Delta := \{(x, x) : x \in X\}$ is a closed in $X \times X$ equipped with the product topology $\tau \otimes \tau$.
2. Suppose τ is Hausdorff and $f, g : Y \to X$ are continuous maps. If $\overline{\{f = g\}}^Y = Y$ then $f = g$. **Hint:** make use of the map $f \times g : Y \to X \times X$ defined by $(f \times g)(y) = (f(y), g(y))$.

Exercise 12.3. Given an example of a topological space which has a non-closed compact subset.

Proposition 12.6. *Suppose that X is a compact topological space, Y is a Hausdorff topological space, and $f : X \to Y$ is a continuous bijection then f is a homeomorphism, i.e. $f^{-1} : Y \to X$ is continuous as well.*

Proof. Since closed subsets of compact sets are compact, continuous images of compact subsets are compact and compact subsets of Hausdorff spaces are closed, it follows that $\left(f^{-1}\right)^{-1}(C) = f(C)$ is closed in X for all closed subsets C of X. Thus f^{-1} is continuous. ■

The next two results shows that locally compact Hausdorff spaces have plenty of open sets and plenty of continuous functions.

Proposition 12.7. *Suppose X is a locally compact Hausdorff space and $U \subset_o X$ and $K \sqsubset\sqsubset U$. Then there exists $V \subset_o X$ such that $K \subset V \subset \overline{V} \subset U \subset X$ and $\bar{V}$ is compact. (Compare with Proposition 11.25 above.)*

Proof. By local compactness, for all $x \in K$, there exists $U_x \in \tau_x$ such that $\bar{U}_x$ is compact. Since K is compact, there exists $\Lambda \subset\subset K$ such that $\{U_x\}_{x \in \Lambda}$ is a cover of K. The set $O = U \cap (\cup_{x \in \Lambda} U_x)$ is an open set such that $K \subset O \subset U$ and O is precompact since $\bar{O}$ is a closed subset of the compact set $\cup_{x \in \Lambda} \bar{U}_x$. ($\cup_{x \in \Lambda} \bar{U}_x$. is compact because it is a finite union of compact sets.) So by replacing U by O if necessary, we may assume that $\bar{U}$ is compact. Since $\bar{U}$ is compact and $\mathrm{bd}(U) = \bar{U} \cap U^c$ is a closed subset of $\bar{U}$, $\mathrm{bd}(U)$ is compact. Because $\mathrm{bd}(U) \subset U^c$, it follows that $\mathrm{bd}(U) \cap K = \emptyset$, so by Proposition 12.5, there exists disjoint open sets V and W such that $K \subset V$ and $\mathrm{bd}(U) \subset W$. By replacing V by $V \cap U$ if necessary we may further assume that $K \subset V \subset U$, see Figure 12.1. Because $\bar{U} \cap W^c$ is a closed set containing V and $\mathrm{bd}(U) \cap W^c = \emptyset$,

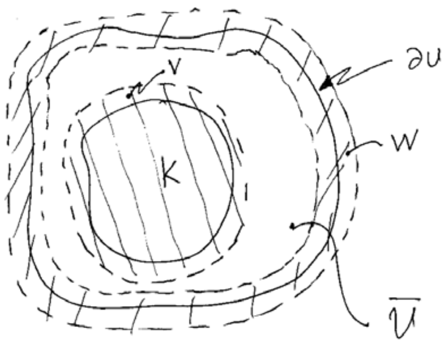

Fig. 12.1. The construction of V.

$$\bar{V} \subset \bar{U} \cap W^c = (U \cup \mathrm{bd}(U)) \cap W^c = U \cap W^c \subset U \subset \bar{U}.$$

Since $\bar{U}$ is compact it follows that $\bar{V}$ is compact and the proof is complete. ■

The following Lemma is analogous to Lemma 11.27.

Lemma 12.8 (Urysohn's Lemma for LCH Spaces). *Let X be a locally compact Hausdorff space and $K \sqsubset\sqsubset U \subset_o X$. Then there exists $f \prec U$ (see Definition 11.26) such that $f = 1$ on K. In particular, if K is compact and C is closed in X such that $K \cap C = \emptyset$, there exists $f \in C_c(X, [0, 1])$ such that $f = 1$ on K and $f = 0$ on C.*

Proof. For notational ease later it is more convenient to construct $g := 1 - f$ rather than f. To motivate the proof, suppose $g \in C(X, [0, 1])$ such that $g = 0$ on K and $g = 1$ on U^c. For $r > 0$, let $U_r = \{g < r\}$. Then for $0 < r < s \leq 1$, $U_r \subset \{g \leq r\} \subset U_s$ and since $\{g \leq r\}$ is closed this implies

$$K \subset U_r \subset \bar{U}_r \subset \{g \leq r\} \subset U_s \subset U.$$

Therefore associated to the function g is the collection open sets $\{U_r\}_{r>0} \subset \tau$ with the property that $K \subset U_r \subset \bar{U}_r \subset U_s \subset U$ for all $0 < r < s \le 1$ and $U_r = X$ if $r > 1$. Finally let us notice that we may recover the function g from the sequence $\{U_r\}_{r>0}$ by the formula

$$g(x) = \inf\{r > 0 : x \in U_r\}. \tag{12.1}$$

The idea of the proof to follow is to turn these remarks around and define g by Eq. (12.1).

Step 1. (Construction of the U_r.) Let

$$\mathbb{D} := \left\{k2^{-n} : k = 1, 2, \ldots, 2^{-1}, n = 1, 2, \ldots\right\}$$

be the dyadic rationals in $(0, 1]$. Use Proposition 12.7 to find a precompact open set U_1 such that $K \subset U_1 \subset \bar{U}_1 \subset U$. Apply Proposition 12.7 again to construct an open set $U_{1/2}$ such that

$$K \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_1$$

and similarly use Proposition 12.7 to find open sets $U_{1/2}, U_{3/4} \subset_o X$ such that

$$K \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_{3/4} \subset \bar{U}_{3/4} \subset U_1.$$

Likewise there exists open set $U_{1/8}, U_{3/8}, U_{5/8}, U_{7/8}$ such that

$$\begin{aligned} K &\subset U_{1/8} \subset \bar{U}_{1/8} \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_{3/8} \subset \bar{U}_{3/8} \subset U_{1/2} \\ &\subset \bar{U}_{1/2} \subset U_{5/8} \subset \bar{U}_{5/8} \subset U_{3/4} \subset \bar{U}_{3/4} \subset U_{7/8} \subset \bar{U}_{7/8} \subset U_1. \end{aligned}$$

Continuing this way inductively, one shows there exists precompact open sets $\{U_r\}_{r \in \mathbb{D}} \subset \tau$ such that

$$K \subset U_r \subset \overline{U}_r \subset U_s \subset U_1 \subset \bar{U}_1 \subset U$$

for all $r, s \in \mathbb{D}$ with $0 < r < s \le 1$.

Step 2. Let $U_r := X$ if $r > 1$ and define

$$g(x) = \inf\{r \in \mathbb{D} \cup (1, \infty) : x \in U_r\},$$

see Figure 12.2. Then $g(x) \in [0, 1]$ for all $x \in X$, $g(x) = 0$ for $x \in K$ since $x \in K \subset U_r$ for all $r \in \mathbb{D}$. If $x \in U_1^c$, then $x \notin U_r$ for all $r \in \mathbb{D}$ and hence $g(x) = 1$. Therefore $f := 1 - g$ is a function such that $f = 1$ on K and $\{f \neq 0\} = \{g \neq 1\} \subset U_1 \subset \bar{U}_1 \subset U$ so that $\mathrm{supp}(f) = \overline{\{f \neq 0\}} \subset \bar{U}_1 \subset U$ is a compact subset of U. Thus it only remains to show f, or equivalently g, is continuous.

Since $\mathcal{E} = \{(\alpha, \infty), (-\infty, \alpha) : \alpha \in \mathbb{R}\}$ generates the standard topology on $\mathbb{R}$, to prove g is continuous it suffices to show $\{g < \alpha\}$ and $\{g > \alpha\}$ are open sets for all $\alpha \in \mathbb{R}$. But $g(x) < \alpha$ iff there exists $r \in \mathbb{D} \cup (1, \infty)$ with $r < \alpha$ such that $x \in U_r$. Therefore

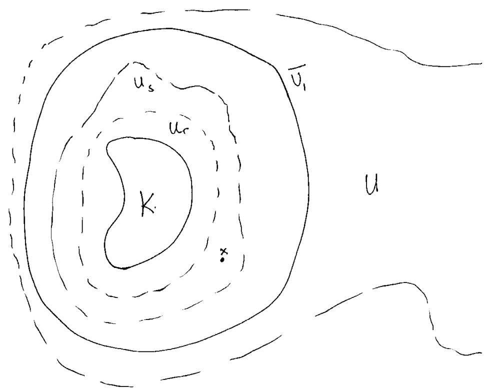

Fig. 12.2. Determining g from $\{U_r\}$.

$$\{g < \alpha\} = \bigcup \{U_r : r \in \mathbb{D} \cup (1, \infty) \ni r < \alpha\}$$

which is open in X. If $\alpha \ge 1$, $\{g > \alpha\} = \emptyset$ and if $\alpha < 0$, $\{g > \alpha\} = X$. If $\alpha \in (0, 1)$, then $g(x) > \alpha$ iff there exists $r \in \mathbb{D}$ such that $r > \alpha$ and $x \notin U_r$. Now if $r > \alpha$ and $x \notin U_r$ then for $s \in \mathbb{D} \cap (\alpha, r)$, $x \notin \bar{U}_s \subset U_r$. Thus we have shown that

$$\{g > \alpha\} = \bigcup \left\{ \left(\overline{U}_s\right)^c : s \in \mathbb{D} \ni s > \alpha \right\}$$

which is again an open subset of X. ■

Theorem 12.9 (Locally Compact Tietz Extension Theorem). *Let (X, τ) be a locally compact Hausdorff space, $K \sqsubset\sqsubset U \subset_o X$, $f \in C(K, \mathbb{R})$, $a = \min f(K)$ and $b = \max f(K)$. Then there exists $F \in C(X, [a, b])$ such that $F|_K = f$. Moreover given $c \in [a, b]$, F can be chosen so that $\mathrm{supp}(F - c) = \overline{\{F \neq c\}} \subset U$.*

The proof of this theorem is similar to Theorem 7.4 and will be left to the reader, see Exercise 12.5.

12.1 Locally compact form of Urysohn's Metrization Theorem

Notation 12.10 *Let $Q := [0, 1]^{\mathbb{N}}$ denote the (infinite dimensional)* ***unit cube*** *in $\mathbb{R}^{\mathbb{N}}$. For $a, b \in Q$ let*

$$d(a, b) := \sum_{n=1}^{\infty} \frac{1}{2^n} |a_n - b_n|. \tag{12.2}$$

The metric introduced in Exercise 11.19 would be defined, in this context, as $\tilde{d}(a,b) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|a_n - b_n|}{1+|a_n-b_n|}$. Since $1 \le 1+|a_n - b_n| \le 2$, it follows that $\tilde{d} \le d \le 2d$. So the metrics d and $\tilde{d}$ are equivalent and in particular the topologies induced by d and $\tilde{d}$ are the same. By Exercises 10.26, the d – topology on Q is the same as the product topology and by Tychonoff's Theorem 11.34 or by Exercise 11.19, (Q, d) is a compact metric space.

Theorem 12.11. *To every separable metric space (X, ρ), there exists a continuous injective map $G : X \to Q$ such that $G : X \to G(X) \subset Q$ is a homeomorphism. In short, any separable metrizable space X is homeomorphic to a subset of (Q, d).*

Remark 12.12. Notice that if we let $\rho'(x, y) := d(G(x), G(y))$, then ρ' induces the same topology on X as ρ and $G : (X, \rho') \to (Q, d)$ is isometric.

Proof. Let $D = \{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X,

$$\phi(t) := \begin{cases} 1 & \text{if } \quad t \le 0 \\ 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } \quad t \ge 1, \end{cases}$$

(see Figure 12.3) and for $m, n \in \mathbb{N}$ let

$$f_{m,n}(x) := 1 - \phi\left(m\rho(x_n, x)\right).$$

Then $f_{m,n} = 0$ if $\rho(x, x_n) < 1/m$ and $f_{m,n} = 1$ if $\rho(x, x_n) > 2/m$. Let

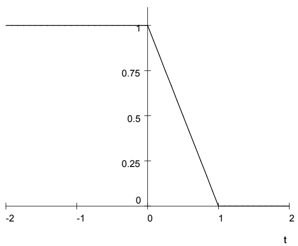

Fig. 12.3. The graph of the function ϕ.

$\{g_k\}_{k=1}^{\infty}$ be an enumeration of $\{f_{m,n} : m, n \in \mathbb{N}\}$ and define $G : X \to Q$ by

$$G(x) = (g_1(x), g_2(x), \dots) \in Q.$$

We will now show $G : X \to G(X) \subset Q$ is a homeomorphism. To show G is injective suppose $x, y \in X$ and $\rho(x, y) = \delta \ge 1/m$. In this case we may

find $x_n \in X$ such that $\rho(x, x_n) \le \frac{1}{2m}$, $\rho(y, x_n) \ge \delta - \frac{1}{2m} \ge \frac{1}{2m}$ and hence $f_{4m,n}(y) = 1$ while $f_{4m,n}(y) = 0$. From this it follows that $G(x) \ne G(y)$ if $x \ne y$ and hence G is injective. The continuity of G is a consequence of the continuity of each of the components g_i of G. So it only remains to show $G^{-1} : G(X) \to X$ is continuous. Given $a = G(x) \in G(X) \subset Q$ and $\varepsilon > 0$, choose $m \in \mathbb{N}$ and $x_n \in X$ such that $\rho(x_n, x) < \frac{1}{2m} < \frac{\varepsilon}{2}$. Then $f_{m,n}(x) = 0$ and for $y \notin B(x_n, \frac{2}{m})$, $f_{m,n}(y) = 1$. So if k is chosen so that $g_k = f_{m,n}$, we have shown that for

$$d(G(y), G(x)) \ge 2^{-k} \text{ for } y \notin B(x_n, 2/m)$$

or equivalently put, if

$$d(G(y), G(x)) < 2^{-k} \text{ then } y \in B(x_n, 2/m) \subset B(x, 1/m) \subset B(x, \varepsilon).$$

This shows that if $G(y)$ is sufficiently close to $G(x)$ then $\rho(y, x) < \varepsilon$, i.e. G^{-1} is continuous at $a = G(x)$. ■

Theorem 12.13 (Urysohn Metrization Theorem for LCH's). *Every second countable locally compact Hausdorff space, (X, τ), is metrizable, i.e. there is a metric ρ on X such that $\tau = \tau_\rho$. Moreover, ρ may be chosen so that X is isometric to a subset $Q_0 \subset Q$ equipped with the metric d in Eq. (12.2). In this metric X is totally bounded and hence the completion of X (which is isometric to $\bar{Q}_0 \subset Q$) is compact. (Also see Theorem 12.44.)*

Proof. Let $\mathcal{B}$ be a countable base for τ and set

$$\Gamma := \{(U, V) \in \mathcal{B} \times \mathcal{B} \mid \bar{U} \subset V \text{ and } \bar{U} \text{ is compact}\}.$$

To each $O \in \tau$ and $x \in O$ there exist $(U, V) \in \Gamma$ such that $x \in U \subset V \subset O$. Indeed, since $\mathcal{B}$ is a base for τ, there exists $V \in \mathcal{B}$ such that $x \in V \subset O$. Now apply Proposition 12.7 to find $U' \subset_o X$ such that $x \in U' \subset \bar{U}' \subset V$ with $\bar{U}'$ being compact. Since $\mathcal{B}$ is a base for τ, there exists $U \in \mathcal{B}$ such that $x \in U \subset U'$ and since $\bar{U} \subset \bar{U}'$, $\bar{U}$ is compact so $(U, V) \in \Gamma$. In particular this shows that $\mathcal{B}' := \{U \in \mathcal{B} : (U, V) \in \Gamma \text{ for some } V \in \mathcal{B}\}$ is still a base for τ. If Γ is a finite, then $\mathcal{B}'$ is finite and τ only has a finite number of elements as well. Since (X, τ) is Hausdorff, it follows that X is a finite set. Letting $\{x_n\}_{n=1}^{N}$ be an enumeration of X, define $T : X \to Q$ by $T(x_n) = e_n$ for $n = 1, 2, \dots, N$ where $e_n = (0, 0, \dots, 0, 1, 0, \dots)$, with the 1 occurring in the n^{th} spot. Then $\rho(x, y) := d(T(x), T(y))$ for $x, y \in X$ is the desired metric.

So we may now assume that Γ is an infinite set and let $\{(U_n, V_n)\}_{n=1}^{\infty}$ be an enumeration of Γ. By Urysohn's Lemma 12.8 there exists $f_{U,V} \in C(X, [0, 1])$ such that $f_{U,V} = 0$ on $\bar{U}$ and $f_{U,V} = 1$ on V^c. Let $\mathcal{F} := \{f_{U,V} \mid (U, V) \in \Gamma\}$ and set $f_n := f_{U_n, V_n}$ – an enumeration of $\mathcal{F}$. We will now show that

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x) - f_n(y)|$$

is the desired metric on X. The proof will involve a number of steps.

1. (ρ is a metric on X.) It is routine to show ρ satisfies the triangle inequality and ρ is symmetric. If $x, y \in X$ are distinct points then there exists $(U_{n_0}, V_{n_0}) \in \Gamma$ such that $x \in U_{n_0}$ and $V_{n_0} \subset O := \{y\}^c$. Since $f_{n_0}(x) = 0$ and $f_{n_0}(y) = 1$, it follows that $\rho(x, y) \geq 2^{-n_0} > 0$.
2. (Let $\tau_0 = \tau(f_n : n \in \mathbb{N})$, then $\tau = \tau_0 = \tau_\rho$.) As usual we have $\tau_0 \subset \tau$. Since, for each $x \in X$, $y \to \rho(x, y)$ is τ_0 – continuous (being the uniformly convergent sum of continuous functions), it follows that $B_x(\varepsilon) := \{y \in X : \rho(x, y) < \varepsilon\} \in \tau_0$ for all $x \in X$ and $\varepsilon > 0$. Thus $\tau_\rho \subset \tau_0 \subset \tau$. Suppose that $O \in \tau$ and $x \in O$. Let $(U_{n_0}, V_{n_0}) \in \Gamma$ be such that $x \in U_{n_0}$ and $V_{n_0} \subset O$. Then $f_{n_0}(x) = 0$ and $f_{n_0} = 1$ on O^c. Therefore if $y \in X$ and $f_{n_0}(y) < 1$, then $y \in O$ so $x \in \{f_{n_0} < 1\} \subset O$. This shows that O may be written as a union of elements from τ_0 and therefore $O \in \tau_0$. So $\tau \subset \tau_0$ and hence $\tau = \tau_0$. Moreover, if $y \in B_x(2^{-n_0})$ then $2^{-n_0} > \rho(x, y) \geq 2^{-n_0} f_{n_0}(y)$ and therefore $x \in B_x(2^{-n_0}) \subset \{f_{n_0} < 1\} \subset O$. This shows O is ρ – open and hence $\tau_\rho \subset \tau_0 \subset \tau \subset \tau_\rho$.
3. (X is isometric to some $Q_0 \subset Q$.) Let $T : X \to Q$ be defined by $T(x) = (f_1(x), f_2(x), \dots, f_n(x), \dots)$. Then T is an isometry by the very definitions of d and ρ and therefore X is isometric to $Q_0 := T(X)$. Since Q_0 is a subset of the compact metric space (Q, d), Q_0 is totally bounded and therefore X is totally bounded.

■

BRUCE: Add Stone Chech Compactification results.

12.2 Partitions of Unity

Definition 12.14. *Let (X, τ) be a topological space and $X_0 \subset X$ be a set. A collection of sets $\{B_\alpha\}_{\alpha \in A} \subset 2^X$ is **locally finite** on X_0 if for all $x \in X_0$, there is an open neighborhood $N_x \in \tau$ of x such that $\#\{\alpha \in A : B_\alpha \cap N_x \neq \emptyset\} < \infty$.*

Definition 12.15. *Suppose that $\mathcal{U}$ is an open cover of $X_0 \subset X$. A collection $\{\phi_i\}_{i=1}^N \subset C(X, [0, 1])$ ($N = \infty$ is allowed here) is a **partition of unity** on X_0 subordinate to the cover $\mathcal{U}$ if:*

1. *for all i there is a $U \in \mathcal{U}$ such that $\operatorname{supp}(\phi_i) \subset U$,*
2. *the collection of sets, $\{\operatorname{supp}(\phi_i)\}_{i=1}^N$, is locally finite on X_0, and*
3. *$\sum_{i=1}^N \phi_i = 1$ on X_0. (Notice by (2), that for each $x \in X_0$ there is a neighborhood N_x such that $\phi_i|_{N_x}$ is not identically zero for only a finite number of terms. So the sum is well defined and we say the sum is **locally finite**.)*

Proposition 12.16 (Partitions of Unity: The Compact Case). *Suppose that X is a locally compact Hausdorff space, $K \subset X$ is a compact set and $\mathcal{U} = \{U_j\}_{j=1}^n$ is an open cover of K. Then there exists a partition of unity $\{h_j\}_{j=1}^n$ of K such that $h_j \prec U_j$ for all $j = 1, 2, \dots, n$.*

Proof. For all $x \in K$ choose a precompact open neighborhood, V_x, of x such that $\overline{V}_x \subset U_j$. Since K is compact, there exists a finite subset, Λ, of K such that $K \subset \bigcup_{x \in \Lambda} V_x$. Let

$$F_j = \cup \left\{\bar{V}_x : x \in \Lambda \text{ and } \overline{V}_x \subset U_j\right\}.$$

Then F_j is compact, $F_j \subset U_j$ for all j, and $K \subset \cup_{j=1}^n F_j$. By Urysohn's Lemma 12.8 there exists $f_j \prec U_j$ such that $f_j = 1$ on F_j. We will now give two methods to finish the proof.

Method 1. Let $h_1 = f_1$, $h_2 = f_2(1 - h_1) = f_2(1 - f_1)$,

$$h_3 = f_3(1 - h_1 - h_2) = f_3(1 - f_1 - (1 - f_1)f_2) = f_3(1 - f_1)(1 - f_2)$$

and continue on inductively to define

$$h_k = (1 - h_1 - \dots - h_{k-1})f_k = f_k \cdot \prod_{j=1}^{k-1}(1 - f_j) \,\forall\, k = 2, 3, \dots, n \tag{12.3}$$

and to show

$$(1 - h_1 - \dots - h_n) = \prod_{j=1}^{n}(1 - f_j). \tag{12.4}$$

From these equations it clearly follows that $h_j \in C_c(X, [0, 1])$ and that $\operatorname{supp}(h_j) \subset \operatorname{supp}(f_j) \subset U_j$, i.e. $h_j \prec U_j$. Since $\prod_{j=1}^n (1 - f_j) = 0$ on K, $\sum_{j=1}^n h_j = 1$ on K and $\{h_j\}_{j=1}^n$ is the desired partition of unity.

Method 2. Let $g := \sum_{j=1}^n f_j \in C_c(X)$. Then $g \geq 1$ on K and hence $K \subset \{g > \frac{1}{2}\}$. Choose $\phi \in C_c(X, [0, 1])$ such that $\phi = 1$ on K and $\operatorname{supp}(\phi) \subset \{g > \frac{1}{2}\}$ and define $f_0 := 1 - \phi$. Then $f_0 = 0$ on K, $f_0 = 1$ if $g \leq \frac{1}{2}$ and therefore,

$$f_0 + f_1 + \dots + f_n = f_0 + g > 0$$

on X. The desired partition of unity may be constructed as

$$h_j(x) = \frac{f_j(x)}{f_0(x) + \dots + f_n(x)}.$$

Indeed $\operatorname{supp}(h_j) = \operatorname{supp}(f_j) \subset U_j$, $h_j \in C_c(X, [0, 1])$ and on K,

$$h_1 + \dots + h_n = \frac{f_1 + \dots + f_n}{f_0 + f_1 + \dots + f_n} = \frac{f_1 + \dots + f_n}{f_1 + \dots + f_n} = 1.$$

■

Proposition 12.17. *Let (X, τ) be a locally compact and σ – compact Hausdorff space. Suppose that $\mathcal{U} \subset \tau$ is an open cover of X. Then we may construct two locally finite open covers $\mathcal{V} = \{V_i\}_{i=1}^N$ and $\mathcal{W} = \{W_i\}_{i=1}^N$ of X ($N = \infty$ is allowed here) such that:*

1. $W_i \subset \bar{W}_i \subset V_i \subset \bar{V}_i$ *and* $\bar{V}_i$ *is compact for all* i.
2. *For each* i *there exist* $U \in \mathcal{U}$ *such that* $\bar{V}_i \subset U$.

Proof. By Remark 11.24, there exists an open cover of $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ of X such that $G_n \subset \bar{G}_n \subset G_{n+1}$. Then $X = \cup_{k=1}^{\infty}(\bar{G}_k \setminus \bar{G}_{k-1})$, where by convention $G_{-1} = G_0 = \emptyset$. For the moment fix $k \geq 1$. For each $x \in \bar{G}_k \setminus G_{k-1}$, let $U_x \in \mathcal{U}$ be chosen so that $x \in U_x$ and by Proposition 12.7 choose an open neighborhood N_x of x such that $\bar{N}_x \subset U_x \cap (G_{k+1} \setminus \bar{G}_{k-2})$, see Figure 12.4 below. Since $\{N_x\}_{x \in \bar{G}_k \setminus G_{k-1}}$ is an open cover of the compact set

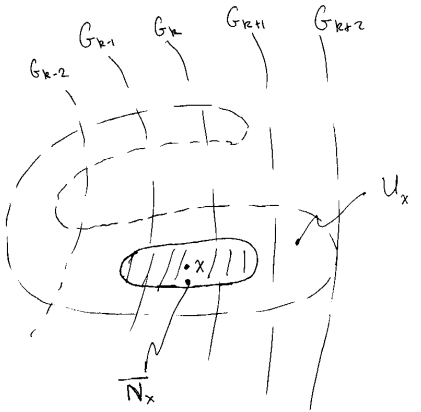

Fig. 12.4. Constructing the $\{W_i\}_{i=1}^{N}$.

$\bar{G}_k \setminus G_{k-1}$, there exist a finite subset $\Gamma_k \subset \{N_x\}_{x \in \bar{G}_k \setminus G_{k-1}}$ which also covers $\bar{G}_k \setminus G_{k-1}$. By construction, for each $W \in \Gamma_k$, there is a $U \in \mathcal{U}$ such that $\bar{W} \subset U \cap (G_{k+1} \setminus \bar{G}_{k-2})$. Apply Proposition 12.7 one more time to find, for each $W \in \Gamma_k$, an open set V_W such that $\bar{W} \subset V_W \subset \bar{V}_W \subset U \cap (G_{k+1} \setminus \bar{G}_{k-2})$. We now choose and enumeration $\{W_i\}_{i=1}^{N}$ of the countable open cover $\cup_{k=1}^{\infty}\Gamma_k$ of X and define $V_i = V_{W_i}$. Then the collection $\{W_i\}_{i=1}^{N}$ and $\{V_i\}_{i=1}^{N}$ are easily checked to satisfy all the conclusions of the proposition. In particular notice that for each k that the set of i's such that $V_i \cap G_k \neq \emptyset$ is finite. ■

Theorem 12.18 (Partitions of Unity in locally and σ – compact spaces). *Let* (X, τ) *be locally compact,* σ *– compact and Hausdorff and let* $\mathcal{U} \subset \tau$ *be an open cover of* X. *Then there exists a partition of unity of* $\{h_i\}_{i=1}^{N}$ *(*$N = \infty$ *is allowed here) subordinate to the cover* $\mathcal{U}$ *such that* $\operatorname{supp}(h_i)$ *is compact for all* i.

Proof. Let $\mathcal{V} = \{V_i\}_{i=1}^{N}$ and $\mathcal{W} = \{W_i\}_{i=1}^{N}$ be open covers of X with the properties described in Proposition 12.17. By Urysohn's Lemma 12.8, there

exists $f_i \prec V_i$ such that $f_i = 1$ on $\bar{W}_i$ for each i. As in the proof of Proposition 12.16 there are two methods to finish the proof.

Method 1. Define $h_1 = f_1$, h_j by Eq. (12.3) for all other j. Then as in Eq. (12.4)

$$1 - \sum_{j=1}^{N} h_j = \prod_{j=1}^{N}(1 - f_j) = 0$$

since for $x \in X$, $f_j(x) = 1$ for some j. As in the proof of Proposition 12.16, it is easily checked that $\{h_i\}_{i=1}^{N}$ is the desired partition of unity.

Method 2. Let $f := \sum_{i=1}^{N} f_i$, a locally finite sum, so that $f \in C(X)$. Since $\{W_i\}_{i=1}^{\infty}$ is a cover of X, $f \geq 1$ on X so that $1/f \in C(X))$ as well. The functions $h_i := f_i/f$ for $i = 1, 2, \ldots, N$ give the desired partition of unity. ■

Lemma 12.19. *Let* (X, τ) *be a locally compact Hausdorff space.*

1. *A subset* $E \subset X$ *is closed iff* $E \cap K$ *is closed for all* $K \sqsubset\sqsubset X$.
2. *Let* $\{C_\alpha\}_{\alpha \in A}$ *be a locally finite collection of closed subsets of* X, *then* $C = \cup_{\alpha \in A} C_\alpha$ *is closed in* X. *(Recall that in general closed sets are only closed under finite unions.)*

Proof. 1. Since compact subsets of Hausdorff spaces are closed, $E \cap K$ is closed if E is closed and K is compact. Now suppose that $E \cap K$ is closed for all compact subsets $K \subset X$ and let $x \in E^c$. Since X is locally compact, there exists a precompact open neighborhood, V, of x.[1] By assumption $E \cap \bar{V}$ is closed so $x \in \left(E \cap \bar{V}\right)^c$ – an open subset of X. By Proposition 12.7 there exists an open set U such that $x \in U \subset \bar{U} \subset \left(E \cap \bar{V}\right)^c$, see Figure 12.5. Let $W := U \cap V$. Since

$$W \cap E = U \cap V \cap E \subset U \cap \bar{V} \cap E = \emptyset,$$

and W is an open neighborhood of x and $x \in E^c$ was arbitrary, we have shown E^c is open hence E is closed.

2. Let K be a compact subset of X and for each $x \in K$ let N_x be an open neighborhood of x such that $\#\{\alpha \in A : C_\alpha \cap N_x \neq \emptyset\} < \infty$. Since K is compact, there exists a finite subset $\Lambda \subset K$ such that $K \subset \cup_{x \in \Lambda} N_x$. Letting $\Lambda_0 := \{\alpha \in A : C_\alpha \cap K \neq \emptyset\}$, then

$$\#(\Lambda_0) \leq \sum_{x \in \Lambda} \#\{\alpha \in A : C_\alpha \cap N_x \neq \emptyset\} < \infty$$

and hence $K \cap (\cup_{\alpha \in A} C_\alpha) = K \cap (\cup_{\alpha \in \Lambda_0} C_\alpha)$. The set $(\cup_{\alpha \in \Lambda_0} C_\alpha)$ is a finite union of closed sets and hence closed. Therefore, $K \cap (\cup_{\alpha \in A} C_\alpha)$ is closed and by item 1. it follows that $\cup_{\alpha \in A} C_\alpha$ is closed as well. ■

[1] If X were a metric space we could finish the proof as follows. If there does not exist an open neighborhood of x which is disjoint from E, then there would exists $x_n \in E$ such that $x_n \to x$. Since $E \cap \bar{V}$ is closed and $x_n \in E \cap \bar{V}$ for all large n, it follows (see Exercise 6.4) that $x \in E \cap \bar{V}$ and in particular that $x \in E$. But we chose $x \in E^c$.

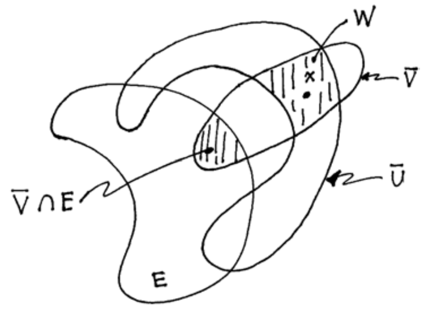

Fig. 12.5. Showing E^c is open.

Corollary 12.20. *Let (X,τ) be a locally compact and σ – compact Hausdorff space and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A} \subset \tau$ be an open cover of X. Then there exists a partition of unity of $\{h_\alpha\}_{\alpha\in A}$ subordinate to the cover $\mathcal{U}$ such that $\mathrm{supp}(h_\alpha) \subset U_\alpha$ for all $\alpha \in A$. (Notice that we do not assert that h_α has compact support. However if $\bar{U}_\alpha$ is compact then $\mathrm{supp}(h_\alpha)$ will be compact.)*

Proof. By the σ – compactness of X, we may choose a countable subset, $\{\alpha_i\}_{i<N}$ ($N = \infty$ allowed here), of A such that $\{U_i := U_{\alpha_i}\}_{i<N}$ is still an open cover of X. Let $\{g_j\}_{j<N}$ be a partition of unity subordinate to the cover $\{U_i\}_{i<N}$ as in Theorem 12.18. Define $\tilde{\Gamma}_k := \{j : \mathrm{supp}(g_j) \subset U_k\}$ and $\Gamma_k := \tilde{\Gamma}_k \setminus \left(\cup_{j=1}^{k-1}\tilde{\Gamma}_k\right)$, where by convention $\tilde{\Gamma}_0 = \emptyset$. Then

$$\{i \in \mathbb{N} : i < N\} = \bigcup_{k=1}^{\infty} \tilde{\Gamma}_k = \coprod_{k=1}^{\infty} \Gamma_k.$$

If $\Gamma_k = \emptyset$ let $h_k := 0$ otherwise let $h_k := \sum_{j\in\Gamma_k} g_j$, a locally finite sum. Then $\sum_{k=1}^{\infty} h_k = \sum_{j=1}^{N} g_j = 1$ and the sum $\sum_{k=1}^{\infty} h_k$ is still locally finite. (Why?) Now for $\alpha = \alpha_k \in \{\alpha_i\}_{i=1}^N$, let $h_\alpha := h_k$ and for $\alpha \notin \{\alpha_i\}_{i=1}^N$ let $h_\alpha := 0$. Since

$$\{h_k \neq 0\} = \cup_{j\in\Gamma_k}\{g_j \neq 0\} \subset \cup_{j\in\Gamma_k}\mathrm{supp}(g_j) \subset U_k$$

and, by Item 2. of Lemma 12.19, $\cup_{j\in\Gamma_k}\mathrm{supp}(g_j)$ is closed, we see that

$$\mathrm{supp}(h_k) = \overline{\{h_k \neq 0\}} \subset \cup_{j\in\Gamma_k}\mathrm{supp}(g_j) \subset U_k.$$

Therefore $\{h_\alpha\}_{\alpha\in A}$ is the desired partition of unity. ■

Corollary 12.21. *Let (X,τ) be a locally compact and σ – compact Hausdorff space and A, B be disjoint closed subsets of X. Then there exists $f \in C(X,[0,1])$ such that $f = 1$ on A and $f = 0$ on B. In fact f can be chosen so that $\mathrm{supp}(f) \subset B^c$.*

Proof. Let $U_1 = A^c$ and $U_2 = B^c$, then $\{U_1, U_2\}$ is an open cover of X. By Corollary 12.20 there exists $h_1, h_2 \in C(X,[0,1])$ such that $\mathrm{supp}(h_i) \subset U_i$ for $i = 1,2$ and $h_1 + h_2 = 1$ on X. The function $f = h_2$ satisfies the desired properties. ■

12.3 $C_0(X)$ and the Alexanderov Compactification

Definition 12.22. *Let (X,τ) be a topological space. A continuous function $f : X \to \mathbb{C}$ is said to* ***vanish at infinity*** *if $\{|f| \geq \varepsilon\}$ is compact in X for all $\varepsilon > 0$. The functions, $f \in C(X)$, vanishing at infinity will be denoted by $C_0(X)$.*

Proposition 12.23. *Let X be a topological space, $BC(X)$ be the space of bounded continuous functions on X with the supremum norm topology. Then*

1. *$C_0(X)$ is a closed subspace of $BC(X)$.*
2. *If we further assume that X is a locally compact Hausdorff space, then $C_0(X) = \overline{C_c(X)}$.*

Proof.

1. If $f \in C_0(X)$, $K_1 := \{|f| \geq 1\}$ is a compact subset of X and therefore $f(K_1)$ is a compact and hence bounded subset of $\mathbb{C}$ and so $M := \sup_{x\in K_1}|f(x)| < \infty$. Therefore $\|f\|_\infty \leq M \vee 1 < \infty$ showing $f \in BC(X)$. Now suppose $f_n \in C_0(X)$ and $f_n \to f$ in $BC(X)$. Let $\varepsilon > 0$ be given and choose n sufficiently large so that $\|f - f_n\|_\infty \leq \varepsilon/2$. Since
$$|f| \leq |f_n| + |f - f_n| \leq |f_n| + \|f - f_n\|_\infty \leq |f_n| + \varepsilon/2,$$
$$\{|f| \geq \varepsilon\} \subset \{|f_n| + \varepsilon/2 \geq \varepsilon\} = \{|f_n| \geq \varepsilon/2\}.$$
Because $\{|f| \geq \varepsilon\}$ is a closed subset of the compact set $\{|f_n| \geq \varepsilon/2\}$, $\{|f| \geq \varepsilon\}$ is compact and we have shown $f \in C_0(X)$.
2. Since $C_0(X)$ is a closed subspace of $BC(X)$ and $C_c(X) \subset C_0(X)$, we always have $\overline{C_c(X)} \subset C_0(X)$. Now suppose that $f \in C_0(X)$ and let $K_n := \{|f| \geq \frac{1}{n}\} \sqsubset\sqsubset X$. By Lemma 12.8 we may choose $\phi_n \in C_c(X,[0,1])$ such that $\phi_n \equiv 1$ on K_n. Define $f_n := \phi_n f \in C_c(X)$. Then
$$\|f - f_n\|_u = \|(1-\phi_n)f\|_\infty \leq \frac{1}{n} \to 0 \text{ as } n \to \infty.$$
This shows that $f \in \overline{C_c(X)}$.

■

Proposition 12.24 (Alexanderov Compactification). *Suppose that (X,τ) is a non-compact locally compact Hausdorff space. Let $X^* = X \cup \{\infty\}$, where $\{\infty\}$ is a new symbol not in X. The collection of sets,*

$$\tau^* = \tau \cup \{X^* \setminus K : K \sqsubset\sqsubset X\} \subset 2^{X^*},$$

is a topology on X^* *and* (X^*, τ^*) *is a compact Hausdorff space. Moreover* $f \in C(X)$ *extends continuously to* X^* *iff* $f = g + c$ *with* $g \in C_0(X)$ *and* $c \in \mathbb{C}$ *in which case the extension is given by* $f(\infty) = c$.

Proof. 1. (τ^* is a topology.) Let $\mathcal{F} := \{F \subset X^* : X^* \setminus F \in \tau^*\}$, i.e. $F \in \mathcal{F}$ iff F is a compact subset of X or $F = F_0 \cup \{\infty\}$ with F_0 being a closed subset of X. Since the finite union of compact (closed) subsets is compact (closed), it is easily seen that $\mathcal{F}$ is closed under finite unions. Because arbitrary intersections of closed subsets of X are closed and closed subsets of compact subsets of X are compact, it is also easily checked that $\mathcal{F}$ is closed under arbitrary intersections. Therefore $\mathcal{F}$ satisfies the axioms of the closed subsets associated to a topology and hence τ^* is a topology.

2. ((X^*, τ^*) is a Hausdorff space.) It suffices to show any point $x \in X$ can be separated from ∞. To do this use Proposition 12.7 to find an open precompact neighborhood, U, of x. Then U and $V := X^* \setminus \bar{U}$ are disjoint open subsets of X^* such that $x \in U$ and $\infty \in V$.

3. ((X^*, τ^*) is compact.) Suppose that $\mathcal{U} \subset \tau^*$ is an open cover of X^*. Since $\mathcal{U}$ covers ∞, there exists a compact set $K \subset X$ such that $X^* \setminus K \in \mathcal{U}$. Clearly X is covered by $\mathcal{U}_0 := \{V \setminus \{\infty\} : V \in \mathcal{U}\}$ and by the definition of τ^* (or using (X^*, τ^*) is Hausdorff), $\mathcal{U}_0$ is an open cover of X. In particular $\mathcal{U}_0$ is an open cover of K and since K is compact there exists $\Lambda \subset\subset \mathcal{U}$ such that $K \subset \cup\{V \setminus \{\infty\} : V \in \Lambda\}$. It is now easily checked that $\Lambda \cup \{X^* \setminus K\} \subset \mathcal{U}$ is a finite subcover of X^*.

4. (Continuous functions on $C(X^*)$ statements.) Let $i : X \to X^*$ be the inclusion map. Then i is continuous and open, i.e. $i(V)$ is open in X^* for all V open in X. If $f \in C(X^*)$, then $g = f|_X - f(\infty) = f \circ i - f(\infty)$ is continuous on X. Moreover, for all $\varepsilon > 0$ there exists an open neighborhood $V \in \tau^*$ of ∞ such that

$$|g(x)| = |f(x) - f(\infty)| < \varepsilon \text{ for all } x \in V.$$

Since V is an open neighborhood of ∞, there exists a compact subset, $K \subset X$, such that $V = X^* \setminus K$. By the previous equation we see that $\{x \in X : |g(x)| \geq \varepsilon\} \subset K$, so $\{|g| \geq \varepsilon\}$ is compact and we have shown g vanishes at ∞.

Conversely if $g \in C_0(X)$, extend g to X^* by setting $g(\infty) = 0$. Given $\varepsilon > 0$, the set $K = \{|g| \geq \varepsilon\}$ is compact, hence $X^* \setminus K$ is open in X^*. Since $g(X^* \setminus K) \subset (-\varepsilon, \varepsilon)$ we have shown that g is continuous at ∞. Since g is also continuous at all points in X it follows that g is continuous on X^*. Now it $f = g + c$ with $c \in \mathbb{C}$ and $g \in C_0(X)$, it follows by what we just proved that defining $f(\infty) = c$ extends f to a continuous function on X^*. ■

Example 12.25. Let X be an uncountable set and τ be the discrete topology on X. Let $(X^* = X \cup \{\infty\}, \tau^*)$ be the one point compactification of X. The smallest dense subset of X^* is the uncountable set X. Hence X^* is a compact but non-separable and hence non-metrizable space.

The next proposition gathers a number of results involving countability assumptions which have appeared in the exercises.

Proposition 12.26 (Summary). *Let* (X, τ) *be a topological space.*

1. *If* (X, τ) *is second countable, then* (X, τ) *is separable; see Exercise 10.9.*
2. *If* (X, τ) *is separable and metrizable then* (X, τ) *is second countable; see Exercise 10.10.*
3. *If* (X, τ) *is locally compact and metrizable then* (X, τ) *is* σ – *compact iff* (X, τ) *is separable; see Exercises 11.9 and 11.10.*
4. *If* (X, τ) *is locally compact and second countable, then* (X, τ) *is* σ *- compact, see Exercise 11.7.*
5. *If* (X, τ) *is locally compact and metrizable, then* (X, τ) *is* σ – *compact iff* (X, τ) *is separable, see Exercises 11.8 and 11.9.*

12.4 Stone-Weierstrass Theorem

We now wish to generalize Theorem 8.34 to more general topological spaces. We will first need some definitions.

Definition 12.27. *Let* X *be a topological space and* $\mathcal{A} \subset C(X) = C(X, \mathbb{R})$ *or* $C(X, \mathbb{C})$ *be a collection of functions. Then*

1. $\mathcal{A}$ *is said to* ***separate points*** *if for all distinct points* $x, y \in X$ *there exists* $f \in \mathcal{A}$ *such that* $f(x) \neq f(y)$.
2. $\mathcal{A}$ *is an* ***algebra*** *if* $\mathcal{A}$ *is a vector subspace of* $C(X)$ *which is closed under pointwise multiplication.*
3. $\mathcal{A}$ *is called a* ***lattice*** *if* $f \vee g := \max(f, g)$ *and* $f \wedge g = \min(f, g) \in \mathcal{A}$ *for all* $f, g \in \mathcal{A}$.
4. $\mathcal{A} \subset C(X)$ *is closed under conjugation if* $\bar{f} \in \mathcal{A}$ *whenever* $f \in \mathcal{A}$.[2]

Remark 12.28. If X is a topological space such that $C(X, \mathbb{R})$ separates points then X is Hausdorff. Indeed if $x, y \in X$ and $f \in C(X, \mathbb{R})$ such that $f(x) \neq f(y)$, then $f^{-1}(J)$ and $f^{-1}(I)$ are disjoint open sets containing x and y respectively when I and J are disjoint intervals containing $f(x)$ and $f(y)$ respectively.

Lemma 12.29. *If* $\mathcal{A} \subset C(X, \mathbb{R})$ *is a closed algebra then* $|f| \in \mathcal{A}$ *for all* $f \in \mathcal{A}$ *and* $\mathcal{A}$ *is a lattice.*

Proof. Let $f \in \mathcal{A}$ and let $M = \sup_{x \in X} |f(x)|$. Using Theorem 8.34 or Exercise 12.10, there are polynomials $p_n(t)$ such that

$$\lim_{n \to \infty} \sup_{|t| \leq M} ||t| - p_n(t)| = 0.$$

[2] This is of course no restriction when $C(X) = C(X, \mathbb{R})$.

By replacing p_n by $p_n - p_n(0)$ if necessary we may assume that $p_n(0) = 0$. Since $\mathcal{A}$ is an algebra, it follows that $f_n = p_n(f) \in \mathcal{A}$ and $|f| \in \mathcal{A}$, because $|f|$ is the uniform limit of the f_n's. Since

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \text{ and}$$
$$f \wedge g = \frac{1}{2}(f + g - |f - g|),$$

we have shown $\mathcal{A}$ is a lattice. ■

Lemma 12.30. *Let $\mathcal{A} \subset C(X, \mathbb{R})$ be an algebra which separates points and $x, y \in X$ be distinct points such that*

$$\exists\ f, g \in \mathcal{A} \ \ni\ f(x) \neq 0 \text{ and } g(y) \neq 0. \tag{12.5}$$

Then

$$V := \{(f(x), f(y)) : f \in \mathcal{A}\} = \mathbb{R}^2. \tag{12.6}$$

Proof. It is clear that V is a non-zero subspace of $\mathbb{R}^2$. If $\dim(V) = 1$, then $V = \text{span}(a, b)$ with $a \neq 0$ and $b \neq 0$ by the assumption in Eq. (12.5). Since $(a, b) = (f(x), f(y))$ for some $f \in \mathcal{A}$ and $f^2 \in \mathcal{A}$, it follows that $(a^2, b^2) = (f^2(x), f^2(y)) \in V$ as well. Since $\dim V = 1$, (a, b) and (a^2, b^2) are linearly dependent and therefore

$$0 = \det\begin{pmatrix} a & a^2 \\ b & b^2 \end{pmatrix} = ab^2 - ba^2 = ab(b - a)$$

which implies that $a = b$. But this the implies that $f(x) = f(y)$ for all $f \in \mathcal{A}$, violating the assumption that $\mathcal{A}$ separates points. Therefore we conclude that $\dim(V) = 2$, i.e. $V = \mathbb{R}^2$. ■

Theorem 12.31 (Stone-Weierstrass Theorem). *Suppose X is a compact Hausdorff space and $\mathcal{A} \subset C(X, \mathbb{R})$ is a **closed** subalgebra which separates points. For $x \in X$ let*

$$\mathcal{A}_x := \{f(x) : f \in \mathcal{A}\} \text{ and}$$
$$\mathcal{I}_x = \{f \in C(X, \mathbb{R}) : f(x) = 0\}.$$

Then either one of the following two cases hold.

1. *$\mathcal{A}_x = \mathbb{R}$ for all $x \in X$, i.e. for all $x \in X$ there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.*[3]
2. *There exists a unique point $x_0 \in X$ such that $\mathcal{A}_{x_0} = \{0\}$.*

Moreover in case (1) $\mathcal{A} = C(X, \mathbb{R})$ and in case (2) $\mathcal{A} = \mathcal{I}_{x_0} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$.

[3] If $\mathcal{A}$ contains the constant function 1, then this hypothesis holds.

Proof. If there exists x_0 such that $\mathcal{A}_{x_0} = \{0\}$ (x_0 is unique since $\mathcal{A}$ separates points) then $\mathcal{A} \subset \mathcal{I}_{x_0}$. If such an x_0 exists let $\mathcal{C} = \mathcal{I}_{x_0}$ and if $\mathcal{A}_x = \mathbb{R}$ for all x, set $\mathcal{C} = C(X, \mathbb{R})$. Let $f \in \mathcal{C}$, then by Lemma 12.30, for all $x, y \in X$ such that $x \neq y$ there exists $g_{xy} \in \mathcal{A}$ such that $f = g_{xy}$ on $\{x, y\}$.[4] The basic idea of the proof is contained in the following identity,

$$f(z) = \inf_{x \in X} \sup_{y \in X} g_{xy}(z) \text{ for all } z \in X. \tag{12.7}$$

To prove this identity, let $g_x := \sup_{y \in X} g_{xy}$ and notice that $g_x \geq f$ since $g_{xy}(y) = f(y)$ for all $y \in X$. Moreover, $g_x(x) = f(x)$ for all $x \in X$ since $g_{xy}(x) = f(x)$ for all x. Therefore,

$$\inf_{x \in X} \sup_{y \in X} g_{xy} = \inf_{x \in X} g_x = f.$$

The rest of the proof is devoted to replacing the inf and the sup above by min and max over finite sets at the expense of Eq. (12.7) becoming only an approximate identity.

Claim. Given $\varepsilon > 0$ and $x \in X$ there exists $g_x \in \mathcal{A}$ such that $g_x(x) = f(x)$ and $f < g_x + \varepsilon$ on X.

To prove the claim, let V_y be an open neighborhood of y such that $|f - g_{xy}| < \varepsilon$ on V_y so in particular $f < \varepsilon + g_{xy}$ on V_y. By compactness, there exists $\Lambda \subset\subset X$ such that $X = \bigcup_{y \in \Lambda} V_y$. Set

$$g_x(z) = \max\{g_{xy}(z) : y \in \Lambda\},$$

then for any $y \in \Lambda$, $f < \varepsilon + g_{xy} < \varepsilon + g_x$ on V_y and therefore $f < \varepsilon + g_x$ on X. Moreover, by construction $f(x) = g_x(x)$, see Figure 12.6 below. We now will finish the proof of the theorem. For each $x \in X$, let U_x be a neighborhood of x such that $|f - g_x| < \varepsilon$ on U_x. Choose $\Gamma \subset\subset X$ such that $X = \bigcup_{x \in \Gamma} U_x$ and define

$$g = \min\{g_x : x \in \Gamma\} \in \mathcal{A}.$$

Then $f < g + \varepsilon$ on X and for $x \in \Gamma$, $g_x < f + \varepsilon$ on U_x and hence $g < f + \varepsilon$ on U_x. Since $X = \bigcup_{x \in \Gamma} U_x$, we conclude

$$f < g + \varepsilon \text{ and } g < f + \varepsilon \text{ on } X,$$

i.e. $|f - g| < \varepsilon$ on X. Since $\varepsilon > 0$ is arbitrary it follows that $f \in \bar{\mathcal{A}} = \mathcal{A}$. ■

Theorem 12.32 (Complex Stone-Weierstrass Theorem). *Let X be a compact Hausdorff space. Suppose $\mathcal{A} \subset C(X, \mathbb{C})$ is closed in the uniform topology, separates points, and is closed under conjugation. Then either $\mathcal{A} = C(X, \mathbb{C})$ or $\mathcal{A} = \mathcal{I}_{x_0}^{\mathbb{C}} := \{f \in C(X, \mathbb{C}) : f(x_0) = 0\}$ for some $x_0 \in X$.*

[4] If $\mathcal{A}_{x_0} = \{0\}$ and $x = x_0$ or $y = x_0$, then g_{xy} exists merely by the fact that $\mathcal{A}$ separates points.

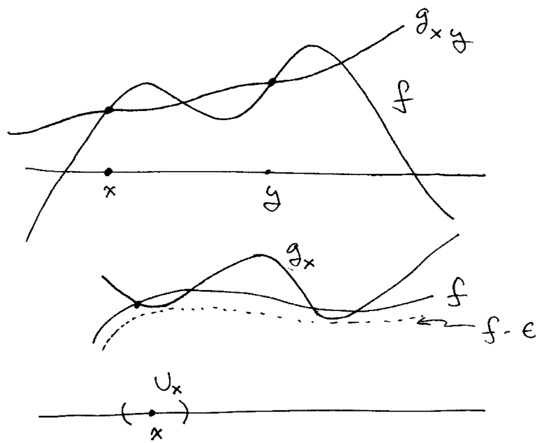

Fig. 12.6. Constructing the funtions g_x.

Proof. Since

$$\operatorname{Re} f = \frac{f+\bar{f}}{2} \text{ and } \operatorname{Im} f = \frac{f-\bar{f}}{2i},$$

Re f and Im f are both in $\mathcal{A}$. Therefore

$$\mathcal{A}_{\mathbb{R}} = \{\operatorname{Re} f, \operatorname{Im} f : f \in \mathcal{A}\}$$

is a real sub-algebra of $C(X,\mathbb{R})$ which separates points. Therefore either $\mathcal{A}_{\mathbb{R}} = C(X,\mathbb{R})$ or $\mathcal{A}_{\mathbb{R}} = \mathcal{I}_{x_0} \cap C(X,\mathbb{R})$ for some x_0 and hence $\mathcal{A} = C(X,\mathbb{C})$ or $\mathcal{I}_{x_0}^{\mathbb{C}}$ respectively. ■

As an easy application, Theorems 12.31 and 12.32 imply Theorem 8.34 and Corollary 8.36 respectively. Here are a couple of more applications.

Example 12.33. Let $f \in C([a,b])$ be a positive function which is injective. Then functions of the form $\sum_{k=1}^{N} a_k f^k$ with $a_k \in \mathbb{C}$ and $N \in \mathbb{N}$ are dense in $C([a,b])$. For example if $a = 1$ and $b = 2$, then one may take $f(x) = x^\alpha$ for any $\alpha \neq 0$, or $f(x) = e^x$, etc.

Exercise 12.4. Let (X,d) be a separable compact metric space. Show that $C(X)$ is also separable. **Hint:** Let $E \subset X$ be a countable dense set and then consider the algebra, $\mathcal{A} \subset C(X)$, generated by $\{d(x,\cdot)\}_{x\in E}$.

12.5 Locally Compact Version of Stone-Weierstrass Theorem

Theorem 12.34. *Let X be non-compact locally compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C_0(X,\mathbb{R})$ which separates points. Then either $\mathcal{A} = C_0(X,\mathbb{R})$ or there exists $x_0 \in X$ such that $\mathcal{A} = \{f \in C_0(X,\mathbb{R}) : f(x_0) = 0\}$.*

Proof. There are two cases to consider. Case 1. There is no point $x_0 \in X$ such that $\mathcal{A} \subset \{f \in C_0(X,\mathbb{R}) : f(x_0) = 0\}$. In this case let $X^* = X \cup \{\infty\}$ be the one point compactification of X. Because of Proposition 12.24 to each $f \in \mathcal{A}$ there exists a unique extension $\tilde{f} \in C(X^*,\mathbb{R})$ such that $f = \tilde{f}|_X$ and moreover this extension is given by $\tilde{f}(\infty) = 0$. Let $\widetilde{\mathcal{A}} := \{\tilde{f} \in C(X^*,\mathbb{R}) : f \in \mathcal{A}\}$. Then $\widetilde{\mathcal{A}}$ is a closed (you check) sub-algebra of $C(X^*,\mathbb{R})$ which separates points. An application of Theorem 12.31 implies $\widetilde{\mathcal{A}} = \{F \in C(X^*,\mathbb{R}) \ni F(\infty) = 0\}$ and therefore by Proposition 12.24 $\mathcal{A} = \{F|_X : F \in \widetilde{\mathcal{A}}\} = C_0(X,\mathbb{R})$. Case 2. There exists $x_0 \in X$ such $\mathcal{A} \subset \{f \in C_0(X,\mathbb{R}) : f(x_0) = 0\}$. In this case let $Y := X \setminus \{x_0\}$ and $\mathcal{A}_Y := \{f|_Y : f \in \mathcal{A}\}$. Since X is locally compact, one easily checks $\mathcal{A}_Y \subset C_0(Y,\mathbb{R})$ is a closed subalgebra which separates points. By Case 1. it follows that $\mathcal{A}_Y = C_0(Y,\mathbb{R})$. So if $f \in C_0(X,\mathbb{R})$ and $f(x_0) = 0$, $f|_Y \in C_0(Y,\mathbb{R}) = \mathcal{A}_Y$, i.e. there exists $g \in \mathcal{A}$ such that $g|_Y = f|_Y$. Since $g(x_0) = f(x_0) = 0$, it follows that $f = g \in \mathcal{A}$ and therefore $\mathcal{A} = \{f \in C_0(X,\mathbb{R}) : f(x_0) = 0\}$. ■

Example 12.35. Let $X = [0,\infty)$, $\lambda > 0$ be fixed, $\mathcal{A}$ be the algebra generated by $t \to e^{-\lambda t}$. So the general element $f \in \mathcal{A}$ is of the form $f(t) = p(e^{-\lambda t})$, where $p(x)$ is a polynomial. Since $\mathcal{A} \subset C_0(X,\mathbb{R})$ separates points and $e^{-\lambda t} \in \mathcal{A}$ is pointwise positive, $\bar{\mathcal{A}} = C_0(X,\mathbb{R})$. See Theorem 22.9 for an application of this result.

12.6 *More on Separation Axioms: Normal Spaces

(This section may safely be omitted on the first reading.)

Definition 12.36 ($T_0 - T_2$ Separation Axioms). *Let (X,τ) be a topological space. The topology τ is said to be:*

1. *T_0 if for $x \neq y$ in X there exists $V \in \tau$ such that $x \in V$ and $y \notin V$ or V such that $y \in V$ but $x \notin V$.*
2. *T_1 if for every $x, y \in X$ with $x \neq y$ there exists $V \in \tau$ such that $x \in V$ and $y \notin V$. Equivalently, τ is T_1 iff all one point subsets of X are closed.*[5]
3. *T_2 if it is Hausdorff.*

[5] If one point subsets are closed and $x \neq y$ in X then $V := \{x\}^c$ is an open set containing y but not x. Conversely if τ is T_1 and $x \in X$ there exists $V_y \in \tau$ such that $y \in V_y$ and $x \notin V_y$ for all $y \neq x$. Therefore, $\{x\}^c = \cup_{y\neq x} V_y \in \tau$.

Note T_2 implies T_1 which implies T_0. The topology in Example 12.1 is T_0 but not T_1. If X is a finite set and τ is a T_1 – topology on X then $\tau = 2^X$. To prove this let $x \in X$ be fixed. Then for every $y \neq x$ in X there exists $V_y \in \tau$ such that $x \in V_y$ while $y \notin V_y$. Thus $\{x\} = \cap_{y \neq x} V_y \in \tau$ showing τ contains all one point subsets of X and therefore all subsets of X. So we have to look to infinite sets for an example of T_1 topology which is not T_2.

Example 12.37. Let X be any infinite set and let $\tau = \{A \subset X : \#(A^c) < \infty\} \cup \{\emptyset\}$ – the so called **cofinite** topology. This topology is T_1 because if $x \neq y$ in X, then $V = \{x\}^c \in \tau$ with $x \notin V$ while $y \in V$. This topology however is not T_2. Indeed if $U, V \in \tau$ are open sets such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ then $U \subset V^c$. But this implies $\#(U) < \infty$ which is impossible unless $U = \emptyset$ which is impossible since $x \in U$.

The uniqueness of limits of sequences which occurs for Hausdorff topologies (see Remark 12.3) need not occur for T_1 – spaces. For example, let $X = \mathbb{N}$ and τ be the cofinite topology on X as in Example 12.37. Then $x_n = n$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ **for all** $x \in \mathbb{N}$. For the most part we will avoid these pathologies in the future by only considering Hausdorff topologies.

Definition 12.38 (Normal Spaces: T_4 – Separation Axiom). *A topological space (X, τ) is said to be **normal** or T_4 if:*

1. *X is Hausdorff and*
2. *if for any two closed disjoint subsets $A, B \subset X$ there exists disjoint open sets $V, W \subset X$ such that $A \subset V$ and $B \subset W$.*

Example 12.39. By Lemma 6.15 and Corollary 12.21 it follows that metric spaces and topological spaces which are locally compact, σ – compact and Hausdorff (in particular compact Hausdorff spaces) are normal. Indeed, in each case if A, B are disjoint closed subsets of X, there exists $f \in C(X, [0,1])$ such that $f = 1$ on A and $f = 0$ on B. Now let $U = \left\{f > \frac{1}{2}\right\}$ and $V = \{f < \frac{1}{2}\}$.

Remark 12.40. A topological space, (X, τ), is normal iff for any $C \subset W \subset X$ with C being closed and W being open there exists an open set $U \subset_o X$ such that

$$C \subset U \subset \bar{U} \subset W.$$

To prove this first suppose X is normal. Since W^c is closed and $C \cap W^c = \emptyset$, there exists disjoint open sets U and V such that $C \subset U$ and $W^c \subset V$. Therefore $C \subset U \subset V^c \subset W$ and since V^c is closed, $C \subset U \subset \bar{U} \subset V^c \subset W$.

For the converse direction suppose A and B are disjoint closed subsets of X. Then $A \subset B^c$ and B^c is open, and so by assumption there exists $U \subset_o X$ such that $A \subset U \subset \bar{U} \subset B^c$ and by the same token there exists $W \subset_o X$ such that $\bar{U} \subset W \subset \bar{W} \subset B^c$. Taking complements of the last expression implies

$$B \subset \bar{W}^c \subset W^c \subset \bar{U}^c.$$

Let $V = \bar{W}^c$. Then $A \subset U \subset_o X$, $B \subset V \subset_o X$ and $U \cap V \subset U \cap W^c = \emptyset$.

Theorem 12.41 (Urysohn's Lemma for Normal Spaces). *Let X be a normal space. Assume A, B are disjoint closed subsets of X. Then there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. To make the notation match Lemma 12.8, let $U = A^c$ and $K = B$. Then $K \subset U$ and it suffices to produce a function $f \in C(X, [0,1])$ such that $f = 1$ on K and $\mathrm{supp}(f) \subset U$. The proof is now identical to that for Lemma 12.8 except we now use Remark 12.40 in place of Proposition 12.7. ∎

Theorem 12.42 (Tietze Extension Theorem). *Let (X, τ) be a normal space, D be a closed subset of X, $-\infty < a < b < \infty$ and $f \in C(D, [a,b])$. Then there exists $F \in C(X, [a,b])$ such that $F|_D = f$.*

Proof. The proof is identical to that of Theorem 7.4 except we now use Theorem 12.41 in place of Lemma 6.15. ∎

Corollary 12.43. *Suppose that X is a normal topological space, $D \subset X$ is closed, $F \in C(D, \mathbb{R})$. Then there exists $F \in C(X)$ such that $F|_D = f$.*

Proof. Let $g = \arctan(f) \in C(D, (-\frac{\pi}{2}, \frac{\pi}{2}))$. Then by the Tietze extension theorem, there exists $G \in C(X, [-\frac{\pi}{2}, \frac{\pi}{2}])$ such that $G|_D = g$. Let $B := G^{-1}(\{-\frac{\pi}{2}, \frac{\pi}{2}\}) \sqsubset X$, then $B \cap D = \emptyset$. By Urysohn's lemma (Theorem 12.41) there exists $h \in C(X, [0,1])$ such that $h \equiv 1$ on D and $h = 0$ on B and in particular $hG \in C(D, (-\frac{\pi}{2}, \frac{\pi}{2}))$ and $(hG)|_D = g$. The function $F := \tan(hG) \in C(X)$ is an extension of f. ∎

Theorem 12.44 (Urysohn Metrization Theorem for Normal Spaces). *Every second countable normal space, (X, τ), is metrizable, i.e. there is a metric ρ on X such that $\tau = \tau_\rho$. Moreover, ρ may be chosen so that X is isometric to a subset $Q_0 \subset Q$ (Q is as in Notation 12.10) equipped with the metric d in Eq. (12.2). In this metric X is totally bounded and hence the completion of X (which is isometric to $\bar{Q}_0 \subset Q$) is compact.*

Proof. (The proof here will be very similar to the proof of Theorem 12.13.) Let $\mathcal{B}$ be a countable base for τ and set

$$\Gamma := \{(U,V) \in \mathcal{B} \times \mathcal{B} \mid \bar{U} \subset V\}.$$

To each $O \in \tau$ and $x \in O$ there exist $(U,V) \in \Gamma$ such that $x \in U \subset V \subset O$. Indeed, since $\mathcal{B}$ is a base for τ, there exists $V \in \mathcal{B}$ such that $x \in V \subset O$. Because $\{x\} \cap V^c = \emptyset$, there exists disjoint open sets $\widetilde{U}$ and W such that $x \in \widetilde{U}$, $V^c \subset W$ and $\widetilde{U} \cap W = \emptyset$. Choose $U \in \mathcal{B}$ such that $x \in U \subset \widetilde{U}$. Since $U \subset \widetilde{U} \subset W^c$, $\overline{U} \subset W^c \subset V$ and hence $(U,V) \in \Gamma$. See Figure 12.7 below. In particular this shows that

$$\mathcal{B}_0 := \{U \in \mathcal{B} : (U,V) \in \Gamma \text{ for some } V \in \mathcal{B}\}$$

is still a base for τ.

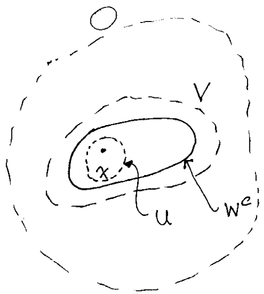

Fig. 12.7. Constructing $(U, V) \in \Gamma$.

If Γ is a finite set, the previous comment shows that τ only has a finite number of elements as well. Since (X, τ) is Hausdorff, it follows that X is a finite set. Letting $\{x_n\}_{n=1}^N$ be an enumeration of X, define $T : X \to Q$ by $T(x_n) = e_n$ for $n = 1, 2, \ldots, N$ where $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$, with the 1 occurring in the n^{th} spot. Then $\rho(x, y) := d(T(x), T(y))$ for $x, y \in X$ is the desired metric.

So we may now assume that Γ is an infinite set and let $\{(U_n, V_n)\}_{n=1}^\infty$ be an enumeration of Γ. By Urysohn's Lemma for normal spaces (Theorem 12.41) there exists $f_{U,V} \in C(X, [0, 1])$ such that $f_{U,V} = 0$ on $\bar{U}$ and $f_{U,V} = 1$ on V^c. Let $\mathcal{F} := \{f_{U,V} \mid (U, V) \in \Gamma\}$ and set $f_n := f_{U_n,V_n}$ – an enumeration of $\mathcal{F}$. The proof that

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x) - f_n(y)|$$

is the desired metric on X now follows exactly as the corresponding argument in the proof of Theorem 12.13. ■

12.7 Exercises

Exercise 12.5. Prove Theorem 12.9. **Hints:**

1. By Proposition 12.7, there exists a precompact open set V such that $K \subset V \subset \bar{V} \subset U$. Now suppose that $f : K \to [0, \alpha]$ is continuous with $\alpha \in (0, 1]$ and let $A := f^{-1}([0, \frac{1}{3}\alpha])$ and $B := f^{-1}([\frac{2}{3}\alpha, 1])$. Appeal to Lemma 12.8 to find a function $g \in C(X, [0, \alpha/3])$ such that $g = \alpha/3$ on B and $\text{supp}(g) \subset V \setminus A$.
2. Now follow the argument in the proof of Theorem 7.4 to construct $F \in C(X, [a, b])$ such that $F|_K = f$.

3. For $c \in [a, b]$, choose $\phi \prec U$ such that $\phi = 1$ on K and replace F by $F_c := \phi F + (1 - \phi)c$.

Exercise 12.6 (Sterographic Projection). Let $X = \mathbb{R}^n$, $X^* := X \cup \{\infty\}$ be the one point compactification of X, $S^n := \{y \in \mathbb{R}^{n+1} : |y| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$ and $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Define $f : S^n \to X^*$ by $f(N) = \infty$, and for $y \in S^n \setminus \{N\}$ let $f(y) = b \in \mathbb{R}^n$ be the unique point such that $(b, 0)$ is on the line containing N and y, see Figure 12.8 below. Find a formula for f and show $f : S^n \to X^*$ is a homeomorphism. (So the one point compactification of $\mathbb{R}^n$ is homeomorphic to the n sphere.)

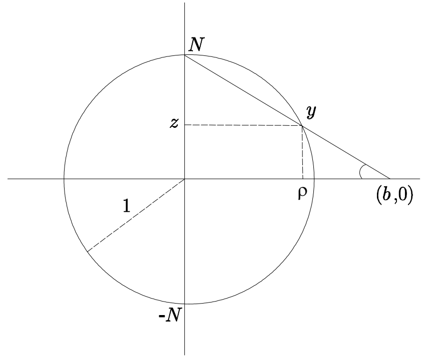

Fig. 12.8. Sterographic projection and the one point compactification of $\mathbb{R}^n$.

Exercise 12.7. Let (X, τ) be a locally compact Hausdorff space. Show (X, τ) is separable iff (X^*, τ^*) is separable.

Exercise 12.8. Show by example that there exists a locally compact metric space (X, d) such that the one point compactification, $(X^* := X \cup \{\infty\}, \tau^*)$, is **not** metrizable. **Hint**: use exercise 12.7.

Exercise 12.9. Suppose (X, d) is a locally compact and σ – compact metric space. Show the one point compactification, $(X^* := X \cup \{\infty\}, \tau^*)$, is metrizable.

Exercise 12.10. Let $M < \infty$, show there are polynomials $p_n(t)$ such that

$$\lim_{n\to\infty} \sup_{|t|\le M} ||t| - p_n(t)| = 0$$

using the following outline.

1. Let $f(x) = \sqrt{1-x}$ for $|x| \le 1$ and use Taylor's theorem with integral remainder (see Eq. A.15 of Appendix A), or analytic function theory if you know it, to show there are constants[6] $c_n > 0$ for $n \in \mathbb{N}$ such that
$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n \text{ for all } |x| < 1. \tag{12.8}$$
2. Let $q_m(x) := 1 - \sum_{n=1}^{m} c_n x^n$. Use (12.8) to show $\sum_{n=1}^{\infty} c_n = 1$ and conclude from this that
$$\lim_{m\to\infty} \sup_{|x|\le 1} |\sqrt{1-x} - q_m(x)| = 0. \tag{12.9}$$
3. Let $1 - x = t^2/M^2$, i.e. $x = 1 - t^2/M^2$, then
$$\lim_{m\to\infty} \sup_{|t|\le M} \left| \frac{|t|}{M} - q_m(1 - t^2/M^2)\right| = 0$$
so that $p_m(t) := M q_m(1 - t^2/M^2)$ are the desired polynomials.

Exercise 12.11. Given a continuous function $f : \mathbb{R} \to \mathbb{C}$ which is 2π - periodic and $\varepsilon > 0$. Show there exists a trigonometric polynomial, $p(\theta) = \sum_{n=-N}^{n} \alpha_n e^{in\theta}$, such that $|f(\theta) - P(\theta)| < \varepsilon$ for all $\theta \in \mathbb{R}$. **Hint:** show that there exists a unique function $F \in C(S^1)$ such that $f(\theta) = F(e^{i\theta})$ for all $\theta \in \mathbb{R}$.

Remark 12.45. Exercise 12.11 generalizes to 2π – periodic functions on $\mathbb{R}^d$, i.e. functions such that $f(\theta + 2\pi e_i) = f(\theta)$ for all $i = 1, 2, \ldots, d$ where $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$. A trigonometric polynomial $p(\theta)$ is a function of $\theta \in \mathbb{R}^d$ of the form
$$p(\theta) = \sum_{n\in\Gamma} \alpha_n e^{in\cdot\theta}$$
where Γ is a finite subset of $\mathbb{Z}^d$. The assertion is again that these trigonometric polynomials are dense in the 2π – periodic functions relative to the supremum norm.

[6] In fact $\alpha_n := \frac{(2n-3)!!}{2^n n!}$, but this is not needed.

13

Baire Category Theorem

Definition 13.1. *Let (X, τ) be a topological space. A set $E \subset X$ is said to be **nowhere dense** if $\left(\bar{E}\right)^o = \emptyset$ i.e. $\bar{E}$ has empty interior.*

Notice that E is nowhere dense is equivalent to
$$X = \left(\left(\bar{E}\right)^o\right)^c = \overline{\left(\bar{E}\right)^c} = \overline{(E^c)^o}.$$
That is to say E is nowhere dense iff E^c has dense interior.

13.1 Metric Space Baire Category Theorem

Theorem 13.2 (Baire Category Theorem). *Let (X, ρ) be a complete metric space.*

1. *If $\{V_n\}_{n=1}^{\infty}$ is a sequence of dense open sets, then $G := \bigcap_{n=1}^{\infty} V_n$ is dense in X.*
2. *If $\{E_n\}_{n=1}^{\infty}$ is a sequence of nowhere dense sets, then $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bar{E}_n \subsetneq X$ and in particular $X \neq \bigcup_{n=1}^{\infty} E_n$.*

Proof. 1) We must shows that $\bar{G} = X$ which is equivalent to showing that $W \cap G \neq \emptyset$ for all non-empty open sets $W \subset X$. Since V_1 is dense, $W \cap V_1 \neq \emptyset$ and hence there exists $x_1 \in X$ and $\varepsilon_1 > 0$ such that
$$\overline{B(x_1, \varepsilon_1)} \subset W \cap V_1.$$
Since V_2 is dense, $B(x_1, \varepsilon_1) \cap V_2 \neq \emptyset$ and hence there exists $x_2 \in X$ and $\varepsilon_2 > 0$ such that
$$\overline{B(x_2, \varepsilon_2)} \subset B(x_1, \varepsilon_1) \cap V_2.$$
Continuing this way inductively, we may choose $\{x_n \in X \text{ and } \varepsilon_n > 0\}_{n=1}^{\infty}$ such that

$$\overline{B(x_n, \varepsilon_n)} \subset B(x_{n-1}, \varepsilon_{n-1}) \cap V_n \ \forall n.$$

Furthermore we can clearly do this construction in such a way that $\varepsilon_n \downarrow 0$ as $n \uparrow \infty$. Hence $\{x_n\}_{n=1}^\infty$ is Cauchy sequence and $x = \lim_{n\to\infty} x_n$ exists in X since X is complete. Since $\overline{B(x_n, \varepsilon_n)}$ is closed, $x \in \overline{B(x_n, \varepsilon_n)} \subset V_n$ so that $x \in V_n$ for all n and hence $x \in G$. Moreover, $x \in \overline{B(x_1, \varepsilon_1)} \subset W \cap V_1$ implies $x \in W$ and hence $x \in W \cap G$ showing $W \cap G \neq \emptyset$. 2) The second assertion is equivalently to showing

$$\emptyset \neq \left(\bigcup_{n=1}^\infty \bar{E}_n \right)^c = \bigcap_{n=1}^\infty \left(\bar{E}_n\right)^c = \bigcap_{n=1}^\infty \left(E_n^c\right)^o.$$

As we have observed, E_n is nowhere dense is equivalent to $(E_n^c)^o$ being a dense open set, hence by part 1), $\bigcap_{n=1}^\infty (E_n^c)^o$ is dense in X and hence not empty. ■

13.2 Locally Compact Hausdorff Space Baire Category Theorem

Here is another version of the Baire Category theorem when X is a locally compact Hausdorff space.

Proposition 13.3. *Let X be a locally compact Hausdorff space.*

1. *If $\{V_n\}_{n=1}^\infty$ is a sequence of dense open sets, then $G := \bigcap_{n=1}^\infty V_n$ is dense in X.*
2. *If $\{E_n\}_{n=1}^\infty$ is a sequence of nowhere dense sets, then $X \neq \bigcup_{n=1}^\infty E_n$.*

Proof. As in the previous proof, the second assertion is a consequence of the first. To finish the proof, if suffices to show $G \cap W \neq \emptyset$ for all open sets $W \subset X$. Since V_1 is dense, there exists $x_1 \in V_1 \cap W$ and by Proposition 12.7 there exists $U_1 \subset_o X$ such that $x_1 \in U_1 \subset \bar{U}_1 \subset V_1 \cap W$ with $\bar{U}_1$ being compact. Similarly, there exists a non-empty open set U_2 such that $U_2 \subset \bar{U}_2 \subset U_1 \cap V_2$. Working inductively, we may find non-empty open sets $\{U_k\}_{k=1}^\infty$ such that $U_k \subset \bar{U}_k \subset U_{k-1} \cap V_k$. Since $\cap_{k=1}^n \bar{U}_k = \bar{U}_n \neq \emptyset$ for all n, the finite intersection characterization of $\bar{U}_1$ being compact implies that

$$\emptyset \neq \cap_{k=1}^\infty \bar{U}_k \subset G \cap W.$$

■

Definition 13.4. *A subset $E \subset X$ is **meager** or of the **first category** if $E = \bigcup_{n=1}^\infty E_n$ where each E_n is nowhere dense. And a set $R \subset X$ is called **residual** if R^c is meager.*

Remarks 13.5 *For those readers that already know some measure theory may want to think of meager as being the topological analogue of sets of measure 0 and residual as being the topological analogue of sets of full measure. (This analogy should not be taken too seriously, see Exercise 19.19.)*

1. *R is residual iff R contains a countable intersection of dense open sets. Indeed if R is a residual set, then there exists nowhere dense sets $\{E_n\}$ such that*
$$R^c = \cup_{n=1}^\infty E_n \subset \cup_{n=1}^\infty \bar{E}_n.$$
Taking complements of this equation shows that
$$\cap_{n=1}^\infty \bar{E}_n^c \subset R,$$
i.e. R contains a set of the form $\cap_{n=1}^\infty V_n$ with each V_n $(= \bar{E}_n^c)$ being an open dense subset of X.
Conversely, if $\cap_{n=1}^\infty V_n \subset R$ with each V_n being an open dense subset of X, then $R^c \subset \cup_{n=1}^\infty V_n^c$ and hence $R^c = \cup_{n=1}^\infty E_n$ where each $E_n = R^c \cap V_n^c$, is a nowhere dense subset of X.
2. *A countable union of meager sets is meager and any subset of a meager set is meager.*
3. *A countable intersection of residual sets is residual.*

Remarks 13.6 *The Baire Category Theorems may now be stated as follows. If X is a complete metric space or X is a locally compact Hausdorff space, then*

1. *all residual sets are dense in X and*
2. *X is not meager.*

It should also be remarked that incomplete metric spaces may be meager. For example, let $X \subset C([0,1])$ be the subspace of polynomial functions on $[0,1]$ equipped with the supremum norm. Then $X = \cup_{n=1}^\infty E_n$ where $E_n \subset X$ denotes the subspace of polynomials of degree less than or equal to n. You are asked to show in Exercise 13.1 below that E_n is nowhere dense for all n. Hence X is meager and the empty set is residual in X.

Here is an application of Theorem 13.2.

Theorem 13.7. *Let $\mathcal{N} \subset C([0,1],\mathbb{R})$ be the set of nowhere differentiable functions. (Here a function f is said to be differentiable at 0 if $f'(0) := \lim_{t\downarrow 0} \frac{f(t)-f(0)}{t}$ exists and at 1 if $f'(1) := \lim_{t\uparrow 0} \frac{f(1)-f(t)}{1-t}$ exists.) Then $\mathcal{N}$ is a residual set so the "generic" continuous functions is nowhere differentiable.*

Proof. If $f \notin \mathcal{N}$, then $f'(x_0)$ exists for some $x_0 \in [0,1]$ and by the definition of the derivative and compactness of $[0,1]$, there exists $n \in \mathbb{N}$ such that $|f(x) - f(x_0)| \leq n|x - x_0| \ \forall\, x \in [0,1]$. Thus if we define

$$E_n := \{f \in C([0,1]) : \exists\, x_0 \in [0,1] \ni \ |f(x) - f(x_0)| \leq n|x - x_0| \ \forall\, x \in [0,1]\},$$

then we have just shown $\mathcal{N}^c \subset E := \cup_{n=1}^\infty E_n$. So to finish the proof it suffices to show (for each n) E_n is a closed subset of $C([0,1],\mathbb{R})$ with empty interior. 1) To prove E_n is closed, let $\{f_m\}_{m=1}^\infty \subset E_n$ be a sequence of functions such that there exists $f \in C([0,1],\mathbb{R})$ such that $\|f - f_m\|_\infty \to 0$ as $m \to \infty$. Since $f_m \in E_n$, there exists $x_m \in [0,1]$ such that

$$|f_m(x) - f_m(x_m)| \le n|x - x_m| \ \forall\ x \in [0,1]. \tag{13.1}$$

Since $[0,1]$ is a compact metric space, by passing to a subsequence if necessary, we may assume $x_0 = \lim_{m\to\infty} x_m \in [0,1]$ exists. Passing to the limit in Eq. (13.1), making use of the uniform convergence of $f_n \to f$ to show $\lim_{m\to\infty} f_m(x_m) = f(x_0)$, implies

$$|f(x) - f(x_0)| \le n|x - x_0| \ \forall\ x \in [0,1]$$

and therefore that $f \in E_n$. This shows E_n is a closed subset of $C([0,1],\mathbb{R})$. 2) To finish the proof, we will show $E_n^0 = \emptyset$ by showing for each $f \in E_n$ and $\varepsilon > 0$ given, there exists $g \in C([0,1],\mathbb{R}) \setminus E_n$ such that $\|f - g\|_\infty < \varepsilon$. We now construct g. Since $[0,1]$ is compact and f is continuous there exists $N \in \mathbb{N}$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever $|y - x| < 1/N$. Let k denote the piecewise linear function on $[0,1]$ such that $k(\frac{m}{N}) = f(\frac{m}{N})$ for $m = 0, 1, \ldots, N$ and $k''(x) = 0$ for $x \notin \pi_N := \{m/N : m = 0, 1, \ldots, N\}$. Then it is easily seen that $\|f - k\|_u < \varepsilon/2$ and for $x \in (\frac{m}{N}, \frac{m+1}{N})$ that

$$|k'(x)| = \frac{|f(\frac{m+1}{N}) - f(\frac{m}{N})|}{\frac{1}{N}} < N\varepsilon/2.$$

We now make k "rougher" by adding a small wiggly function h which we define as follows. Let $M \in \mathbb{N}$ be chosen so that $4\varepsilon M > 2n$ and define h uniquely by $h(\frac{m}{M}) = (-1)^m \varepsilon/2$ for $m = 0, 1, \ldots, M$ and $h''(x) = 0$ for $x \notin \pi_M$. Then $\|h\|_\infty < \varepsilon$ and $|h'(x)| = 4\varepsilon M > 2n$ for $x \notin \pi_M$. See Figure 13.1 below. Finally define $g := k + h$. Then

$$\|f - g\|_\infty \le \|f - k\|_\infty + \|h\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and

$$|g'(x)| \ge |h'(x)| - |k'(x)| > 2n - n = n \ \forall x \notin \pi_M \cup \pi_N.$$

It now follows from this last equation and the mean value theorem that for any $x_0 \in [0,1]$,

$$\left|\frac{g(x) - g(x_0)}{x - x_0}\right| > n$$

for all $x \in [0,1]$ sufficiently close to x_0. This shows $g \notin E_n$ and so the proof is complete. ∎

Here is an application of the Baire Category Theorem in Proposition 13.3.

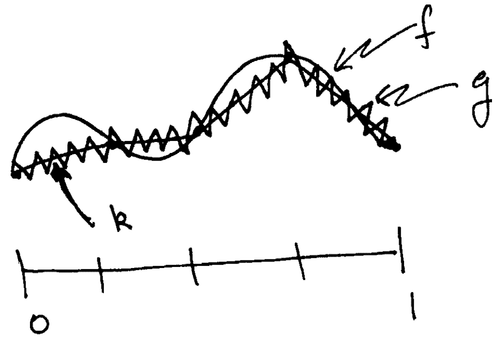

Fig. 13.1. Constgructing a rough approximation, g, to a continuous function f.

Proposition 13.8. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f'(x)$ exists for all $x \in \mathbb{R}$. Let*

$$U := \cup_{\varepsilon>0} \left\{ x \in \mathbb{R} : \sup_{|y|<\varepsilon} |f'(x+y)| < \infty \right\}.$$

Then U is a dense open set. (It is not true that $U = \mathbb{R}$ in general, see Example 31.35 below.)

Proof. It is easily seen from the definition of U that U is open. Let $W \subset_o \mathbb{R}$ be an open subset of $\mathbb{R}$. For $k \in \mathbb{N}$, let

$$\begin{aligned} E_k &:= \left\{ x \in W : |f(y) - f(x)| \le k\,|y - x| \text{ when } |y - x| \le \frac{1}{k} \right\} \\ &= \bigcap_{z:|z|\le k^{-1}} \{x \in W : |f(x+z) - f(x)| \le k\,|z|\}, \end{aligned}$$

which is a closed subset of $\mathbb{R}$ since f is continuous. Moreover, if $x \in W$ and $M = |f'(x)|$, then

$$\begin{aligned} |f(y) - f(x)| &= |f'(x)\,(y - x) + o\,(y - x)| \\ &\le (M + 1)\,|y - x| \end{aligned}$$

for y close to x. (Here $o(y - x)$ denotes a function such that $\lim_{y\to x} o(y - x)/(y - x) = 0$.) In particular, this shows that $x \in E_k$ for all k sufficiently large. Therefore $W = \cup_{k=1}^\infty E_k$ and since W is not meager by the Baire category Theorem in Proposition 13.3, some E_k has non-empty interior. That is there exists $x_0 \in E_k \subset W$ and $\varepsilon > 0$ such that

$$J := (x_0 - \varepsilon, x_0 + \varepsilon) \subset E_k \subset W.$$

For $x \in J$, we have $|f(x+z) - f(x)| \le k|z|$ provided that $|z| \le k^{-1}$ and therefore that $|f'(x)| \le k$ for $x \in J$. Therefore $x_0 \in U \cap W$ showing U is dense. ∎

Remark 13.9. This proposition generalizes to functions $f : \mathbb{R}^n \to \mathbb{R}^m$ in an obvious way.

For our next application of Theorem 13.2, let $X := BC^\infty((-1,1))$ denote the set of smooth functions f on $(-1,1)$ such that f and all of its derivatives are bounded. In the metric

$$\rho(f,g) := \sum_{k=0}^{\infty} 2^{-k} \frac{\left\|f^{(k)} - g^{(k)}\right\|_\infty}{1 + \left\|f^{(k)} - g^{(k)}\right\|_\infty} \text{ for } f, g \in X,$$

X becomes a complete metric space.

Theorem 13.10. *Given an increasing sequence of positive numbers $\{M_n\}_{n=1}^\infty$, the set*

$$\mathcal{F} := \left\{ f \in X : \limsup_{n\to\infty} \left| \frac{f^{(n)}(0)}{M_n} \right| \ge 1 \right\}$$

is dense in X. In particular, there is a dense set of $f \in X$ such that the power series expansion of f at 0 has zero radius of convergence.

Proof. Step 1. Let $n \in \mathbb{N}$. Choose $g \in C_c^\infty((-1,1))$ such that $\|g\|_\infty < 2^{-n}$ while $g'(0) = 2M_n$ and define

$$f_n(x) := \int_0^x dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \dots \int_0^{t_2} dt_1 g(t_1).$$

Then for $k < n$,

$$f_n^{(k)}(x) = \int_0^x dt_{n-k-1} \int_0^{t_{n-k-1}} dt_{n-k-2} \dots \int_0^{t_2} dt_1 g(t_1),$$

$f^{(n)}(x) = g'(x)$, $f_n^{(n)}(0) = 2M_n$ and $f_n^{(k)}$ satisfies

$$\left\|f_n^{(k)}\right\|_\infty \le \frac{2^{-n}}{(n-1-k)!} \le 2^{-n} \text{ for } k < n.$$

Consequently,

$$\begin{aligned}\rho(f_n, 0) &= \sum_{k=0}^{\infty} 2^{-k} \frac{\left\|f_n^{(k)}\right\|_\infty}{1 + \left\|f_n^{(k)}\right\|_\infty} \\ &\le \sum_{k=0}^{n-1} 2^{-k} 2^{-n} + \sum_{k=n}^{\infty} 2^{-k} \cdot 1 \le 2\left(2^{-n} + 2^{-n}\right) = 4 \cdot 2^{-n}.\end{aligned}$$

Thus we have constructed $f_n \in X$ such that $\lim_{n\to\infty} \rho(f_n, 0) = 0$ while $f_n^{(n)}(0) = 2M_n$ for all n. Step 2. The set

$$G_n := \cup_{m \ge n} \left\{ f \in X : \left|f^{(m)}(0)\right| > M_m \right\}$$

is a dense open subset of X. The fact that G_n is open is clear. To see that G_n is dense, let $g \in X$ be given and define $g_m := g + \varepsilon_m f_m$ where $\varepsilon_m := sgn(g^{(m)}(0))$. Then

$$\left|g_m^{(m)}(0)\right| = \left|g^{(m)}(0)\right| + \left|f_m^{(m)}(0)\right| \ge 2M_m > M_m \text{ for all } m.$$

Therefore, $g_m \in G_n$ for all $m \ge n$ and since

$$\rho(g_m, g) = \rho(f_m, 0) \to 0 \text{ as } m \to \infty$$

it follows that $g \in \bar{G}_n$. Step 3. By the Baire Category theorem, $\cap G_n$ is a dense subset of X. This completes the proof of the first assertion since

$$\begin{aligned}\mathcal{F} &= \left\{ f \in X : \limsup_{n\to\infty} \left| \frac{f^{(n)}(0)}{M_n} \right| \ge 1 \right\} \\ &= \cap_{n=1}^\infty \left\{ f \in X : \left| \frac{f^{(n)}(0)}{M_n} \right| \ge 1 \text{ for some } n \ge m \right\} \supset \cap_{n=1}^\infty G_n.\end{aligned}$$

Step 4. Take $M_n = (n!)^2$ and recall that the power series expansion for f near 0 is given by $\sum_{n=0}^\infty \frac{f_n(0)}{n!} x^n$. This series can not converge for any $f \in \mathcal{F}$ and any $x \ne 0$ because

$$\begin{aligned}\limsup_{n\to\infty} \left| \frac{f_n(0)}{n!} x^n \right| &= \limsup_{n\to\infty} \left| \frac{f_n(0)}{(n!)^2} n! x^n \right| \\ &= \limsup_{n\to\infty} \left| \frac{f_n(0)}{(n!)^2} \right| \cdot \lim_{n\to\infty} n! \left|x^n\right| = \infty\end{aligned}$$

where we have used $\lim_{n\to\infty} n! \left|x^n\right| = \infty$ and $\limsup_{n\to\infty} \left| \frac{f_n(0)}{(n!)^2} \right| \ge 1$. ∎

Remark 13.11. Given a sequence of real number $\{a_n\}_{n=0}^\infty$ there always exists $f \in X$ such that $f^{(n)}(0) = a_n$. To construct such a function f, let $\phi \in C_c^\infty(-1,1)$ be a function such that $\phi = 1$ in a neighborhood of 0 and $\varepsilon_n \in (0,1)$ be chosen so that $\varepsilon_n \downarrow 0$ as $n \to \infty$ and $\sum_{n=0}^\infty |a_n| \varepsilon_n^n < \infty$. The desired function f can then be defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \phi(x/\varepsilon_n) =: \sum_{n=0}^{\infty} g_n(x). \tag{13.2}$$

The fact that f is well defined and continuous follows from the estimate:

$$|g_n(x)| = \left|\frac{a_n}{n!}x^n \phi(x/\varepsilon_n)\right| \leq \frac{\|\phi\|_\infty}{n!} |a_n| \varepsilon_n^n$$

and the assumption that $\sum_{n=0}^\infty |a_n| \varepsilon_n^n < \infty$. The estimate

$$\begin{aligned}|g_n'(x)| &= \left|\frac{a_n}{(n-1)!}x^{n-1}\phi(x/\varepsilon_n) + \frac{a_n}{n!\varepsilon_n}x^n \phi'(x/\varepsilon_n)\right| \\ &\leq \frac{\|\phi\|_\infty}{(n-1)!} |a_n| \varepsilon_n^{n-1} + \frac{\|\phi'\|_\infty}{n!} |a_n| \varepsilon_n^n \\ &\leq (\|\phi\|_\infty + \|\phi'\|_\infty) |a_n| \varepsilon_n^n\end{aligned}$$

and the assumption that $\sum_{n=0}^\infty |a_n| \varepsilon_n^n < \infty$ shows $f \in C^1(-1,1)$ and $f'(x) = \sum_{n=0}^\infty g_n'(x)$. Similar arguments show $f \in C_c^k(-1,1)$ and $f^{(k)}(x) = \sum_{n=0}^\infty g_n^{(k)}(x)$ for all x and $k \in \mathbb{N}$. This completes the proof since, using $\phi(x/\varepsilon_n) = 1$ for x in a neighborhood of 0, $g_n^{(k)}(0) = \delta_{k,n} a_k$ and hence

$$f^{(k)}(0) = \sum_{n=0}^\infty g_n^{(k)}(0) = a_k.$$

13.3 Exercises

Exercise 13.1. Let $(X, \|\cdot\|)$ be an infinite dimensional normed space and $E \subset X$ be a finite dimensional subspace. Show that $E \subset X$ is nowhere dense.

Exercise 13.2. Now suppose that $(X, \|\cdot\|)$ is an infinite dimensional Banach space. Show that X can not have a countable **algebraic** basis. More explicitly, there is no countable subset $S \subset X$ such that every element $x \in X$ may be written as a **finite** linear combination of elements from S. **Hint:** make use of Exercise 13.1 and the Baire category theorem.